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Bifurcation of Degenerate Homoclinic Orbits to Saddle-Center in Reversible Systems**

Xingbo LIU* Deming ZHU*

Abstract The authors study the bifurcation of homoclinic orbits from a degenerate homoclinic orbit in reversible system. The unperturbed system is assumed to have saddle-center type equilibrium whose stable and unstable manifolds intersect in two-dimensional manifolds. A perturbation technique for the detection of symmetric and nonsymmetric homoclinic orbits near the primary homoclinic orbits is developed. Some known results are extended.

Keywords Reversible system, Homoclinic orbits, Saddle-center, Bifurcation
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1 Introduction

In recent years, a new class of solitary waves has been found in a number of examples from nonlinear optics and water wave theory (see [1] and references therein). They are typically presented by homoclinic solutions to saddle-center equilibrium in the associated ordinary differential equation that describes traveling waves in the original partial differential equation. Many important examples lead to a reversible traveling wave ODE, i.e., an ODE that is invariant under time reversal up to some linear involution R that fixes half the phase space variables (see [2, 3] and references therein). The corresponding homoclinic solution is invariant under time-reversal, and therefore it is called symmetric.

In this paper, we consider the bifurcation of symmetric homoclinic orbits to saddle-center equilibrium in reversible systems of ODE.

We consider a reversible system

$$\dot{x} = f(x, \lambda), \quad x \in R^{2n+2},\ \lambda \in R^2,\ f \in C^r,\ r \geq 2, \tag{1.1}$$

which has, for $\lambda = 0$, a symmetric homoclinic orbit Γ asymptotic to a saddle-center $x = 0$. Here reversibility means that there is a linear involution R ($R^2 = I$) with

$$Rf(x, \lambda) = -f(Rx, \lambda).$$

A fundamental characteristic of reversible systems is that if $x(t)$ is a solution, then so is $Rx(-t)$. We call a solution symmetric if $x(t) = Rx(-t)$. It is well-known that an orbit is symmetric

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*Department of Mathematics, East China Normal University, Shanghai 200241, China.
E-mail: xbliu@math.ecnu.edu.cn dmzhu@math.ecnu.edu.cn
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if and only if it intersects the space $\mathrm{Fix}R = \{x \in R^{2n+2} \mid Rx = x\}$ (see [4]). Obviously the equilibrium $x = 0$ is symmetric.

When $\lambda = 0$, we make the following assumption.

Assumption 1.1 *The origin* 0 *is an equilibrium of saddle-center type. More precisely, the Jacobian matrix* $A = D_1 f(0,0)$ *has one pair of purely imaginary eigenvalues* $\pm \mathrm{i}$ *and* $2n$ *hyperbolic eigenvalues:*

$$\sigma(A) = \{\pm \mathrm{i}\} \cup \{\pm \mu\} \cup \{\sigma^{ss}\} \cup \{\sigma^{uu}\},$$

where $\mu \in R^+$, $|\mathrm{Re}\widetilde{\mu}| > \mu$, $\forall\, \widetilde{\mu} \in \sigma^{ss} \cup \sigma^{uu}$, *and* $\sigma^{ss(uu)}$ *denotes the strong stable* (*unstable*) *spectrum of* $D_1 f(0,0)$. *Because of the R-reversibility, we have* $\sigma^{ss} = -\sigma^{uu}$.

Note that $D_1 f(0,0)$ is nonsingular. Therefore, we have, for all sufficiently small λ, a unique equilibrium point x_λ nearby $x = 0$. Without loss of generality, we always assume $f(0,\lambda) = 0$ for $|\lambda| \ll 1$. The spectrum of $D_1 f(0,\lambda)$ contains exactly one pair of purely imaginary eigenvalues as well. This is due to the reversibility which prevents simple eigenvalues from moving off the imaginary axis. So the saddle-center o has n-dimensional stable and unstable manifolds W^s_λ and W^u_λ, and a two-dimensional center manifold W^c_λ filled with symmetric periodic orbits (Liapunov orbits) surrounding the equilibrium. Also from [5], we know that the center manifold, the center-stable manifolds W^{cs}_λ and the center-unstable manifolds W^{cu}_λ respectively are uniquely determined.

