Chin. Ann. Math.
29B(5), 2008, 557–566
DOI: 10.1007/s11401-007-0044-z

Chinese Annals of Mathematics, Series B

The Classification of Proper Holomorphic Mappings Between Special Hartogs Triangles of Different Dimensions***

Zhihua CHEN*   Yang LIU**

Abstract The authors discuss the proper holomorphic mappings between special Hartogs triangles of different dimensions and obtain a corresponding classification theorem.

Keywords Proper holomorphic mapping, Hartogs triangle, Automorphism group
2000 MR Subject Classification 32H30, 32A22

1 Introduction

Proper holomorphic mapping theory dates from 1950s, and there are many good results on it (see [1–9]). The classification of proper holomorphic mappings (see the definition in [1]) is an important and difficult problem, especially between bounded domains of different dimensions (see [2–4]). Assume that $g, f : D_1 \to D_2$ are proper holomorphic mappings, D_1 and D_2 are bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively. If there exist $h_1 \in \mathrm{Aut}(D_1)$ and $h_2 \in \mathrm{Aut}(D_2)$ such that $f = h_2 \circ g \circ h_1$, then f and g are equivalent. Let B^n be the unit ball in $\mathbb{C}^n$. By a classical result of Alexander [5], every proper holomorphic self-mapping of B^n with $n \geq 2$ is equivalent to the identity mapping.

For $1 < n < N$, denote by $\mathrm{Rat}(B^n, B^N)$ the collection of all rational proper holomorphic mappings from B^n to B^N. For $n > 2$, the authors of [2] proved that there are only two equivalence classes in $\mathrm{Rat}(B^n, B^N)$. In [3], the authors got a new gap phenomenon for proper holomorphic mappings from B^n to B^N when $N \leq 3n - 4$. When $N < 2n - 1$ Huang Xiaojun gave the following classical theorem on the classification of proper holomorphic mappings from B^n to B^N.

Theorem A (see [4]) *Let B^n, B^m $(n > 1, n < m < 2n - 1)$ be the unit balls in $\mathbb{C}^n, \mathbb{C}^m$ respectively. Let $f : B^n \to B^m$ be a holomorphic proper mapping that is twice continuously differentiable up to the boundary. Then there exist $\sigma \in \mathrm{Aut}(B^n), \tau \in \mathrm{Aut}(B^m)$ such that*

$$\tau \circ f \circ \sigma(z_1, z_2, \cdots, z_n) = (z_1, z_2, \cdots, z_n, 0, \cdots, 0).$$

From the above theorem, we know that the holomorphic proper mapping from B^n to B^m is unique up the holomorphic automorphisms of B^n and B^m.

Manuscript received January 31, 2007. Published online August 27, 2008.
*Department of Mathematics, Tongji University, Shanghai 200092, China. E-mail: zzzhhc@tongji.edu.cn
**Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China; Department of Mathematics, Tongji University, Shanghai 200092, China. E-mail: liuyang4740@gmail.com
***Project supported by the National Natural Science Foundation of China (No. 10571135) and the Doctoral Program Foundation of the Ministry of Education of China (No. 20050240711).

In this paper, we will discuss the proper holomorphic mappings between special Hartogs triangles of different dimensions and obtain a corresponding classification theorem. The main idea is from [6], in which the authors gave the classification of proper holomorphic mappings between generalized Hartogs triangles of same dimensions.

Firstly, we give the definition of the special Hartogs triangles:

$$\Omega(n_1,m_1)=\Big\{(z,w)\in\mathbb{C}^{n_1+m_1}:0<\sum_{i=1}^{n_1}|z_i|^2<\sum_{j=1}^{m_1}|w_j|^2<1\Big\},$$
$$\Omega(n_2,m_2)=\Big\{(z',w')\in\mathbb{C}^{n_2+m_2}:0<\sum_{i=1}^{n_2}|z_i'|^2<\sum_{j=1}^{m_2}|w_j'|^2<1\Big\},$$

where

$$1<n_1<n_2<\min\{n_1+m_1-1,2n_1-1\},\quad 1<m_1<m_2<2m_1-1, \tag{1.1}$$

and we use the notations

$$|z|^2:=\sum_{i=1}^{n_1}|z_i|^2,\quad |w|^2:=\sum_{j=1}^{m_1}|w_j|^2,\quad |z'|^2:=\sum_{i=1}^{n_2}|z_i'|^2,\quad |w'|^2:=\sum_{j=1}^{m_2}|w_j'|^2.$$

The main result is as follows.

