*Chin. Ann. Math.*
29B(6), 2008, 585–596
DOI: 10.1007/s11401-008-0036-7

**Chinese Annals of Mathematics, Series B**


# A Construction of the Rational Function Sheaves on Elliptic Curves***

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**Abstract** The authors introduce an effective method to construct the rational function sheaf $\mathcal{K}$ on an elliptic curve $\mathbb{E}$, and further study the relationship between $\mathcal{K}$ and any coherent sheaf on $\mathbb{E}$. Finally, it is shown that the category of all coherent sheaves of finite length on $\mathbb{E}$ is completely characterized by $\mathcal{K}$.

**Keywords** Elliptic curve, Coherent sheaf, Rational function sheaf
**2000 MR Subject Classification** 14H52, 14F05, 16G20, 18E15

## 1 Introduction

Let $\mathbb{E}$ be an elliptic curve over an algebraically closed field $k$. A rational function sheaf $\mathcal{K}$ on $\mathbb{E}$ is the constant sheaf having section the function field of $\mathbb{E}$. It is known that $\mathcal{K}$ is a quasi-coherent sheaf, but not a coherent sheaf. By [15] the rational function sheaf $\mathcal{K}$ on $\mathbb{E}$ is the unique big injective sheaf, i.e., $\mathcal{K}$ is the unique indecomposable injective sheaf such that $\mathrm{End}\mathcal{K}$ is a division ring and every quasi-coherent sheaf on $\mathbb{E}$ is a subquotient of a direct sum of copies of $\mathcal{K}$. In particular, each coherent sheaf is a subquotient of a finite direct sum of copies of $\mathcal{K}$, and every simple sheaf is a subquotient of $\mathcal{K}$. In [5], we proved that the rational function sheaf $\mathcal{K}$ is a generic sheaf, i.e., for all coherent sheaves $\mathcal{F}$, both $\mathrm{Hom}(\mathcal{F},\mathcal{K})$ and $\mathrm{Ext}^1(\mathcal{F},\mathcal{K})$ have finite $\mathrm{End}\mathcal{K}$-length. Therefore, it is significant to study the rational function sheaf on $\mathbb{E}$.

C. M. Ringel [14, Proposition 5.2] provided a method to construct the unique indecomposable torsionfree divisible module over the ring of tame representation type. Geigle-Lenzing [7] and Lenzing-Meltzer [11] pointed out that there is a classification of finite dimensional modules over a tubular algebra which is closely related to the Atiyah's classification of vector bundle on an elliptic curve (see [1]). All these encourage us to consider the problems: How many indecomposable torsionfree divisible objects are there in the category of quasi-coherent sheaves on an elliptic curve? And how do we construct them?

It is pleased that many main statements which have been proved in [14] for module categories also hold in the category $\mathrm{Qcoh}\mathbb{E}$ of quasi-coherent sheaves on $\mathbb{E}$ by some corresponding relationship. We show in this paper that $\mathcal{K}$ is the only indecomposable torsionfree divisible object in $\mathrm{Qcoh}\mathbb{E}$, and we can construct the rational function sheaf $\mathcal{K}$ in a way similar to Ringel's method (see [14]). Indeed, the rank functor plays an important role in this construction. Using this construction, we study the relationship between $\mathcal{K}$ and any coherent sheaf on $\mathbb{E}$, and then

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**Project supported by the National Natural Science Foundation of China (No. 10671161) and the Doctoral Program Foundation of the Ministry of Education of China (No. 20060384002).

prove that the category of all coherent sheaves of finite length on $\mathbb{E}$ is completely characterized by $\mathcal{K}$.

## 2 The Category of Coherent Sheaves on an Elliptic Curve

By definition, an elliptic curve $\mathbb{E}$ over an algebraically closed field $k$ is a smooth plane projective curve of genus one admitting a $k$-rational point $p_0$. Every quasi-coherent sheaf on $\mathbb{E}$ is a direct limit of coherent sheaves, and the category Qcoh$\mathbb{E}$ of quasi-coherent sheaves on $\mathbb{E}$ is a locally noetherian Grothendieck category. Hence, the structure of a quasi-coherent sheaf on $\mathbb{E}$ much depends on that of coherent sheaves on $\mathbb{E}$. In this section, we recall some well-known results on the category coh$\mathbb{E}$ of coherent sheaves on $\mathbb{E}$.