Due to our assumption of reversibility, we will have no local bifurcations (around the equilibrium $x = 0$) of fixed points or periodic orbits.

Next, we assume the existence of homoclinic solutions to (1.1) as follows:

Assumption 1.2 *At* $\lambda = 0$, *system* (1.1) *possesses a homoclinic orbit* $\Gamma = \{\gamma(r) : t \in R\}$, $\gamma(t) \to 0$ *as* $t \to \pm\infty$, $R\Gamma = \Gamma$. *Furthermore, we suppose that*

$$\dim(T_{\gamma(0)} W^s \cap T_{\gamma(0)} W^{cu}) = 2,$$

where $T_{\gamma(0)} W^{s(cu)}$ *is the tangent space of the stable* (*center-unstable*) *manifold* $W^{s(cu)}(O)$ *of the nonhyperbolic equilibrium* $x = 0$. *By reversibility, we also have*

$$\dim(T_{\gamma(0)} W^{cs} \cap T_{\gamma(0)} W^{u}) = 2.$$

So Assumption 1.2 ensures that Γ is contained in the intersection of the stable and unstable manifold of O and Γ is a symmetric homoclinic orbit. Homoclinic orbits with such an exponential bound will play a distinguished role in the forthcoming analysis and we call them fast decaying. We study bifurcation of homoclinic orbits from the primary one Γ. Therefore we concentrate on the existence of one-homoclinic orbits (fast decaying) to the origin, that is, orbits which are contained in a tubular neighborhood of Γ and make exactly one winding.

In [6], K. Yagasaki and T. Wagenknecht have analyzed the situation of degenerate homoclinic orbits to a saddle-center which possesses one pair of imaginary eigenvalues. They used an idea similar to that of Melnikov's method to discuss the existence of a symmetric homoclinic orbit to equilibrium. In this paper, we will present a technique to give a more complete description of the bifurcation of one-homoclinic orbits to the saddle-center equilibrium. The ideas come from

Lin's method which has been proved to be a powerful tool for the investigation of the dynamics near connecting orbits. Originally, this method has been used for orbits connecting hyperbolic fixed points (see [7]), and extended to study the bifurcation of nondegenerate homoclinic orbits to a saddle-center (see [8]). The main technical difference to the analysis for homoclinic orbits to hyperbolic equilibrium is that in the nonhyperbolic situation the variational equation along the homoclinic orbit possesses an exponential trichotomy, instead of an exponential dichotomy. This requires a modified approach. Inspired by [8], we will adapt Lin's method to discuss the bifurcation of degenerate homoclinic orbits.

2 Homoclinic Orbits to the Equilibrium

In this section, we give the precise analysis leading to this bifurcation equation. Let $\gamma(0) \in \mathrm{Fix}\,R$. This is always possible because Γ is symmetric and it has exactly one common point with $\mathrm{Fix}\,R$. Then by reversibility, we have

$$f(\gamma(0), 0) \in \mathrm{Fix}\,(-R).$$

The following direct sum decomposition is fundamental for all of our considerations:

$$R^{2n+2} = \mathrm{span}\{f(\gamma(0), 0)\} \oplus U \oplus W^+ \oplus W^- \oplus Z, \tag{2.1}$$

where

$$\begin{aligned} &\mathrm{span}\{f(\gamma(0), 0)\} \oplus U = T_{\gamma(0)}W^s(0) \cap T_{\gamma(0)}W^u(0), \\ &(T_{\gamma(0)}W^s(0) \cap T_{\gamma(0)}W^u(0)) \oplus W^{+(-)} = T_{\gamma(0)}W^{s(u)}(0), \\ &Z \in (T_{\gamma(0)}W^s(0) + T_{\gamma(0)}W^u(0))^{\perp}. \end{aligned} \tag{2.2}$$

