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Hardy-Type Inequalities on H-Type Groups and Anisotropic Heisenberg Groups**

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Abstract The author obtains some weighted Hardy-type inequalities on H-type groups and anisotropic Heisenberg groups. These inequalities generalize some recent results due to N. Garofalo, E. Lanconelli, I. Kombe and P. Niu et al.

Keywords Hardy-type inequalities, H-type groups, Anisotropic Heisenberg groups
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1 Introduction

It is well-known that Hardy's inequality and their generalizations play important roles in many areas of mathematics. The classical Hardy inequality is given by, for $n \geq 3$,

$$\int_{\mathbb{R}^n} |\nabla\Phi(x)|^2 \mathrm{d}x \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{|\Phi(x)|^2}{|x|^2} \mathrm{d}x, \tag{1.1}$$

where $\Phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

In recent years, there has been considerable development in the study of Hardy-type inequalities on sub-Riemannian spaces, and it turns out that these inequalities play important roles in the study of linear and nonlinear partial differential equations on the Carnot groups (see [1, 4–8] and the references therein). In the setting of Heisenberg group $\mathbb{H} = \mathbb{H}^n$, Garofalo and Lanconelli [1] obtained the following Hardy inequality,

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}}\Phi|^2 \mathrm{d}z\mathrm{d}t \geq \Big(\frac{Q-2}{2}\Big)^2 \int_{\mathbb{H}} \Big(\frac{|z|^2}{|z|^4+t^2}\Big)|\Phi|^2 \mathrm{d}z\mathrm{d}t, \tag{1.2}$$

where $\Phi \in C_0^\infty(\mathbb{H}\backslash\{0\})$, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}$, $\nabla_{\mathbb{H}}\Phi = (X_1\Phi, X_2\Phi, \cdots, X_n\Phi, Y_1\Phi, \cdots, Y_n\Phi)$ and $X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}$ are the left-invariant vector fields associated to the Kohn Laplacian. Niu et al [3] obtained the L^p version of the inequality (1.2) by using the Picone type identity for the p-sub-Laplacian. Recently, in [2] Kombe investigates the existence and the explicit determination of constant C and weight $w(x)$ on the Carnot group such that the Hardy-type inequality

$$\int_{\mathbb{G}} w(x)|\nabla_{\mathbb{G}}\Phi(x)|^2 \mathrm{d}x \geq C\int_{\mathbb{G}} w(x)|\Phi(x)|^2 \mathrm{d}x \tag{1.3}$$

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holds for all $\Phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Here the special weight $w(x)$ is related to the fundamental solution of sub-Laplacian $\triangle_{\mathbb{G}}$ on the Carnot group $\mathbb{G}$.

In this paper, we obtain an L^p version of the weighted Hardy-type inequality (1.3) on H-type groups and anisotropic Heisenberg groups. Our result is an extension of the corresponding inequalities in [1–3]. We mention that a similar Hardy inequality on H-type groups was proved in a recent paper [9]. However their conditions on the power of the weight is different from ours (see Remark 3.2). In particular we answer their question on the best constant. We also note that there are some Hardy-type inequalities on the Heisenberg groups in [4]. Here we prove a similar Hardy-type inequality for all H-type groups by using different methods based on the potential analysis for the weighted p-sub-Laplacian.

The plan of the paper is as follows. In Section 2 we recall some notations and some results about the fundamental solution associated with a weighted p-sub-Laplacian on H-type group. Section 3 is devoted to the proof of the Hardy-type inequality on H-type groups. In the final Section 4 we prove the Hardy-type inequality on the anisotropic Heisenberg group.

We thank professor Han for sending us a copy of their paper [9].

2 Preliminaries

We recall that a simply connected nilpotent group $\mathbb{G}$ is of Heisenberg type, or of H-type, if its Lie algebra $\mathfrak{n} = V \oplus \mathfrak{t}$ is of step-two, $[V, V] \subset \mathfrak{t}$, and if there is an inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{n}$ such that the linear map

$$J : \mathfrak{t} \to \mathrm{End}(V),$$

defined by the condition

$$\langle J_t(u), v \rangle = \langle t, [u, v] \rangle, \quad u, v \in V, \ z \in \mathfrak{t},$$

satisfies

$$J_t^2 = -|t|^2 \mathrm{Id}$$

for all $t \in \mathfrak{t}$, where $|t|^2 = \langle t, t \rangle$.