Theorem 1.1 *Let $\Omega(n_1,m_1)$ and $\Omega(n_2,m_2)$ be Hartogs triangles with the dimensional assumption* (1.1). *Let $F:\Omega(n_1,m_1)\to\Omega(n_2,m_2)$ be a proper holomorphic mapping that is twice continuously differentiable up to the boundary. Then there exist $\sigma\in\mathrm{Aut}(\Omega(n_1,m_1))$ and $\tau\in\mathrm{Aut}(\Omega(n_2,m_2))$, such that*

$$\tau\circ F\circ\sigma(z,w)=(z_1,\cdots,z_{n_1},\underbrace{0,\cdots,0}_{n_2-n_1},w_1,\cdots,w_{m_1},\underbrace{0,\cdots,0}_{m_2-m_1}).$$

2 Main Lemmas

Let $F=(F_1,F_2):\Omega(n_1,m_1)\to\Omega(n_2,m_2)$ be a proper holomorphic mapping, where $F_1=(f_1,\cdots,f_{n_2})$, $F_2=(f_{n_2+1},\cdots,f_{n_2+m_2})$.

Let $\partial\Omega(n_1,m_1)=A\cup B\cup C$, where

$$\begin{aligned}A&=\{\,(z,w)\in\mathbb{C}^{n_1+m_1}\mid |z|^2-|w|^2=0,|z|^2\neq0,|w|^2\neq1\},\\ B&=\{\,(z,w)\in\mathbb{C}^{n_1+m_1}\mid |w|^2=1\},\\ C&=\{\,0\in\mathbb{C}^{n_1+m_1}\}.\end{aligned}$$

It is obvious that $A\cap B=B\cap C=A\cap C=\emptyset$.

Similarly, $\partial\Omega(n_2,m_2)=A'\cup B'\cup C'$, where

$$\begin{aligned}A'&=\{\,(z',w')\in\mathbb{C}^{n_2+m_2}\mid |z'|^2-|w'|^2=0,|z'|^2\neq0,|w'|^2\neq1\},\\ B'&=\{\,(z',w')\in\mathbb{C}^{n_2+m_2}\mid |w'|^2=1\},\\ C'&=\{\,0\in\mathbb{C}^{n_2+m_2}\}.\end{aligned}$$

We also have $A'\cap B'=B'\cap C'=A'\cap C'=\emptyset$.

Lemma 2.1 *$F=(F_1,F_2):\Omega(n_1,m_1)\to\Omega(n_2,m_2)$ is a proper holomorphic mapping that is twice continuously differentiable up to the boundary. Then $F(B)\subset B'$.*

Proof Since F is proper, and twice continuously differentiable up to the boundary, we have $F(B)\subset\partial\Omega(n_2,m_2)$.

If there exists $x_0\in B$, such that $F(x_0)\in A'$, then by the continuity of F, there exist an open set U of x_0 in $\mathbb{C}^{n_1+m_1}$ and an open set V of $F(x_0)$ in $\mathbb{C}^{n_2+m_2}$, such that $F(U)\subset V$.

Let

$$S=\{(z,w)\in\partial\Omega(n_1,m_1):\operatorname{rank}(F')<n_1+m_1\},$$

where F' is the Jacobian matrix of F. Select $x_1\in B\backslash S$. Then we can find a suitable open set U_1 of x_1, such that $F|U_1:U_1\to F(U_1)$ has maximum rank. Then $(|z'|^2-|w'|^2)\circ F$ and $|w|^2-1$ are local defining functions of $U_1\cap B$.

The coefficient matrices of the Levi-forms of $|w|^2-1$ and $(|z'|^2-|w'|^2)\circ F$ are respectively

$$\begin{pmatrix}0&0\\0&I_{m_1}\end{pmatrix}\quad\text{and}\quad(F')^t\begin{pmatrix}I_{n_1}&0\\0&-I_{m_2}\end{pmatrix}(\overline{F'}),\tag{2.1}$$

where $(F')^t$ is an $(n_1+m_1)\times(n_2+m_2)$ matrix which defines on $U_1\cap B$ with maximum rank. Then $(F')^t$ can be expressed as the following

$$(F')^t=R(0,I_{m_1+n_1})V,\tag{2.2}$$

where R and V are $(n_1+m_1)\times(n_1+m_1)$ and $(n_2+m_2)\times(n_2+m_2)$ nonsingular matrices respectively, $I_{n_1+m_1}$ is the $(n_1+m_1)\times(n_1+m_1)$ unit matrix.

Let

$$V=\begin{pmatrix}V_1&V_2\\V_3&V_4\end{pmatrix},$$

where V_1 is an $(n_2+m_2-n_1-m_1)\times n_2$ matrix, V_4 is an $(n_1+m_1)\times(m_2)$ matrix.