This means $\dim Z = 4$ and $\dim U = 1$. We define the transversal section $\sum$ by

$$\sum = \gamma(0) + \{U \oplus W^+ \oplus W^- \oplus Z\}. \tag{2.3}$$

Due to the reversibility of the vector field, we have $RW^s = W^u$. Hence $RT_{\gamma(0)}W^s = T_{\gamma(0)}W^u$. Then we have

$$RW^+ = W^-, \quad R(U) = U.$$

Obviously, if $\eta^+ \in W^+$, then $\eta^+ + R\eta^+ \in \mathrm{Fix}\,R \cap (W^+ \oplus W^-)$, and $\eta^+ - R\eta^+ \in \mathrm{Fix}\,(-R) \cap (W^+ \oplus W^-)$. We obtain the following result:

Lemma 2.1 *The space $W^+ \oplus W^-$ contains $(n-2)$-dimensional subspace of both* $\mathrm{Fix}\,R$ *and* $\mathrm{Fix}\,(-R)$, *where* $\dim W^{+(-)} = n - 2$.

A consequence of (2.3) is

$$RZ = Z, \quad Z = (Z \cap \mathrm{Fix}\,R) \oplus (Z \cap \mathrm{Fix}\,(-R)). \tag{2.4}$$

Taking into consideration that $\dim U = 1$, we have $R|_U = -\mathrm{id}$ or $R|_U = \mathrm{id}$. In the first case, i.e., $R|_U = -\mathrm{id}$, U is a subspace of $\mathrm{Fix}\,(-R)$. Because of $R^{2n+2} = \mathrm{Fix}\,R \oplus \mathrm{Fix}\,(-R)$ and Lemma 2.1, we obtain

Lemma 2.2 *The space Z is a direct sum of a one-dimensional subspace of* $\mathrm{Fix}(-R)$ *and a three-dimensional subspace of* $\mathrm{Fix}\,R$.

Remark 2.1 In the second case, i.e., $R|_U = \mathrm{id}$, U is a subspace of $\mathrm{Fix}(R)$ and Z is a direct sum of a two-dimensional subspace of $\mathrm{Fix}(-R)$ and a two-dimensional subspace of $\mathrm{Fix}\,R$.

Following Lin's method for hyperbolic fixed points, we would look for solutions $\gamma^{\pm}$ to (1.1) defined on $R^{\pm}$ which start in $\sum$ with a difference lying in a certain space and approach O for $t \to \pm\infty$. Since, in our case, O is a non-hyperbolic fixed point, we will first detect exponentially decaying solutions. So let us choose α such that $0 < \alpha < \mu$ (see Assumption 1.1 for the definition of μ), and look for solutions $\gamma^{\pm}$ that fulfil

$$\begin{aligned}
&(\mathrm{i})\ \ \text{The orbits of } \gamma^{\pm} \text{ are near } \Gamma,\\
&(\mathrm{ii})\ \ \gamma^{+}(0),\ \gamma^{-}(0) \in \Sigma,\\
&(\mathrm{iii})\ \ \sup\{\mathrm{e}^{\pm\alpha t}\|\gamma^{\pm}(\mathrm{t})\| : \mathrm{t} \in \mathrm{R}^{\pm}\} < \infty,\\
&(\mathrm{vi})\ \ \gamma^{+}(0) - \gamma^{-}(0) \in \Sigma.
\end{aligned} \tag{P_γ}$$

Indeed, we consider $\gamma^{\pm}$ as perturbations of γ. For that we define functions $v^{\pm}(\,\cdot\,)$ on $R^{\pm}$ by

$$\gamma^{\pm}(t) = \gamma(t) + v^{\pm}(t), \quad t \in R^{\pm}. \tag{2.5}$$

We will formulate an equivalent problem to (P_γ) for $v^{\pm}$. First, the functions $v^{\pm}$ have to satisfy the equation

$$\dot{v} = D_1 f(\gamma(t), 0)v + h(t, v, \lambda), \tag{2.6}$$

where $h(t, v, \lambda) = f(\gamma(t) + v, \lambda) - f(\gamma(t), 0) - D_1 f(\gamma(t), 0)v$, $h(\,\cdot\,, 0, 0) = 0$, $D_2 h(\,\cdot\,, 0, 0) = 0$.