**Lemma 2.1** (see [10]) *Let $\mathcal{H} = \mathrm{coh}\mathbb{E}$ be the category of coherent sheaves on $\mathbb{E}$.*

(1) *$\mathcal{H}$ is an Abelian, Ext-finite, noetherian, hereditary and Krull-Schmidt $k$-category.*

(2) *$\mathcal{H}$ is a 1-Calabi-Yau category, that is, for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is an isomorphism $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{DExt}^1(\mathcal{G},\mathcal{F})$, where $\mathrm{D} = \mathrm{Hom}_k(-,k)$.*

(3) *$\mathcal{H} = \mathcal{H}_+ \bigvee \mathcal{H}_0$, that is, each indecomposable object of $\mathcal{H}$ lies either in $\mathcal{H}_+$ or in $\mathcal{H}_0$, and there are no nonzero morphisms from $\mathcal{H}_0$ to $\mathcal{H}_+$, where $\mathcal{H}_+$ denotes the full subcategory of $\mathcal{H}$ consisting of all objects which do not have a simple subobject, and $\mathcal{H}_0$ denotes the full subcategory of $\mathcal{H}$ consisting of all objects of finite length.*

**Remark 2.1** From [13], we know that Qcoh$\mathbb{E}$ is also hereditary.

There is an additive function $\mathrm{rk} : \mathcal{H} \to \mathbb{Z}$, called rank function, separating the objects of $\mathcal{H}_+$ and $\mathcal{H}_0$, that is, an object in $\mathcal{H}_+$ has rank $> 0$ and in $\mathcal{H}_0$ has rank 0. Objects of $\mathcal{H}_+$ are called bundles and those of rank one are called line bundles. In particular, $\mathcal{O}_{\mathbb{E}}$ is a line bundle. It is known that if $\mathcal{L}$ is a line bundle and $\mathcal{S}$ is a simple sheaf, then $\mathrm{Hom}(\mathcal{L},\mathcal{S}) \cong k$ (see [10]).

**Lemma 2.2** (see [10]) *Line bundles have the following properties.*

(1) *Each nonzero morphism from a line bundle to any bundle is a monomorphism. In particular, the endomorphism ring of a line bundle is isomorphic to $k$.*

(2) *Each bundle $\mathcal{F}$ with rank $n$ has a line bundle filtration, that is, a chain*

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n = \mathcal{F}$$

*of subobjects of $\mathcal{F}$, satisfying each quotient $\mathcal{F}_{i+1}/\mathcal{F}_i$ is isomorphic to a line bundle.*

**Lemma 2.3** (see [10]) *$\mathcal{H}_0$ has the following characteristics.*

(1) *$\mathcal{H}_0$ is a hereditary Abelian length category with Serre duality.*

(2) *$\mathcal{H}_0$ is uniserial, and decomposes into a coproduct $\coprod\limits_{x\in\mathbb{E}} \mathcal{U}_x$ of connected uniserial subcategories, whose associated quivers are homogeneous tubes, and the mouth of each homogeneous tube is a simple sheaf.*

For objects $\mathcal{F}$, $\mathcal{G} \in \mathcal{H}$, we define

$$\langle \mathcal{F},\mathcal{G}\rangle = \dim_k \mathrm{Hom}(\mathcal{F},\mathcal{G}) - \dim_k \mathrm{Ext}^1(\mathcal{F},\mathcal{G}).$$

Then the slope of a coherent sheaf $\mathcal{F}$ is an element in $\mathbb{Q} \cup \{\infty\}$ defined as $\mu(\mathcal{F}) = \frac{\chi(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$, where $\chi(\mathcal{F}) = \langle \mathcal{O}_{\mathbb{E}},\mathcal{F}\rangle$.