Groups of H-type were introduced by Kaplan in [10] as direct generalizations of Heisenberg groups, and they have been studied quite extensively (see [11–14] and the references therein).

We identify $\mathbb{G}$ with its Lie algebra $\mathfrak{n}$ via the exponential map, $\exp : V \oplus \mathfrak{t} \to \mathbb{G}$. For $g \in \mathbb{G}$, we write $g = (x(g), t(g)) \in V \oplus \mathfrak{t}$, and let

$$K(g) = (|x(g)|^4 + 16|t(g)|^2)^{\frac{1}{4}}. \tag{2.1}$$

Then $K(g)$ defines a homogeneous norm in G which is smooth outside the origin. Note that $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$ where $m = \dim V$ and $k = \dim \mathfrak{t}$.

Let $\{X_j\}_1^m$ be an orthonormal basis of V. Denote $\nabla_{\mathbb{G}} u = (X_1 u, X_2 u, \cdots, X_m u)$. Let $w \geq 0$ be a weight on $\mathbb{G}$. For a domain Ω in $\mathbb{G}$ and $p \geq 1$, we denote $HW^{1,p}_{\mathrm{loc}}(\Omega, w) = \{u \in L^p_{\mathrm{loc}}(\Omega, w) : \nabla_{\mathbb{G}} u \in L^p_{\mathrm{loc}}(\Omega, w)\}$ and $HW^{1,p}_0(\Omega, w)$ the closure of $C_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_{\mathbb{G}} u|^p w)^{\frac{1}{p}}$.

A function $u \in HW^{1,p}_{\mathrm{loc}}(\Omega, w)$ is said to be a weak solution of the weighted p-sub-Laplacian equation

$$-L_{p,w} u = f, \quad f \in L^1_{\mathrm{loc}}(\Omega)$$

if

$$\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle w = \int_\Omega f\varphi$$

for all test function $\varphi \in C_0^\infty(\Omega)$. In case $u \in C^2(\Omega)$ some standard computations show that

$$L_{p,w} u \ = \sum_{i=1}^m X_i(|\nabla_{\mathbb{G}} u|^{p-2} w X_i u). \tag{2.2}$$

Note that

$$\sum_{i=1}^m X_i(|\nabla_{\mathbb{G}} u|^{p-2} w X_i u) = 0$$

is the Euler-Lagrange equation for the variational integral $u \mapsto \int_\Omega |\nabla_{\mathbb{G}} u|^p w$.

For H-type groups, the following remarkable explicit fundamental solution of the (unweighted) p-sub-Laplacian L_p were obtained by Capogna, Danielli, Garofalo [15]:

Lemma 2.1 *Let $\mathbb{G}$ be the H-type group with homogeneous dimension $Q = m + 2k$. Then for $1 < p < \infty$, there exist constants C_p, C_Q such that the function*

$$u_p = \begin{cases} C_p \ K^{\frac{p-Q}{p-1}}, & p \neq Q, \\ C_Q \log \dfrac{1}{K}, & p = Q \end{cases}$$

is a fundamental solution for L_p with singularity at the origin $0 \in \mathbb{G}$.

By the similar method in [15], we can obtain a fundamental solution for the following weighted p-sub-Laplacian (see also [16, 17]):

$$L_{p,w} u \ = \sum_{i=1}^m X_i(|\nabla_{\mathbb{G}} u|^{p-2} w X_i u), \quad w = K^\alpha |\nabla_{\mathbb{G}} K|^\beta, \quad \alpha > -Q, \ \beta > -m, \tag{2.3}$$

where K is taken from (2.1).

Lemma 2.2 *Let $\mathbb{G}$ be the H-type group with homogeneous dimension $Q = m+2k$, and $L_{p,w}$ be defined as in* (2.3)*. Then there exist positive constants $C_{p,w}, C_{Q+\alpha,w}$ such that*

$$\Gamma_{p,w} = \begin{cases} C_{p,w} \ K^{\frac{p-Q-\alpha}{p-1}}, & p \neq Q + \alpha, \\ C_{Q+\alpha,w} \log \frac{1}{K}, & p = Q + \alpha \end{cases}$$

is a fundamental solution of $L_{p,w}$ with singularity at the origin $0 \in \mathbb{G}$.

3 Hardy-Type Inequality on H-Type Groups

In this section, we will prove the following Hardy-type inequality on the H-type group $\mathbb{G}$.