In order to satisfy the exponential rate for $\gamma^{\pm}$, we introduce spaces

$$\begin{aligned}
v_\alpha^+ &:= \{v \in C^0([0, \infty], R^{2n+2}) : \sup \mathrm{e}^{\alpha t}\|v(t)\| < \infty,\ t \ge 0\},\\
v_\alpha^- &:= \{v \in C^0([-\infty, 0], R^{2n+2}) : \sup \mathrm{e}^{-\alpha t}\|v(t)\| < \infty,\ t \le 0\}.
\end{aligned}$$

We rewrite (P_γ) by (2.5). Then $v^+(\,\cdot\,)$ and $v^-(\,\cdot\,)$ have to satisfy

$$\begin{aligned}
&(\mathrm{i})\ \ \|v^{\pm}(t)\| \text{ is small for all } t \in R^{\pm},\\
&(\mathrm{ii})\ \ v^{+}(0),\ v^{-}(0) \in U \oplus W^+ \oplus W^- \oplus Z,\\
&(\mathrm{iii})\ \ v^{+}(t) \in v_\alpha^+,\ v^{-}(t) \in v_\alpha^-,\\
&(\mathrm{vi})\ \ v^{+}(0) - v^{-}(0) \in Z.
\end{aligned} \tag{P_v}$$

So the original task of finding solutions to the system (1.1) fulfilling (P_γ) has been turned into the problem of determining solutions to the "nonlinear" variational equation (2.6) satisfying (P_v).

In order to find solutions to (2.6) that fulfil (P_v), we use the fact that the variational equation along the homoclinic orbit Γ

$$\dot{v} = D_1 f(\gamma(t), 0)v \tag{2.7}$$

possesses exponential trichotomies on $R^{\pm}$ (see [8]). This means that there exist projections $P_u^{\pm}(t)$, $P_s^{\pm}(t)$ and $P_c^{\pm}(t)$ such that

$$P_s^{\pm}(t) + P_u^{\pm}(t) + P_c^{\pm}(t) = \mathrm{Id}, \quad t \in R^{\pm}$$

and

$$\Phi(t,s)P_i^{\pm}(s) = P_i^{\pm}(t)\Phi(t,s), \quad i = u, s, c,$$

where $\Phi(\,\cdot\,)$ denotes the transition matrix of (2.7). Moreover, for $t \ge s \ge 0$ and for all α_c with $\mu > \alpha > \alpha_c > 0$, we have

$$\begin{aligned}
&\|\Phi(t,s)P_s^+(s)\| \le K\mathrm{e}^{-\alpha(t-s)}, \quad \|\Phi(s,t)P_u^+(t)\| \le K\mathrm{e}^{-\alpha(t-s)},\\
&\|\Phi(t,s)P_c^+(s)\| \le K\mathrm{e}^{\alpha_c(t-s)}, \quad\ \ \|\Phi(s,t)P_c^+(t)\| \le K\mathrm{e}^{\alpha_c(t-s)}.
\end{aligned}$$

Using reversibility, one can define $P_i^-(t)$ such that similar relation holds on R^-. Furthermore, we demand that

$$\operatorname{Im} P_s^+(t) = T_{\gamma(t)}W^s, \quad \operatorname{Im} P_u^-(t) = T_{\gamma(t)}W^u,$$

and we can choose

$$\operatorname{Ker} P_s^+(0) = Z \oplus W^-, \quad \operatorname{Ker} P_u^-(0) = Z \oplus W^+.$$

These results are proved in [8] (see also [7]).