**Lemma 2.4** (Riemann-Roch Formula) *For any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on an elliptic curve $\mathbb{E}$, we have*

$$\langle \mathcal{F}, \mathcal{G} \rangle = \chi(\mathcal{G})\mathrm{rk}(\mathcal{F}) - \chi(\mathcal{F})\mathrm{rk}(\mathcal{G}).$$

*In particular, $\langle \mathcal{F}, \mathcal{G} \rangle = -\langle \mathcal{G}, \mathcal{F} \rangle$.*

A coherent sheaf $\mathcal{F}$ is called stable (resp. semistable) if for any nontrivial exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, $\mu(\mathcal{F}') < \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') \le \mu(\mathcal{F})$) holds.

**Lemma 2.5** (see [1, 2]) *Any indecomposable coherent sheaf $\mathcal{F}$ on $\mathbb{E}$ is semistable. If two semistable coherent sheaves $\mathcal{F}$, $\mathcal{H} \in \mathrm{coh}\mathbb{E}$ satisfy $\mu(\mathcal{F}) > \mu(\mathcal{H})$, then* $\mathrm{Hom}(\mathcal{F}, \mathcal{H}) = 0$.

**Lemma 2.6** (see [2, 4]) *$\mathcal{H}$ has the following detailed description.*

(1) *Let $\mathrm{coh}^{\infty}\mathbb{E}$ be the category of semistable sheaves of slope $\infty$. Then $\mathrm{coh}^{\infty}\mathbb{E}$ is just $\mathcal{H}_0$. The category of simple sheaves is precisely $\{k(x)\}_{x \in \mathbb{E}}$, where $k(x)$ is a skyscraper sheaf supported at $x$ and it is the mouth of a homogeneous tube $\mathcal{T}_x$ which is the associated quiver of $\mathcal{U}_x$.*

(2) *The indecomposable objects of $\mathcal{H}$ are semistable, and*

$$\mathcal{H} = \mathrm{add}\Big( \bigcup_{q \in \mathbb{Q} \cup \{\infty\}} \mathrm{coh}^q\mathbb{E} \Big),$$

*where $\mathrm{coh}^q\mathbb{E} = \{semistable\ sheaves\ of\ slope\ q\}$.*

(3) *For any $p \in \mathbb{Q} \cup \{\infty\}$, there is an equivalence of Abelian categories $\mathrm{coh}^p\mathbb{E} \cong \mathrm{coh}^{\infty}\mathbb{E}$ induced by an autoequivalence of $D^b(\mathrm{coh}\mathbb{E})$.*

## 3 A Construction of the Rational Function Sheaf on Elliptic Curves

First, we extend the notions of torsion sheaves and torsionfree sheaves in [9] to $\mathrm{coh}\mathbb{E}$.

**Definition 3.1** *For a quasi-coherent sheaf $\mathcal{F}$, its torsion part $t\mathcal{F}$ is defined to be the sum of all subsheaves of $\mathcal{F}$ having finite length. If $t\mathcal{F} = \mathcal{F}$, then $\mathcal{F}$ is called a torsion sheaf. If $t\mathcal{F} = 0$, i.e., $\mathrm{Hom}(\mathcal{S}, \mathcal{F}) = 0$ for each simple sheaf $\mathcal{S}$, then $\mathcal{F}$ is called torsionfree.*

It is easy to see that each object in $\mathcal{H}_0$ is torsion and each object in $\mathcal{H}_+$ is torsionfree. And the class of torsion sheaves is closed under quotients and extensions; the class of torsionfree sheaves is closed under subsheaves and extensions.

For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{E}$, the torsion part $t\mathcal{F}$ is always a pure subsheaf of $\mathcal{F}$ (see [6]). In particular, if $\mathcal{F}$ is a coherent sheaf, $t\mathcal{F}$ is a direct summand of $\mathcal{F}$ (see [10]).

By definition, the following lemma is easy.