Setting

$$\eta=(0,\cdots,0,b_1,\cdots,b_{n_1+m_1})_{1\times(n_2+m_2)},$$
$$\eta_1=(b_1,\cdots,b_{n_1+m_1})_{1\times(n_1+m_1)},$$

we consider the equation system:

$$\begin{cases}\eta_1V_3=(\underbrace{0,\cdots,0}_{n_2}),\\ \operatorname{grad}(|w|^2-1)(R^{-1})^t(0\quad I)_{(n_1+m_1)\times(n_2+m_2)}\eta^t=0,\end{cases}\tag{2.3}$$

where

$$\operatorname{grad}(|w|^2-1)=(0,\cdots,0,\overline{w}_1,\cdots,\overline{w}_{m_1}).$$

(2.3) has n_1+m_1 variables and n_2+1 equations. By the assumption condition (1.1), there exist nontrivial solutions of (2.3). Since η_1 is nontrivial and $\eta_1V_3=0$, we have $\eta_1V_4=(d_1,\cdots,d_{m_2})\neq 0$.

Set

$$\xi=\eta\begin{pmatrix}0\\I\end{pmatrix}_{(n_2+m_2)\times(n_1+m_1)}R^{-1}=(c_1,\cdots,c_{n_1+m_1}).$$

From (2.3), we get $\operatorname{grad}(|w|^2-1)\xi^t=0$. On the other hand, by the coefficient matrices of Levi-forms of $|w|^2-1$, we have

$$L(|w|^2-1)(\xi,\xi)=(c_1,\cdots,c_{n_1+m_1})\begin{pmatrix}0 & 0\\ 0 & I_{m_1}\end{pmatrix}(\overline{c}_1,\cdots,\overline{c}_{n_1+m_1})^T\geq 0. \tag{2.4}$$

From the expression of ξ,

$$\xi(F')^t=\eta\begin{pmatrix}0 & 0\\ V_3 & V_4\end{pmatrix}=(\underbrace{0,\cdots,0,d_1,\cdots,d_{m_2}}_{n_2+m_2}), \tag{2.5}$$

then

$$L((|z'|^2-|w'|^2)\circ F)(\xi,\xi)=\xi(F')^t\begin{pmatrix}I_{n_2} & 0\\ 0 & -I_{m_2}\end{pmatrix}(\overline{F'})\overline{\xi^t}=-\sum_{j=1}^{m_2}|d_j|^2<0. \tag{2.6}$$

But it is impossible. Because $|w|^2-1$ and $(|z'|^2-|w'|^2)\circ F$ are local defining functions of B, their Levi-forms $L(|w|^2-1), L((|z'|^2-|w'|^2)\circ F)$ considered as Hermitian quadratic form on $T^{(1,0)}(B)$ are only different by a positive factor, so (2.4), (2.6) deduce contradiction. Therefore the assumption $F(x_0)\in A'$ is impossible.

We now only need to verify that when $x_0\in B$, $F(x_0)\in C'$ is impossible. If $x_0\in B$, $F(x_0)=0$, and there exists an open set U of x_0, such that $F(B\cap U)\equiv 0$. Since $B\cap U$ is a real manifold of dimension $2(n_1+m_1)-1$ in $\Omega(n_1,m_1)$, we have $F\equiv 0$ on $\Omega(n_1,m_1)$, which is impossible by the definition of F. Otherwise there exists an open set U of x_0. By the continuity of F, $F((B\backslash S)\cap U)\cap A'\neq\emptyset$ is impossible, so we have proved that if $x_0\in B$, $F(x_0)\notin C'$. Thus, we complete the proof of Lemma 2.1.

Lemma 2.2 *$F=(F_1,F_2)$ is a proper holomorphic mapping as in Lemma* 2.1*. Then F_2 is independent of $z=(z_1,\cdots,z_{n_1})$.*

Proof Let $w=(w_1,\cdots,w_{m_1})\in B$. From Lemma 2.1, we have $F(B)\subset B'$, i.e.,

$$\sum_{j=1}^{m_2}\mid F_{n_2+j}(z,w)\mid^2=1. \tag{2.7}$$

Operating $\sum\limits_{k=1}^{n_1}\frac{\partial^2}{\partial z_k\partial\overline{z}_k}$ to (2.7), we have

$$\sum_{k=1}^{n_1}\sum_{j=1}^{m_2}\left|\frac{\partial F_{n_2+j}(z,w)}{\partial z_k}\right|^2=0.$$

Therefore

$$\frac{\partial F_{n_2+j}(z,w)}{\partial z_k}\equiv 0,\quad 1\leq k\leq n_1,\ 1\leq j\leq m_2$$

on B. Since B is a real manifold of dimension $2(n_1+m_1)-1$ in $\Omega(n_1,m_1)$, and $\frac{\partial F_{n_2+j}}{\partial z_k}$ is a holomorphic function, we have $\frac{\partial F_{n_2+j}}{\partial z_k}\equiv 0$, $1\leq k\leq n_1$, $1\leq j\leq m_2$, on $\Omega(n_1,m_1)$, which means that F_2 is independent of z.