Solutions to (2.6) satisfy the following fixed point problem

$$\begin{aligned}
v^+(t) &= \Phi(t,0)\eta^+ + \int_0^t \Phi(t,s)P_s^+(s)h(s,v^+,\lambda)\mathrm{d}s\\
&\quad - \int_t^\infty \Phi(t,s)(\mathrm{Id} - P_s^+(s))h(s,v^+,\lambda)\mathrm{d}s,\\
v^-(t) &= \Phi(t,0)\eta^- - \int_t^0 \Phi(t,s)P_u^-(s)h(s,v^-,\lambda)\mathrm{d}s\\
&\quad + \int_{-\infty}^t \Phi(t,s)(\mathrm{Id} - P_u^-(s))h(s,v^-,\lambda)\mathrm{d}s,
\end{aligned} \tag{2.8}$$

where $\eta^+ \in T_{\gamma(0)}W^s$, $\eta^- \in T_{\gamma(0)}W^u$. Using the definition of h and Taylor expansion of f, we obtain the following estimate:

$$\|h(t,v,\lambda)\| \le K_1\|v\|^2 + K_2\|\lambda\|(\|\gamma(t)\| + \|v\|). \tag{2.9}$$

Hence, $v^\pm \in v_\alpha^\pm$ implies $h(\,\cdot\,, v^\pm(\,\cdot\,), \lambda) \in v_\alpha^\pm$. Now we consider the function defined by the right-hand side of (2.8):

$$\begin{aligned}
L^+(t,h) &:= \int_0^t \Phi(t,s)P_s^+(s)h(s,v^+,\lambda)\mathrm{d}s - \int_t^\infty \Phi(t,s)(\mathrm{Id} - P_s^+(s))h(s,v^+,\lambda)\mathrm{d}s,\\
L^-(t,h) &:= -\int_t^0 \Phi(t,s)P_u^-(s)h(s,v^-,\lambda)\mathrm{d}s + \int_{-\infty}^t \Phi(t,s)(\mathrm{Id} - P_u^-(s))h(s,v^-,\lambda)\mathrm{d}s.
\end{aligned}$$

The norm of $L^+(t,h)$ can be estimated as

$$\begin{aligned}
\|L^+(t,h)\| \le{}& \Big\|\int_0^t \Phi(t,s)P_s^+(s)h(s,v^+,\lambda)\mathrm{d}s\Big\| + \Big\|\int_t^\infty \Phi(t,s)P_u^+(s)h(s,v^+,\lambda)\mathrm{d}s\Big\|\\
&+ \Big\|\int_t^\infty \Phi(t,s)P_c^+(s)h(s,v^+,\lambda)\mathrm{d}s\Big\|.
\end{aligned}$$

Now using the exponential trichotomy and $h(\,\cdot\,, v^+(\,\cdot\,), \lambda) \in v_\alpha^+$, we can obtain

$$\Big\| \int_t^\infty \Phi(t,s) P_c^+(s) h(s, v^+, \lambda) \mathrm{d}s \Big\| \le K \mathrm{e}^{-\alpha_c t} \int_t^\infty \mathrm{e}^{(\alpha_c - \overline{\alpha})s} \mathrm{d}s \le \widehat{K} \mathrm{e}^{-\overline{\alpha} t}$$

for some positive constants K and $\widehat{K}$. Similar estimations hold for $L^-(t,h)$.

Remark 2.2 This estimation is not possible if $h(\,\cdot\,, v^\pm, \lambda)$ is only bounded.

Now we see that the right-hand side of (2.8) is a map

$$T_{\gamma(0)} W^{s(u)} \times R^2 \times v_\alpha^\pm \to v_\alpha^\pm.$$

Therefore, the exponentially bounded solutions to (2.6) are exactly the solutions to (2.8) considered in $v_\alpha^\pm$. By the implicit function theorem, this problem can be solved around $(\eta^\pm, v^\pm, \lambda) = (0,0,0)$ for $v^\pm = v^\pm(\eta^\pm, \lambda)$. So the problem ((2.6), (P_v)(i), (iii)) has been solved.