**Lemma 3.1** *In $\mathrm{Qcoh}\mathbb{E}$, there is no nonzero morphism from a torsion sheaf to a torsionfree sheaf.*

**Proof** Suppose that there exist a torsion sheaf $\mathcal{F}$ and a torsionfree sheaf $\mathcal{E}$ satisfying $\mathrm{Hom}(\mathcal{F}, \mathcal{E}) \neq 0$. Each $0 \neq f \in \mathrm{Hom}(\mathcal{F}, \mathcal{E})$ induces two short exact sequences:

$$0 \longrightarrow \mathrm{Ker}f \longrightarrow \mathcal{F} \longrightarrow \mathrm{Im}f \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}f \longrightarrow \mathcal{E} \longrightarrow \mathrm{Coker}f \longrightarrow 0.$$

The first short exact sequence implies that $\mathrm{Im}f$ is torsion, but the second one implies that $\mathrm{Im}f$ is torsionfree. Then $\mathrm{Im}f = t\mathrm{Im}f = 0$, a contradiction.

Let $\mathcal{F}$ be a coherent sheaf, and $\mathcal{S}$ be a simple sheaf. Then the dimension of $\mathrm{Ext}^1(\mathcal{S},\mathcal{F})$ as $\mathrm{End}\mathcal{S}$-vector space is finite. We set

$$e_{_{\mathcal{S}\mathcal{F}}} = \dim \mathrm{Ext}^1(\mathcal{S},\mathcal{F})_{\mathrm{End}\mathcal{S}}.$$

Since $\mathrm{End}\mathcal{S} \cong k$, we write $e_{_{\mathcal{S}\mathcal{F}}} = \dim_k \mathrm{Ext}^1(\mathcal{S},\mathcal{F})$.

The following lemma shows that [14, Lemma 5.2] also holds in $\mathrm{coh}\mathbb{E}$ by some corresponding relationships.

**Lemma 3.2** *Let $\mathcal{F}$ be a bundle, and $\mathcal{S}$ be a simple sheaf. If there exists an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \oplus_m \mathcal{S} \longrightarrow 0,$$

*where $\mathcal{F}'$ is a bundle and $\oplus_m\mathcal{S}$ denotes the direct sum of $m$ copies of $\mathcal{S}$, then $m \le e_{_{\mathcal{S}\mathcal{F}}}$. Conversely, for $m \le e_{_{\mathcal{S}\mathcal{F}}}$, there exists such an exact sequence with $\mathcal{F}'$ being a bundle.*

**Proof** See the proof of [14, Lemma 5.2], and we only need to replace $P, X$ and $S$ by $\mathcal{F}, \mathcal{F}'$ and $\mathcal{S}$ respectively.

Next, we extend the notion of divisible in [14, Definition 4.6] to $\mathrm{Qcoh}\mathbb{E}$.

**Definition 3.2** *A quasi-coherent sheaf $\mathcal{F}$ is called divisible if $\mathrm{Ext}^1(\mathcal{S},\mathcal{F}) = 0$ for all simple sheaves $\mathcal{S}$.*

It is easy to check that the class of divisible sheaves is closed under quotients and extensions. And then the class of torsionfree divisible sheaves is closed under direct summands.

**Lemma 3.3** *A quasi-coherent sheaf $\mathcal{I}$ is divisible if and only if it is an injective sheaf.*

**Proof** By definition, an injective sheaf is obviously a divisible sheaf. And by Baer's test, it is not hard to see that the "only if" part holds.

Now we can show that [14, Lemma 5.1] also holds in $\mathrm{Qcoh}\mathbb{E}$.