Theorem 3.1 *Let $\mathbb{G}$ be the H-type group with homogeneous dimension $Q = m + 2k$ and let $\alpha \in \mathbb{R}, \ 1 < p < Q + \alpha, \ \beta > -m, \ \Phi \in C_0^\infty(\mathbb{G} \backslash \{0\})$. Then the following inequality holds:*

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} \Phi|^p K^\alpha |\nabla_{\mathbb{G}} K|^\beta \geq \Big(\frac{Q + \alpha - p}{p}\Big)^p \int_{\mathbb{G}} |\Phi|^p K^{\alpha - p} |\nabla_{\mathbb{G}} K|^{p+\beta}, \tag{3.1}$$

where K is as in (2.1)*. Moreover, the constant $\big(\frac{Q+\alpha-p}{p}\big)^p$ is sharp.*

Remark 3.1 If $p=2,\beta=0$, then our Theorem 3.1 is actually [2, Theorem 3.2]; if $\alpha=\beta=0$ and $\mathbb{G}$ is the Heisenberg group, inequality (3.1) is exactly the inequality (1) in [3].

Proof of Theorem 3.1 We know from Lemma 2.2 that $K^{\frac{p-Q-\alpha}{p-1}}$ is a weak solution of the following equation in $\mathbb{G}\backslash\{0\}$:

$$\sum_{i=1}^{m} X_i(|\nabla_{\mathbb{G}} u|^{p-2}K^{\alpha}|\nabla_{\mathbb{G}}K|^{\beta}X_i u)=0.$$

Take $\varphi=|\Phi|^pK^{Q+\alpha-p}$ as a test function in the above equation. Then

$$\int_{\mathbb{G}}|\nabla_{\mathbb{G}}(K^{\frac{p-Q-\alpha}{p-1}})|^{p-2}K^{\alpha}|\nabla_{\mathbb{G}}K|^{\beta}\nabla_{\mathbb{G}}(K^{\frac{p-Q-\alpha}{p-1}})\cdot\nabla_{\mathbb{G}}(|\Phi|^pK^{Q+\alpha-p})=0, \tag{3.2}$$

which implies

$$\begin{aligned}&p\int_{\mathbb{G}}K^{1+\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta-2}|\Phi|^{p-2}\Phi\nabla_{\mathbb{G}}K\cdot\nabla_{\mathbb{G}}\Phi\\&+(Q+\alpha-p)\int_{\mathbb{G}}K^{\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta}|\Phi|^p=0.\end{aligned} \tag{3.3}$$

We get by (3.3)

$$\frac{Q+\alpha-p}{p}\int_{\mathbb{G}}K^{\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta}|\Phi|^p\le\int_{\mathbb{G}}K^{1+\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta-1}|\Phi|^{p-1}|\nabla_{\mathbb{G}}\Phi|. \tag{3.4}$$

By Hölder inequality,

$$\begin{aligned}&\int_{\mathbb{G}}K^{1+\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta-1}|\Phi|^{p-1}|\nabla_{\mathbb{G}}\Phi|\\&=\int_{\mathbb{G}}K^{(\alpha-p)\frac{p-1}{p}}|\nabla_{\mathbb{G}}K|^{(p+\beta)\frac{p-1}{p}}|\Phi|^{p-1}\cdot K^{\frac{\alpha}{p}}|\nabla_{\mathbb{G}}K|^{\frac{\beta}{p}}|\nabla_{\mathbb{G}}\Phi|\\&\le\Big(\int_{\mathbb{G}}K^{\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta}|\Phi|^p\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{G}}K^{\alpha}|\nabla_{\mathbb{G}}K|^{\beta}|\nabla_{\mathbb{G}}\Phi|^p\Big)^{\frac{1}{p}}.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we get the inequality (3.1).