Now regarding the requirements on $v^\pm(\eta^\pm, \lambda)(0)$ in ((P_v)(ii), (iv)) and writing U as $U = \operatorname{span}\{u_0\}$, we decompose both $v^+(\eta^+, \lambda)(0)$ and $v^-(\eta^+, \lambda)(0)$ as follows:

$$\begin{aligned} v^+(\eta^+, \lambda)(0) &= \eta^+ - \int_0^\infty \Phi(0,s)(\mathrm{Id} - P_s^+(s)) h(s, v^+, \lambda) \mathrm{d}s \\ &= \rho_1 u_0 + w^+ + \widetilde{w}^-(\eta^+, \lambda) + z^+(\eta^+, \lambda), \\ v^-(\eta^-, \lambda)(0) &= \eta^- + \int_{-\infty}^0 \Phi(0,s)(\mathrm{Id} - P_u^-(s)) h(s, v^-, \lambda) \mathrm{d}s \\ &= \rho_2 u_0 + w^- + \widetilde{w}^+(\eta^-, \lambda) + z^-(\eta^-, \lambda), \end{aligned} \tag{2.10}$$

where $w^{+(-)}$, $\widetilde{w}^\pm \in W^{+(-)}$, $z^\pm \in Z$, $\rho_1 u_0 + w^+ = \eta^+$, $\rho_2 u_0 + w^- = \eta^-$, $\rho_i u_0 \in U$, $i = 1, 2$. Moreover, in (P_v) we demand

$$v^+(\eta^+, \lambda)(0) - v^-(\eta^-, \lambda)(0) \in Z.$$

Therefore,

$$w^+ = \widetilde{w}^+(\eta^-, \lambda), \quad w^- = \widetilde{w}^-(\eta^+, \lambda), \quad \rho_1 = \rho_2 = \rho \tag{2.11}$$

have to be satisfied. Together with $h(\,\cdot\,, 0, 0) = 0$ and $D_2 h(\,\cdot\,, 0, 0) = 0$, we obtain

$$\begin{aligned} &\widetilde{w}^+(0,0) = 0, \quad D_1 \widetilde{w}^+(0,0) = 0, \\ &\widetilde{w}^-(0,0) = 0, \quad D_1 \widetilde{w}^-(0,0) = 0. \end{aligned}$$

By implicit function theorem, we can solve (2.11) for

$$w^+ = w^+(\rho, \lambda), \quad w^- = w^-(\rho, \lambda). \tag{2.12}$$

We thus obtain the following lemma in complete analogy to [8, Lemma 2.7].

Lemma 2.3 *For each sufficiently small λ, there is a pair $(\gamma^+(\rho, \lambda), \gamma^-(\rho, \lambda))$ of solutions to* (1.1) *satisfying the properties (P_γ).*

We find homoclinic orbits asymptotic to the fixed point $x = 0$ with some minimal exponential rate by solving the following bifurcation equation

$$\xi^\infty(\rho, \lambda) = \gamma^+(\rho, \lambda)(0) - \gamma^-(\rho, \lambda)(0) = 0, \tag{2.13}$$

which satisfies

$$\xi^{\infty}(0,0)=0, \quad D_1\xi^{\infty}(0,0)=0.$$

The reversibility of f and the symmetry of the homoclinic orbit $\gamma(\cdot)$ imply

$$\begin{aligned} RD_1f(\gamma(t),0) &= -D_1f(\gamma(-t),0)R, \\ Rh(t,x,\lambda) &= -h(-t,Rx,\lambda). \end{aligned} \tag{2.14}$$

This means equation (2.6) is reversible. Similarly to [8], we have

Lemma 2.4 *The solutions* $v^{\pm}$ *to the fixed point equation* (2.8) *satisfy*

$$\begin{aligned} Rv^{+}(\eta^{+},\lambda)(t) &= v^{-}(R\eta^{+},\lambda)(-t), \\ Rv^{-}(\eta^{-},\lambda)(t) &= v^{+}(R\eta^{-},\lambda)(-t). \end{aligned}$$

Proof The lemma is an immediate consequence of the reversibility of (2.7). For the transition matrix Φ of this equation, we have

$$R\Phi(t,s)=\Phi(-t,-s)R. \tag{2.15}$$

Then we get

$$RP^{+}(s)=P^{-}(-s)R.$$

Together with the uniqueness of the solution to (2.8), the lemma can be proved.