**Lemma 3.4** *Let $\mathcal{E}, \mathcal{F}$ be torsionfree divisible sheaves, $\mathcal{E}' \subseteq \mathcal{E}$, $\mathcal{F}' \subseteq \mathcal{F}$ be subsheaves such that $\mathcal{E}/\mathcal{E}'$ and $\mathcal{F}/\mathcal{F}'$ are torsion sheaves. Then any homomorphism $\varphi' : \mathcal{E}' \to \mathcal{F}'$ has a unique extension $\varphi : \mathcal{E} \to \mathcal{F}$. In particular, if $\varphi'$ is an isomorphism, then its extension $\varphi$ is an isomorphism.*

**Proof** Consider the following two short exact sequences

$$0 \longrightarrow \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\pi} \mathcal{E}/\mathcal{E}' \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\beta} \mathcal{F} \xrightarrow{\sigma} \mathcal{F}/\mathcal{F}' \longrightarrow 0.$$

Forming the pushout of $\alpha$ and $\beta\varphi'$ induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r, "\alpha"] \arrow[d, "\beta\varphi'"'] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\gamma"'] & \mathcal{E}/\mathcal{E}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r, "\alpha'"] & \mathcal{F}'' \arrow[r, "\pi'"] & \mathcal{E}/\mathcal{E}' \arrow[r] & 0.
\end{tikzcd}$$

According to Lemma 3.3, $\mathcal{F}$ is an injective sheaf. Then $\mathrm{Ext}^1(\mathcal{E}/\mathcal{E}', \mathcal{F}) = 0$. This shows that there exists a homomorphism $\delta : \mathcal{F}'' \to \mathcal{F}$ such that $\delta\alpha' = \mathrm{id}_{\mathcal{F}}$. Let $\varphi = \delta\gamma$. Then $\varphi$ is an extension of $\varphi'$ and $\varphi\alpha = \delta\gamma\alpha = \delta\alpha'\beta\varphi' = \beta\varphi'$.

In order to prove the uniqueness, it is sufficient to show that the extension of zero homomorphism must be zero. If $\varphi' = 0$, then $\varphi\alpha = \beta\varphi' = 0$. Thus, there exists a unique homomorphism $\theta : \mathcal{E}/\mathcal{E}' \to \mathcal{F}$ such that $\varphi = \theta\pi$. However, $\mathcal{E}/\mathcal{E}'$ is torsion and $\mathcal{F}$ is torsionfree, so $\theta = 0$, and then $\varphi = 0$.

Now assume that $\varphi'$ is an isomorphism. Since $\varphi\alpha = \beta\varphi'$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r, "\alpha"] \arrow[d, "\varphi'"'] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \mathcal{E}/\mathcal{E}' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & \mathcal{F}/\mathcal{F}' \arrow[r] & 0.
\end{tikzcd}$$

Since $\varphi'$ is an isomorphism, there exists a homomorphism $\psi' : \mathcal{F}' \to \mathcal{E}'$ such that $\psi'\varphi' = \mathrm{id}_{\mathcal{E}'}$ and $\varphi'\psi' = \mathrm{id}_{\mathcal{F}'}$. Using a similar argument as above, we see that $\psi'$ has a unique extension $\psi : \mathcal{F} \to \mathcal{E}$ such that $\psi\beta = \alpha\psi'$. Therefore, $\psi'\varphi'$ has an extension $\psi\varphi$. But $\psi'\varphi' = \mathrm{id}_{\mathcal{E}'}$ has an extension $\mathrm{id}_{\mathcal{E}}$, so $\psi\varphi = \mathrm{id}_{\mathcal{E}}$ holds by the uniqueness of the extension. Similarly, we have $\varphi\psi = \mathrm{id}_{\mathcal{F}}$, and then $\varphi$ is an isomorphism.

Under the previous groundwork, and by a little change of the proof of [14, Proposition 5.2], it is not hard to obtain the following theorem.