3 Proof of Main Results

Proof of Theorem 1.1 Step 1 Fix w_0 such that $|w_0|^2 = 1$. From Lemmas 2.1 and 2.2, we can get $|F_2(w_0)|^2 = 1$. Set

$$F_{w_0} : \{z \in \mathbb{C}^{n_1} : 0 < |z|^2 < |w_0|^2 = 1\} \to \{z' \in \mathbb{C}^{n_2} : 0 < |z'|^2 < |F_2(w_0)|^2 = 1\},$$

which is a proper holomorphic mapping. Since $F_{w_0} : B \to B'$, we have $\forall z_n \to 0$, where $z_n \in \{z \in \mathbb{C}^{n_1} : 0 < |z|^2 < |w_0|^2 = 1\}$, $F_{w_0}(z_n) \to 0$. Otherwise, $F_{w_0}(z_n) \to B'$. By Hartogs extension theorem, we can extend F_{w_0} and we will still use F_{w_0} to denote this extended mapping:

$$F_{w_0} : \{z \in \mathbb{C}^{n_1} : |z|^2 < |w_0|^2 = 1\} \to \{z' \in \mathbb{C}^{n_2} : |z'|^2 \le |F_2(w_0)|^2 = 1\}.$$

If $F_{w_0}(z_n) \to B'$, then $|F_{w_0}(0)| = 1$, which contradicts the maximum principle. So $|F_{w_0}(0)| = 0$.

By Theorem A, we have

$$F_{w_0} = \theta_2(\underbrace{\theta_1 z}_{n_1}, \underbrace{0, \cdots, 0}_{n_2 - n_1}), \tag{3.1}$$

where $\theta_1 \in \mathrm{Aut}(B_{n_1}), \theta_2 \in \mathrm{Aut}(B_{n_2})$. Using the representation of automorphism of the unit ball, and $\theta_1(z^0) = 0, \theta_2(u^0) = 0, z^0 \in \mathbb{C}^{n_1}, u^0 \in \mathbb{C}^{n_2}$, we have

$$\theta_1 : (z_1, \cdots, z_{n_1}) \to (u_1, \cdots, u_{n_1}) \quad \text{and} \quad u_j = \frac{\sum\limits_{k=1}^{n_1} q_{jk}(z_k - z_k^0)}{\Big(1 - \sum\limits_{k=1}^{n_1} \overline{z_k^0} z_k\Big) R_1},$$

where

$$\begin{aligned} &z^0 = (z_1^0, \cdots, z_{n_1}^0) \in \mathbb{C}^{n_1}, \quad Q = (q_{jk})_{1 \le j,k \le n_1}, \\ &\overline{Q}(I - \overline{z^0}^t z^0) Q^t = I_{n_1}, \quad \overline{R}_1(1 - z^0 \overline{z^0}^t) R_1 = 1; \\ &\theta_2 : (u_1, \cdots, u_{n_1}, 0, \cdots, 0) \to (f_1, \cdots, f_{n_2}) \quad \text{and} \quad f_j = \frac{\sum\limits_{k=1}^{n_1} q_{jk}^*(u_k - u_k^0)}{\Big(1 - \sum\limits_{k=1}^{n_1} \overline{u_k^0} u_k\Big) R_2}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &u^0 = (u_1^0, \cdots, u_{n_2}^0) \in \mathbb{C}^{n_2}, \quad Q^* = (q_{jk}^*)_{1 \le j,k \le n_2}, \\ &\overline{Q^*}(I - \overline{u^0}^t u^0) Q^{*t} = I_{n_2}, \quad \overline{R}_2(1 - u^0 \overline{u^0}^t) R_2 = 1. \end{aligned} \tag{3.3}$$

Set

$$\lambda_1 = \Big(1 - \sum_{k=1}^{n_1} \overline{z_k^0} z_k\Big) R_1, \quad \lambda_2 = \Big(1 - \sum_{k=1}^{n_1} \overline{u_k^0} u_k\Big) R_2.$$

By the properties of $F_{w_0}(0, w_0) = 0$, we have

$$u^0 = \Big(\frac{-z^0 Q^t}{R_1}, 0\Big). \tag{3.4}$$

Then from (3.1), (3.2), $F^t_{w_0}$ can be expressed as follows:

$$\begin{aligned}F^t_{w_0} &= \frac{1}{\lambda_1\lambda_2}Q^*\begin{pmatrix}Q(z^t-{z^0}^t)+Q{z^0}^t(1-\overline{z^0}z^t)\\ 0\end{pmatrix}\\ &= \frac{1}{\lambda_1\lambda_2}Q^*\begin{pmatrix}(Qz^t-Q{z^0}^t\overline{z^0}z^t)\\ 0\end{pmatrix}=\frac{1}{\lambda_1\lambda_2}Q^*\begin{pmatrix}Q(I-{z^0}^t\overline{z^0})z^t\\ 0\end{pmatrix}\\ &= \frac{1}{\lambda_1\lambda_2}Q^*\begin{pmatrix}\overline{Q^t}^{-1}z^t\\ 0\end{pmatrix}. \qquad (3.5)\end{aligned}$$