Next we show that the constant $\big(\frac{Q+\alpha-p}{p}\big)^p$ in (3.1) is sharp. Given $\varepsilon>0$, we consider the following radial function:

$$u_{\varepsilon}=\begin{cases}C_{\varepsilon}, & 0\le K\le 1,\\ C_{\varepsilon}K^{\frac{p-Q-\alpha}{p}-\varepsilon}, & K>1,\end{cases}$$

where $C_{\varepsilon}=\big(\frac{Q+\alpha-p}{p}+\varepsilon\big)^{-1}$. It is easy to see that

$$\nabla_{\mathbb{G}}(u_{\varepsilon})=\begin{cases}0, & 0\le K<1,\\ -K^{-\frac{Q+\alpha+p\varepsilon}{p}}\nabla_{\mathbb{G}}K, & K>1.\end{cases}$$

For u_{ε}, we have

$$\begin{aligned}\int_{\mathbb{G}}K^{\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta}|u_{\varepsilon}|^p&=C_{\varepsilon}^p\Big(\int_{B_1}K^{\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta}+\int_{\mathbb{G}\backslash B_1}K^{-Q-p\varepsilon}|\nabla_{\mathbb{G}}K|^{p+\beta}\Big)\\&=C_{\varepsilon}^p\Big(\int_{B_1}K^{\alpha-p}|\nabla_{\mathbb{G}}K|^{p+\beta}+\int_{\mathbb{G}}K^{\alpha}|\nabla_{\mathbb{G}}K|^{\beta}|\nabla_{\mathbb{G}}u_{\varepsilon}|^p\Big),\end{aligned} \tag{3.6}$$

where $B_1 = \{g \in \mathbb{G} : K(g) \le 1\}$. Since $|\nabla_{\mathbb{G}} K| = \frac{|x|}{K} \le 1, K^{\alpha-p}$ has homogeneity $\alpha - p > -Q$, it is easy to calculate that the first integral $\int_{B_1} K^{\alpha-p}|\nabla_{\mathbb{G}} K|^p$ in (3.6) is finite by the polar coordinate integration formula for H-type group (see [18, Proposition 1.15]). Letting $\varepsilon \to 0$, we conclude the argument.

Remark 3.2 There is a similar Hardy inequality on H-type groups proved in [9, Theorem 5.1], but their conditions on the weight and the constant are different from ours. The condition $\alpha > p - Q$ in our theorem is equivalent to $\beta < \frac{Q-p}{p(p-1)}$ in theirs, whereas their conditions are $\beta < 0$ and $1 < p < Q$. So our result admits larger range on the power of the weight. Furthermore we allow the case $p \ge Q$ and we find the best constant.

An immediate consequence of Theorem 3.1 is the following corollary.

Corollary 3.1 *Let $\mathbb{G}$ be the H-type group with homogeneous dimension $Q = m + 2k$, and let $u \in C_0^\infty(\mathbb{G}\backslash\{0\}), \frac{1}{p} + \frac{1}{q} = 1$ $(1 < p < Q)$. Then we have*

$$\Big(\int_{\mathbb{G}} |x(g)|^q |u(g)|^q \mathrm{d}g\Big)^{\frac{1}{q}} \Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u(g)|^p \mathrm{d}g\Big)^{\frac{1}{p}} \ge \frac{Q-p}{p} \int_{\mathbb{G}} \frac{|x(g)|^2}{K^2(g)} |u(g)|^2 \mathrm{d}g.$$

4 Hardy-Type Inequality on Anisotropic Heisenberg Groups

Let $a = (a_1, a_2, \cdots, a_n),\ a_1, a_2, \cdots, a_n > 0$. Let $H = H(a) = \mathbb{R}^{2n} \oplus \mathbb{R}$ be the corresponding anisotropic Heisenberg group with the product

$$(\zeta, t) \circ (\eta, s) = \Big(\zeta + \eta, t + s + 2\sum_{j=1}^n a_j(\zeta_{j+n}\eta_j - \zeta_j \eta_{j+n})\Big).$$

The corresponding left-invariant vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} - 2a_j y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2a_j x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n,$$

where $(x, y, t) \in \mathbb{R}^{2n} \oplus \mathbb{R}, x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n)$. It is easy to see that $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ satisfy the Heisenberg commutation relations

$$[X_j, Y_j] = -4a_j \frac{\partial}{\partial t},$$

and all other commutators are zero. Denote

$$\nabla_H u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u), \quad \mathrm{div}_H(u_1, \cdots, u_{2n}) = \sum_{j=1}^n (X_j u_j + Y_j u_{n+j}).$$

For further information on the anisotropic Heisenberg group, see e.g. [19, 20]. In this paper we obtain the following Hardy-type inequality on anisotropic Heisenberg group H.