Now we want to investigate $R\xi$. For this we have to distinguish $U\in \mathrm{Fix}(-R)$ and $U\in \mathrm{Fix}\,R$.

First we turn to the case $U\in\mathrm{Fix}(-R)$. From Lemma 2.4 and (2.10), we know

$$\begin{aligned} R\widetilde{w}^{-}(\rho,w^{+},\lambda) &= \widetilde{w}^{+}(-\rho,Rw^{+},\lambda), \\ R\widetilde{w}^{+}(\rho,w^{-},\lambda) &= \widetilde{w}^{-}(-\rho,Rw^{-},\lambda) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} Rz^{+}(\rho,w^{+},\lambda) &= z^{-}(-\rho,Rw^{+},\lambda), \\ Rz^{-}(\rho,w^{-},\lambda) &= z^{+}(-\rho,Rw^{-},\lambda). \end{aligned} \tag{2.17}$$

Exploiting the implicit function theorem applied to (2.11), we get

$$w^{+}(-\rho,\lambda)=Rw^{-}(\rho,\lambda), \tag{2.18}$$

$$w^{-}(-\rho,\lambda)=Rw^{+}(\rho,\lambda). \tag{2.19}$$

Together with (2.17), we obtain

$$R\xi^{\infty}(\rho,\lambda)=-\xi^{\infty}(-\rho,\lambda). \tag{2.20}$$

Before starting our analysis, we use the refinement of decomposition (2.1):

$$R^{2n+2}=\mathrm{span}\{f(\gamma(0),0)\}\oplus U\oplus W^{+}\oplus W^{-}\oplus Y^{c}\oplus \widetilde{Z}, \tag{2.21}$$

where

$$\mathrm{span}\{f(\gamma(0),0)\}\oplus U\oplus Y^{c}=T_{\gamma(0)}W^{cs}\cap T_{\gamma(0)}W^{cu}$$

and

$$Z = \widetilde{Z} \oplus Y^c.$$

Since W^{cs} and W^{cu} intersect transversally in $\gamma(0)$, when restricted to the space

$$\operatorname{span}\{f(\gamma(0),0)\} \oplus U \oplus W^+ \oplus W^- \oplus Y^c,$$

from [9], it is easy to see that $\xi^\infty(\rho,\lambda) \in \widetilde{Z}$. Next, we assume that the primary homoclinic orbits Γ is elementary; that is,

Assumption 2.1 *W^{cu} intersects* $\operatorname{Fix} R$ *transversally at* $\gamma(0)$.

Again counting dimensions, we see that this time $\dim(T_{\gamma(0)}W^{cs} \cap \operatorname{Fix} R) = 1$. Since $RT_{\gamma(0)}W^s = T_{\gamma(0)}W^u$, we have

$$\dim(Y^c \cap \operatorname{Fix} R) = 1.$$

Hence Y^c is spanned by a one-dimensional subspace of $\operatorname{Fix} R$ and a one-dimensional subspace of $\operatorname{Fix}(-R)$. From the direct sum decomposition (2.21) and Lemma 2.2, we get

$$\widetilde{Z} \in \operatorname{Fix} R,$$

which means that $\xi^\infty(\rho,\lambda) \in \operatorname{Fix} R$ since $\xi^\infty(\rho,\lambda) \in \widetilde{Z}$. Together with (2.20), this means that

$$\xi^\infty(-\rho,\lambda) = -\xi^\infty(\rho,\lambda). \tag{2.22}$$

First, we know that

$$\xi^\infty(0,\lambda) \equiv 0. \tag{2.23}$$

From (2.5) and (2.10), we get

$$\begin{aligned} 0 = \xi^\infty(0,\lambda) &= \gamma^+(w^+(0,\lambda),\lambda)(0) - \gamma^-(w^-(0,\lambda),\lambda)(0) \\ &= v^+(w^+(0,\lambda),\lambda)(0) - v^-(w^-(0,\lambda),\lambda)(0) \\ &= w^+(0,\lambda) + \widetilde{w}^-(0,w^+(0,\lambda),\lambda) + z^+(0,w^+(0,\lambda),\lambda) \\ &\quad - w^-(0,\lambda) - \widetilde{w}^+(0,w^-(0,\lambda),\lambda) - z^-(0,w^-(0,\lambda),\lambda). \end{aligned}$$