By (3.3), (3.4),

$$\begin{aligned}\lambda_1\lambda_2 &= (1-\overline{u^0}u^t)R_2(1-\overline{z^0}z^t)R_1\\ &= (1-\overline{z^0}z^t)R_1R_2+\frac{\overline{z^0}\,\overline{Q^t}Q(z^t-{z^0}^t)R_2}{\overline{R}_1}\\ &= \frac{R_2}{\overline{R}_1}(\overline{R}_1(1-\overline{z^0}z^t)R_1+\overline{z^0}\,\overline{Q^t}Q(z^t-{z^0}^t)) \qquad (3.6)\end{aligned}$$

and

$$I-\overline{z^0}^tz^0=\overline{Q}^{-1}{Q^t}^{-1}.$$

Then

$$F^t_{w_0}=\frac{\overline{R}_1}{R_2}\frac{Q^*\begin{pmatrix}\overline{Q^t}^{-1}z^t\\ 0\end{pmatrix}}{\overline{R}_1(1-\overline{z^0}z^t)R_1+\overline{z^0}\,\overline{Q^t}Q(z^t-{z^0}^t)}. \qquad (3.7)$$

Without loss of generality, let

$$z^0=(z^0,0,\cdots,0),\quad z=(e^{i\theta},0,\cdots,0), \qquad (3.8)$$

since (U,I_{m_1}) is an automorphism of $\Omega(n_1,m_1)$, where U is a unitary matrix.

Using (3.2) again, we have

$$\begin{aligned}&|R_1|^2=\frac{1}{1-|z^0|^2},\\ &\overline{Q}\begin{pmatrix}1-|z^0|^2 & 0\\ 0 & I_{n_1-1}\end{pmatrix}Q^t=I, \qquad (3.9)\\ &Q^{-1}\overline{{Q^t}^{-1}}=(\overline{Q^t}Q)^{-1}=\begin{pmatrix}1-|z^0|^2 & 0\\ 0 & I_{n_1-1}\end{pmatrix}.\end{aligned}$$

Then

$$\overline{Q^t}Q=\begin{pmatrix}\frac{1}{1-|z^0|^2} & 0\\ 0 & I_{n_1-1}\end{pmatrix}. \qquad (3.10)$$

Similarly, using (3.3) and (3.4) again, we have

$$\begin{aligned}&|R_2|^2=\frac{1}{1-|u^0|^2}=\frac{1}{1-\frac{\overline{z^0}\,\overline{Q^t}Q{z^0}^t}{|R_1|^2}}=\frac{|R_1|^2}{|R_1|^2-|R_1|^2|z^0|^2}=|R_1|^2,\\ &\overline{Q^*}(I-\overline{u^0}^tu^0){Q^*}^t=\overline{Q^*}\left(I-\begin{pmatrix}\frac{-\overline{Q}\,\overline{{z^0}^t}}{\overline{R}_1}\\ 0\end{pmatrix}\left(\frac{-z^0Q^t}{R_1},0\right)\right){Q^*}^t=I, \qquad (3.11)\\ &({Q^*}^t\overline{Q^*})^{-1}=\overline{{Q^*}^{-1}}{{Q^*}^t}^{-1}=\begin{pmatrix}I_{n_1}-\frac{\overline{Q}\,\overline{{z^0}^t}z^0Q^t}{|R_1|^2} & 0\\ 0 & I_{n_2-n_1}\end{pmatrix}.\end{aligned}$$

Then

$$\overline{Q^*}^t Q^* = \begin{pmatrix} I_{n_1} - \frac{Qz^{0t}\overline{z^0}\,\overline{Q^t}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1}. \tag{3.12}$$

By (3.2),

$$Q\overline{Q^t} - Qz^{0t}\overline{z^0}\,\overline{Q^t} = I_{n_1} \quad \text{and} \quad Qz^{0t}\overline{z^0}\,\overline{Q^t} = Q\overline{Q^t} - I_{n_1}.$$

Then from (3.12), we have

$$\overline{Q^*}^t Q^* = \begin{pmatrix} I_{n_1} - \frac{Q\overline{Q^t} - I_{n_1}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1}. \tag{3.13}$$

On the other hand, by (3.8)–(3.10)

$$\begin{aligned} \overline{R}_1(1-\overline{z^0}z^t)R_1 + \overline{z^0}\,\overline{Q^t}Q(z^t - z^{0t}) &= |R_1|^2(1-\overline{z^0}z^t) + |R_1|^2(\overline{z^0}z^t - \overline{z^0}z^{0t}) \\ &= |R_1|^2(1-\overline{z^0}z^{0t}) = 1. \end{aligned} \tag{3.14}$$

Now we can rewrite (3.7) as follows:

$$F_{w_0}{}^t = \frac{\overline{R}_1}{R_2} Q^* \begin{pmatrix} \overline{Q^t}^{-1} z^t \\ 0 \end{pmatrix}. \tag{3.15}$$