Theorem 4.1 *Let $H = H(a)$ be the anisotropic Heisenberg group with $a_j \le 1,\ j = 1, \cdots, n$. Let $\alpha \in \mathbb{R},\ 2 \le p < Q + \alpha$ and $\Phi \in C_0^\infty(H\backslash\{0\})$. Then the following inequality is valid:*

$$\int_H d^\alpha |\nabla_H \Phi|^p \ge \Big(\frac{Q + \alpha - p}{p}\Big)^p \int_H d^\alpha \frac{\Big(\sum\limits_{j=1}^n a_j^2 |z_j|^2\Big)^{\frac{p}{2}}}{d^{2p}} |\Phi|^p, \tag{4.1}$$

where $d = \Big(\Big(\sum\limits_{j=1}^n a_j |z_j|^2\Big)^2 + t^2\Big)^{\frac{1}{4}},\ Q = 2\sum\limits_{j=1}^n a_j + 2.$

Remark 4.1 If $p = 2, \alpha = 0, a_i = 1$ $(i = 1, \cdots, n)$, then we get the known Hardy inequality on the Heisenberg group (see [1]),

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} \phi|^2 \geq n^2 \int_{\mathbb{H}^n} \frac{|z|^2}{|z|^4 + t^2} \phi^2 \ .$$

For the proof of Theorem 4.1, we need the following lemma which can be proved by similar computation as in [3].

Lemma 4.1 *Let $w \geq 0$ be a weight function in $\Omega \subset H$ and $L_{H,p,w}$ be the following weighted p-sub-Laplacian on H :*

$$L_{H,p,w} v = \mathrm{div}_H (|\nabla_H v|^{p-2} w \nabla_H v). \tag{4.2}$$

Suppose that for some $\lambda > 0$, there exists $v \in C^\infty(\Omega), v > 0$ such that

$$-L_{H,p,w} v \geq \lambda g v^{p-1} \tag{4.3}$$

in the sense of distribution for some $g \geq 0$. Then for any $u \in HW_0^{1,p}(\Omega, w)$, it holds that

$$\int_\Omega |\nabla_H u|^p w \geq \lambda \int_\Omega g |u|^p.$$

Proof of Theorem 4.1 Take $w = d^\alpha$ and $v = d^{\frac{p-Q-\alpha}{p}}$ in Lemma 4.1. Noting that

$$L_{H,p,w} v = \sum_{i=1}^n \left[X_i (|\nabla_H v|^{p-2} w X_i v) + Y_i (|\nabla_H v|^{p-2} w Y_i v) \right],$$

$$X_i d = \frac{\left[\left(\sum\limits_{j=1}^n a_j |z_j|^2 \right) a_i x_i + a_i y_i t \right]}{d^3},$$

$$Y_i d = \frac{\left[\left(\sum\limits_{j=1}^n a_j |z_j|^2 \right) a_i y_i - a_i x_i t \right]}{d^3},$$

$$|\nabla_H d|^2 = \frac{\sum\limits_{j=1}^n a_j^2 |z_j|^2}{d^2},$$

we get

$$\begin{aligned}
-L_{H,p,w} v = &- \sum_{i=1}^n \Big[X_i \Big(d^\alpha \Big| \frac{Q+\alpha-p}{p} d^{-\frac{Q+\alpha}{p}} \Big|^{p-2} |\nabla_H d|^{p-2} \Big(\frac{p-Q-\alpha}{p} \Big) d^{-\frac{Q+\alpha}{p}} X_i d \Big) \\
&+ Y_i \Big(d^\alpha \Big| \frac{Q+\alpha-p}{p} d^{-\frac{Q+\alpha}{p}} \Big|^{p-2} |\nabla_H d|^{p-2} \Big(\frac{p-Q-\alpha}{p} \Big) d^{-\frac{Q+\alpha}{p}} Y_i d \Big) \Big] \\
= &\Big(\frac{Q+\alpha-p}{p} \Big)^{p-1} \Big\{ \Big(\frac{Q+\alpha}{p} + 2 - p - Q \Big) d^{\frac{Q+\alpha}{p}+1-p-Q} \Big(\sum_{j=1}^n a_j^2 |z_j|^2 \Big)^{\frac{p-2}{2}} |\nabla_H d|^2 \\
&+ d^{\frac{Q+\alpha}{p}+2-p-Q} (p-2) \Big(\sum_{j=1}^n a_j^2 |z_j|^2 \Big)^{\frac{p-4}{2}} \sum_{j=1}^n a_j^2 (x_j \cdot X_j d + y_j \cdot Y_j d) \\
&+ d^{\frac{Q+\alpha}{p}+2-p-Q} \Big(\sum_{j=1}^n a_j^2 |z_j|^2 \Big)^{\frac{p-2}{2}} \sum_{j=1}^n (X_j^2 d + Y_j^2 d) \Big\}.
\end{aligned}$$