According to (2.16)–(2.19), we know

$$v^+(w^+(0,\lambda),\lambda)(0) \in \operatorname{Fix} R.$$

Hence

$$\gamma^+(0,\lambda)(0) \in \operatorname{Fix} R.$$

This means that $(0,\lambda)$ is associated with a symmetric homoclinic orbit of (1.1). Because $U \in \operatorname{Fix}(-R)$, we have

$$\begin{aligned} v^+(\rho u_0 + w^+(\rho,\lambda),\lambda)(0) &= \rho u_0 + w^+(\rho,\lambda) + \widetilde{w}^-(\rho,w^+(\rho,\lambda),\lambda) \\ &\quad + z^+(\rho,w^+(\rho,\lambda),\lambda) \notin \operatorname{Fix} R \end{aligned} \tag{2.24}$$

for $\rho \neq 0$. This means that each solution (ρ,λ), $\rho \neq 0$ to $\xi^\infty(\rho,\lambda) = 0$ corresponds to a nonsymmetric homoclinic orbit.

Due to (2.23), we get

$$\xi^\infty(\rho,\lambda) = \rho\phi(\rho,\lambda). \tag{2.25}$$

Obviously, ϕ is even with respect to ρ. Moreover, from (2.13), it follows that

$$\phi(0,0) = 0, \quad D_1\phi(0,0) = 0. \tag{2.26}$$

In order to describe the solution set to $\xi^\infty(\rho,\lambda) = 0$, we impose another transversality condition by assuming that

Assumption 2.2 $\frac{\partial}{\partial\lambda}\phi(0,0)$ *has rank two.*

Combining this with (2.24), we can get the following results:

Theorem 2.1 *Suppose that Assumptions* 1.1–2.2 *are valid for system* (1.1). *Then there exists a unique symmetric homoclinic orbit nearby* Γ. *Moreover, there is a curve* $\kappa \subset R^2$, *such that for each point* $\lambda \in \kappa$, *system* (1.1) *has two nonsymmetric homoclinic orbits. Also, these two homoclinic orbits approach each other for* $\lambda \to 0$ *and merge to the degenerate homoclinic orbit* Γ *for* $\lambda = 0$.

Next, we consider the case $U \subset \mathrm{Fix}\,R$. Instead of (2.16)–(2.20), we get

$$R\xi^\infty(\rho,\lambda) = -\xi^\infty(\rho,\lambda),$$

which means $\xi^\infty(\rho,\lambda) \in \mathrm{Fix}\,(-R)$. Therefore, we get a two-dimensional bifurcation equation

$$\xi^\infty(\rho,\lambda) = 0. \tag{2.27}$$

Let (ρ,λ) be a solution to (2.27). Taking into account (2.10) and (2.16)–(2.19), we see that the corresponding solutions $v^{+(-)}(\rho u_0 + w^{+(-)}(\rho,\lambda),\lambda)(t)$ satisfy

$$Rv^+(\rho,\lambda)(0) = v^-(\rho,\lambda)(0).$$

Hence $v^+(\rho,\lambda)(0) \in \mathrm{Fix}\,R$, which proves that the homoclinic orbit corresponding to (ρ,λ) is symmetric.

Suppose that

Assumption 2.3 $\frac{\partial}{\partial\lambda}\xi^\infty(0,0)$ *has rank two.*

Putting things together, we obtain the following theorem.

Theorem 2.2 *Under Assumptions* 1.1, 1.2 *and* 2.3, *there is a smooth manifold* $\mathcal{H}$ *in the parameter space, such that, for each* $\lambda \in \mathcal{H}$, *there is a one-homoclinic orbit* $\widetilde{\Gamma}$ *to the equilibrium and when* $\lambda \to 0$, $\widetilde{\Gamma} \to \Gamma$. *Moreover, these homoclinic orbits are symmetric.*

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