So

$$\begin{aligned} |F_{w_0}|^2 &= (\overline{z}Q^{-1}, 0)\overline{Q^*}^t Q^* \begin{pmatrix} \overline{Q^{t-1}} z^t \\ 0 \end{pmatrix} \\ &= (\overline{z}Q^{-1}, 0) \begin{pmatrix} I_{n_1} - \frac{Q\overline{Q^t} - I_{n_1}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1} \begin{pmatrix} \overline{Q^{t-1}} z^t \\ 0 \end{pmatrix} \\ &= \overline{z}Q^{-1}\Big(I_{n_1} - \frac{Q\overline{Q^t} - I_{n_1}}{|R_1|^2}\Big)^{-1} \overline{Q^{t-1}} z^t, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &Q^{-1}\Big(I_{n_1} - \frac{Q\overline{Q^t} - I_{n_1}}{|R_1|^2}\Big)^{-1} \overline{Q^{t-1}} \\ &= \Big(\overline{Q^t}\Big(I_{n_1} - \frac{Q\overline{Q^t} - I_{n_1}}{|R_1|^2}\Big)Q\Big)^{-1} = \Big(\overline{Q^t}Q + \frac{\overline{Q^t}Q}{|R_1|^2} - \frac{\overline{Q^t}Q\overline{Q^t}Q}{|R_1|^2}\Big)^{-1} \\ &= \left(1 + \frac{1}{|R_1|^2}\begin{pmatrix} |R_1|^2 & 0 \\ 0 & I_{n_1-1} \end{pmatrix} - \frac{1}{|R_1|^2}\begin{pmatrix} |R_1|^4 & 0 \\ 0 & I_{n_1-1} \end{pmatrix}\right)^{-1} \\ &= \left(\frac{1}{|R_1|^2}\begin{pmatrix} |R_1|^2 & 0 \\ 0 & |R_1|^2 I_{n_1-1} \end{pmatrix}\right)^{-1} = I_{n_1}. \end{aligned} \tag{3.17}$$

Let

$$\gamma^* = \begin{pmatrix} \frac{1}{|R_1|} & 0 & 0 \\ 0 & \sqrt{1 - \frac{1}{|R_1|^2}}\, I_{n_1-1} & 0 \\ 0 & 0 & I_{n_2-n_1} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} \frac{1}{|R_1|} & 0 \\ 0 & \sqrt{1 - \frac{1}{|R_1|^2}}\, I_{n_1-1} \end{pmatrix}. \tag{3.18}$$

From (3.2), (3.10), (3.13),

$$\begin{aligned}\overline{Q^{*t}}Q^* &= \begin{pmatrix} I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1}\end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{|R_1|^2} & 0 & 0 \\ 0 & 1-\frac{1}{|R_1|^2}I_{n_1-1} & 0 \\ 0 & 0 & I_{n_2-n_1}\end{pmatrix}^{-1} \\ &= (\gamma^*\overline{\gamma^*}^t)^{-1} = \overline{\gamma^*}^{t\,-1}\gamma^{*\,-1}. \end{aligned} \tag{3.19}$$

Now, let $\overline{\mathbf{B}^t(w_0)} = Q^*\gamma^*$. Then

$$\mathbf{B}(w_0)\overline{\mathbf{B}^t(w_0)} = \overline{\gamma^*}^t\overline{Q^*}^tQ^*\gamma^* = I_{n_2},$$

which means that $\mathbf{B}(w_0) \in \mathbf{U}(n_2)$, where $\mathbf{U}(n_2)$ is the unitary group of degree n_2.

Since $|R_1|^2 = |R_2|^2$, we have $\frac{\overline{R_1}}{R_2} = \mathrm{e}^{\mathrm{i}\alpha}, \alpha \in \mathbb{R}$. Let $\overline{\mathbf{A}^t(w_0)} = \mathrm{e}^{\mathrm{i}\alpha}\gamma^{-1}\overline{Q^{t\,-1}}$. Then from (3.17) and the definition of γ, we have

$$\mathbf{A}^t(w_0)\overline{\mathbf{A}(w_0)} = I_{n_1},$$

which means that $\mathbf{A}(w_0) \in \mathbf{U}(n_1)$, where $\mathbf{U}(n_1)$ is the unitary group of degree n_1.