By direct computations we have

$$\sum_{j=1}^{n} a_j^2(x_j \cdot X_j d + y_j \cdot Y_j d) = \frac{\Big(\sum\limits_{j=1}^{n} a_j|z_j|^2\Big)\Big(\sum\limits_{j=1}^{n} a_j^3|z_j|^2\Big)}{d^3},$$

$$\sum_{j=1}^{n}(X_j^2 d + Y_j^2 d) = \frac{\Big(2\sum\limits_{j=1}^{n} a_j\Big)\Big(\sum\limits_{j=1}^{n} a_j|z_j|^2\Big) + \sum\limits_{j=1}^{n} a_j^2|z_j|^2}{d^3}.$$

Then

$$-L_{H,p,w}v = \Big(\frac{Q+\alpha-p}{p}\Big)^{p-1} d^{\frac{Q+\alpha}{p}-1-p-Q}\Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)^{\frac{p-4}{2}} \cdot \mathrm{I},$$

where

$$\mathrm{I} = \Big\{\Big(\frac{Q+\alpha-p}{p} + 4 - p - Q\Big)\Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)^2 + (p-2)\Big(\sum_{j=1}^{n} a_j|z_j|^2\Big)\Big(\sum_{j=1}^{n} a_j^3|z_j|^2\Big) + \Big(2\sum_{j=1}^{n} a_j\Big)\Big(\sum_{j=1}^{n} a_j|z_j|^2\Big)\cdot\Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)\Big\}. \tag{4.4}$$

Using Cauchy inequality

$$\Big(\sum_{j=1}^{n} a_j|z_j|^2\Big)\Big(\sum_{j=1}^{n} a_j^3|z_j|^2\Big) \geq \Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)^2,$$

and (since, by assumption, $a_j \leq 1,\ j = 1, \cdots, n$)

$$\sum_{j=1}^{n} a_j|z_j|^2 \geq \sum_{j=1}^{n} a_j^2|z_j|^2,$$

we find

$$\mathrm{I} \geq \frac{Q+\alpha-p}{p}\Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)^2,$$

so for $p \geq 2$,

$$\begin{aligned} -L_{H,p,w}v &\geq \Big(\frac{Q+\alpha-p}{p}\Big)^{p-1}(d^{\frac{p-Q-\alpha}{p}})^{p-1} d^{\alpha-2p}\Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)^{\frac{p}{2}}\Big[\frac{Q+\alpha-p}{p}\Big] \\ &= \Big(\frac{Q+\alpha-p}{p}\Big)^{p} v^{p-1} d^{\alpha}\frac{\Big(\sum\limits_{j=1}^{n} a_j^2|z_j|^2\Big)^{\frac{p}{2}}}{d^{2p}}. \end{aligned} \tag{4.5}$$

Hence Theorem 4.1 follows from Lemma 4.1 with $\lambda = \big(\frac{Q+\alpha-p}{p}\big)^p,\ g = d^{\alpha}\frac{\big(\sum_{j=1}^{n} a_j^2|z_j|^2\big)^{\frac{p}{2}}}{d^{2p}}$.

From Hardy-type inequality (4.1) we have the following corollary.

Corollary 4.1 *Let $H = H(a)$ be an anisotropic Heisenberg group with $a_j \leq 1,\ j = 1, \cdots, n$. $u \in C_0^{\infty}(H\backslash\{0\}), \frac{1}{p} + \frac{1}{q} = 1$ $(2 \leq p < Q)$. Then we have*

$$\Big(\int_H \Big(\sum_{j=1}^{n} a_j^2|z_j|^2\Big)^{\frac{q}{2}}|u|^q\Big)^{\frac{1}{q}}\Big(\int_H |\nabla_H u|^p\Big)^{\frac{1}{p}} \geq \frac{Q-p}{p}\int_H \frac{\sum\limits_{j=1}^{n} a_j^2|z_j|^2}{d^2}|u|^2,$$

where $d = \Big(\Big(\sum_{j=1}^{n} a_j |z_j|^2\Big)^2 + t^2\Big)^{\frac{1}{4}},\ Q = 2\sum_{j=1}^{n} a_j + 2.$

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