Now we can rewrite (3.15) as

$$F^t_{w_0} = \overline{\mathbf{B}^t(w_0)}\begin{pmatrix}\overline{\mathbf{A}^t(w_0)}z^t \\ 0\end{pmatrix}, \quad \text{i.e.,} \quad F_{w_0} = (z\mathbf{A}(w_0), 0)\mathbf{B}(w_0). \tag{3.20}$$

Step 2 From Lemma 2.2, F_2 is independent of z.

$$F_2 : \{w \in \mathbb{C}^{m_1} : 0 < |w|^2 < 1\} \to \{w' \in \mathbb{C}^{m_2} : 0 < |w'|^2 < 1\}$$

is a proper holomorphic mapping. Then by Hartogs extension theorem, we can extend F_2 so that

$$F_2 : \{w \in \mathbb{C}^{m_1} : |w|^2 < 1\} \to \{w' \in \mathbb{C}^{m_2} : |w'|^2 < 1\}$$

with $F_2(0) = 0$. Use Theorem A again,

$$F_2 = \theta'_2(\underbrace{\theta'_1 w}_{m_1}, \underbrace{0, \cdots, 0}_{m_2-m_1}).$$

By the proper properties of F_2, for every $w : |w| = 1, |F_2| = 1$. With the same argument used in Step 1, we have that θ'_1, θ'_2 are unitary transformations.

Now we can assume

$$\theta'(z) = z\mathbf{A}',\ \mathbf{A}' \in \mathbf{U}(m_1), \quad \theta'_1(w) = w\mathbf{B}',\ \mathbf{B}' \in \mathbf{U}(m_2),$$

where $z \in \mathbb{C}^{m_1},\ w \in \mathbb{C}^{m_2}$, $\mathbf{U}(m_1),\ \mathbf{U}(m_2)$ are unitary groups of degree m_1 and m_2 respectively. Then

$$F_2(w) = (w\mathbf{A}', 0)\mathbf{B}'. \tag{3.21}$$

Step 3 From the above expression of F_2, we have $|F_2(w)| = |w|, \forall\, w : |w| = l \le 1$. Now for a given $w : |w| = l \le 1$, set

$$F_1(z, w) : \{z \in \mathbb{C}^{n_1} : 0 < |z|^2 < |w|^2 = l^2\} \to \{z' \in \mathbb{C}^{n_2} : 0 < |z'|^2 < |F_2(w)|^2 = l^2\},$$

which is a proper holomorphic mapping. By Hartogs extension theorem, we can extend F_1 so that

$$F_1(z,w):\{z\in\mathbb{C}^{n_1}:|z|^2<|w|^2=l^2\}\to\{z'\in\mathbb{C}^{n_2}:|z'|^2<|F_2(w)|^2=l^2\}$$

is a proper holomorphic mapping with $F_1(0,w)=0$. It is easy to see that

$$F_1(z,w)=\frac{|F_2(w)|}{|w|}(z\mathbf{A}(w),0)\mathbf{B}(w)=\frac{l}{l}(z\mathbf{A}(w),0)\mathbf{B}(w)=(z\mathbf{A}(w),0)\mathbf{B}(w),$$

where $\mathbf{A}(w)\in\mathbf{U}(n_1)$, $\mathbf{B}(w)\in\mathbf{U}(n_2)$, $\mathbf{U}(n_1),\mathbf{U}(n_2)$ are unitary groups of degree n_1 and n_2 respectively. Since $F_1(z,w)$ is holomorphic on z and w, we have

$$\frac{\mathrm{d}}{\mathrm{d}\overline{w}}(\mathbf{A}(w),0)\mathbf{B}(w)=0.$$

As we know, $\mathbf{A}(w)\overline{\mathbf{A}^t(w)}=I_{n_1}$, $\mathbf{B}(w)\overline{\mathbf{B}^t(w)}=I_{n_2}$. Set

$$\mathbf{B}(w)=\begin{pmatrix}\mathbf{B}_1(w) & \mathbf{B}_2(w)\\ \mathbf{B}_3(w) & \mathbf{B}_4(w)\end{pmatrix},$$

where $\mathbf{B}_1(w)$ is an $n_1\times n_1$ matrix, $\mathbf{B}_2(w)$ is an $n_1\times(n_2-n_1)$ matrix. Then $(\mathbf{A}(w),0)\mathbf{B}(w)=(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))$, and

$$\begin{aligned}&(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))\overline{(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))}^t\\ &=\mathbf{A}(w)\mathbf{B}_1(w)\overline{(\mathbf{A}(w)\mathbf{B}_1(w))}^t+\mathbf{A}(w)\mathbf{B}_2(w)\overline{(\mathbf{A}(w)\mathbf{B}_2(w))}^t\\ &=\mathbf{A}(w)(\mathbf{B}_1(w)\overline{\mathbf{B}_1(w)}^t+\mathbf{B}_2(w)\overline{\mathbf{B}_2(w)}^t)\overline{\mathbf{A}(w)}^t=\mathbf{A}(w)\overline{\mathbf{A}(w)}^t=I_{n_1}.\end{aligned}$$

Set $(\mathbf{A}(w)\mathbf{B}_1(w))=(\varphi_{ij})_{1\le i,j\le n_1}$, and $(\mathbf{A}(w)\mathbf{B}_2(w))=(\psi_{i\alpha})_{1\le i\le n_1,1\le\alpha\le n_2-n_1}$, where φ_{ij}, $\psi_{i\alpha}$ are holomorphic dependent on $w_1,\cdots,w_{m_1}$, and

$$\begin{aligned}&\sum_{1\le i,j\le n_1}|\varphi_{ij}|^2+\sum_{1\le i\le n_1,1\le\alpha\le n_2-n_1}|\psi_{i\alpha}|^2\\ &=\mathrm{tr}[(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))\overline{(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))}^t]\\ &=n_1.\end{aligned}$$

Operating $\sum\limits_{k=1}^{m_1}\frac{\partial^2}{\partial w_k\partial\overline{w}_k}$ on the above equation, we have

$$\sum_{\substack{1\le i,j\le n_1\\1\le k\le m_1}}\Big|\frac{\partial\varphi_{ij}}{\partial w_k}\Big|^2+\sum_{\substack{1\le i\le n_1\\1\le\alpha\le n_2-n_1\\1\le k\le m_1}}\Big|\frac{\partial\psi_{i\alpha}}{\partial w_k}\Big|^2=0,$$

i.e., $\varphi_{ij},\psi_{i\alpha}$ are all independent of $w_1,\cdots,w_{m_1}$, so that $\mathbf{A}(w)\mathbf{B}_1(w)$ and $\mathbf{A}(w)\mathbf{B}_2(w)$ are all independent of $w_1,\cdots,w_{m_1}$. Hence, there exist constant matrices $A\in\mathbf{U}(n_1)$ and $B\in\mathbf{U}(n_2)$ such that $(\mathbf{A}(w),0)\mathbf{B}(w)=(\mathbf{A},0)\mathbf{B}$. Thus

$$F_1(z,w)=(z\mathbf{A},0)\mathbf{B}. \tag{3.22}$$

Step 4 Now it is obvious from (3.21), (3.22) that

$$F=(F_1,F_2):(z,w)\to(\underbrace{(z\mathbf{A},0)\mathbf{B}}_{n_2},\underbrace{(w\mathbf{A}',0)\mathbf{B}'}_{m_2}). \tag{3.23}$$

By the main theorem in [7], any proper holomorphic self-mapping of $\Omega(n_1, m_1)$ or $\Omega(n_2, m_2)$ is automorphism. Without loss of generality, let $\sigma \in \mathrm{Aut}(\Omega(n_1, m_1))$, $\tau \in \mathrm{Aut}(\Omega(n_2, m_2))$, such that

$$\sigma : (z, w) \to (z\mathbf{A}, w\mathbf{A}'),$$
$$\tau : (z', w') \to (z'\mathbf{B}, w'\mathbf{B}'),$$

where $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}'$ are unitary matrices of degree n_1, m_1, n_2, m_2 respectively. Then from (3.23), for the proper holomorphic mapping: $F : \Omega(n_1, m_1) \to \Omega(n_2, m_2)$, that is twice continuously differentiable up to the boundary, there exist σ, τ which are automorphisms of $\Omega(n_1, m_1)$ and $\Omega(n_2, m_2)$ respectively, such that

$$\tau \circ F \circ \sigma(z, w) = (z_1, \cdots, z_{n_1}, \underbrace{0, \cdots, 0}_{n_2 - n_1}, w_1, \cdots, w_{m_1}, \underbrace{0, \cdots, 0}_{m_2 - m_1}).$$

The proof of Theorem 1.1 is completed.

References

[1] Alexander, H., Holomorphic mappings from ball and polydisc, *Math. Ann.*, **209**, 1974, 245–256.

[2] Huang, X. J. and Ji, S. Y, Mapping B^n into B^{2n-1}, *Invent Math.,* **145**(2), 2001, 219–250.

[3] Huang, X. J., Ji, S. Y. and Xu, D. K., A new gap phenomenon for proper holomorphic maps from B^n into B^N, *Math. Res. Letts.*, **13**, 2006, 515–529.

[4] Huang, X. J., On a linearity problem for holomorphic maps between balls in complex spaces of different dimensions, *J. Diff. Geom.,* **51**, 1999, 13–33.

[5] Alexander, H., Proper holomorphic mappings in $\mathbb{C}^n$, *Indiana Univ. Math. J.*, **26**, 1977, 137–146.

[6] Chen, Z. H. and Liu, Y. C., A paper on the classification of the proper mapping between some generalized Hartogs triangles, *Chin. J. Cont. Math.*, **24**, 2003, 215–220.

[7] Chen, Z. H. and Xu, D. K., Rigidity of proper self-mapping on some kinds of generalized Hartogs triangles, *Acta. Math. Sinica*, **18**, 2002, 357–362.

[8] Chen, Z. H. and Xu, D. K., Proper holomorphic mappings between some non- smooth domains, *Chin. Ann. Math.*, **22B**(2), 2001, 177–182.

[9] Coffman, A. and Pan, Y. F., Proper holomorphic maps from domains in $\mathbb{C}^2$ with transverse circle action, *Chin. Ann. Math.*, **28B**(5), 2007, 533–542.