**Theorem 3.1** *Let $\mathcal{F}$ be a bundle. Then there exists a torsionfree divisible sheaf $\mathcal{G}_{\mathcal{F}}$ with an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_{\mathcal{F}} \longrightarrow \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \longrightarrow 0,$$

*where $\mathcal{S}$ runs through all simple sheaves, $\mathcal{S}_\infty$ is the direct limit of the homogeneous tube whose mouth is $\mathcal{S}$.*

**Proof** Let $\mathcal{S}$ be a simple sheaf. According to the structure of $\mathrm{coh}\mathbb{E}$, we have $\mathrm{Ext}^1(\mathcal{S}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{S}) \neq 0$. By Lemma 3.2, there exists a short exact sequence

$$\xi'_{\mathcal{S}} : 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'_{\mathcal{S}} \longrightarrow \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S} \longrightarrow 0,$$

such that $\mathcal{F}'_{\mathcal{S}}$ is a bundle. Since the inclusion $\gamma_{\mathcal{S}} : \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S} \to \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty$ induces an epimorphism $\mathrm{Ext}^1(\oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty, \mathcal{F}) \to \mathrm{Ext}^1(\oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}, \mathcal{F})$, we choose $\xi_{\mathcal{S}} \in \mathrm{Ext}^1(\oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty, \mathcal{F})$ corresponding to $\xi'_{\mathcal{S}}$. Thus, we have the following commutative diagram

$$\begin{tikzcd}
\xi'_{\mathcal{S}} : 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F}'_{\mathcal{S}} \arrow[r, "\pi'_{\mathcal{S}}"] \arrow[d, "\gamma_{\mathcal{S}'}"'] & \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S} \arrow[r] \arrow[d, "\gamma_{\mathcal{S}}"'] & 0 \\
\xi_{\mathcal{S}} : 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}''_{\mathcal{S}} \arrow[r, "\pi_{\mathcal{S}}"] & \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r] & 0.
\end{tikzcd}$$

Let $\xi = (\xi_{\mathcal{S}})_{\mathcal{S}} \in \Pi_{\mathcal{S}} \mathrm{Ext}^1(\oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty, \mathcal{F}) = \mathrm{Ext}^1(\oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty, \mathcal{F})$ be as follows:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}_{\mathcal{F}} \xrightarrow{\pi} \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \longrightarrow 0,$$

where $\mathcal{S}$ runs through all simples. Then for each $\xi_{\mathcal{S}}$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\alpha"] \arrow[d, equal] & \mathcal{G}_{\mathcal{F}} \arrow[r, "\pi"] \arrow[d, "\sigma'_{\mathcal{S}}"'] & \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r] \arrow[d, "\sigma_{\mathcal{S}}"'] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}''_{\mathcal{S}} \arrow[r, "\pi_{\mathcal{S}}"] & \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r] & 0.
\end{tikzcd}$$

In order to show that $\mathcal{G}_{\mathcal{F}}$ is torsionfree, suppose that there is a nonzero monomorphism $\iota : \mathcal{T} \to \mathcal{G}_{\mathcal{F}}$, where $\mathcal{T}$ is a simple sheaf. It is easy to see from $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ that $\pi\iota \neq 0$. According to the structure of coh$\mathbb{E}$, we know $\pi\iota \in \mathrm{Hom}(\mathcal{T}, \oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}_\infty)$. Thus, there is a monomorphism $\iota'' : \mathcal{T} \to \oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}$ such that $\gamma_{\mathcal{T}}\iota'' = \sigma_{\mathcal{T}}\pi\iota$. By the universal property of pullback, there exists a nonzero homomorphism $f : \mathcal{T} \to \mathcal{F}'_{\mathcal{T}}$ such that $\gamma'_{\mathcal{T}} f = \sigma'_{\mathcal{T}}\iota$ and $\pi'_{\mathcal{T}} f = \iota''$, a contradiction. Thus we conclude that $\mathcal{G}_{\mathcal{F}}$ is torsionfree.

Now we are going to show that $\mathcal{G}_{\mathcal{F}}$ is divisible. Suppose that there exists a simple sheaf $\mathcal{T}$ such that $\mathrm{Ext}^1(\mathcal{T}, \mathcal{G}_{\mathcal{F}}) \neq 0$. Then a nonzero element $\xi$ in $\mathrm{Ext}^1(\mathcal{T}, \mathcal{G}_{\mathcal{F}})$,

$$\xi : 0 \longrightarrow \mathcal{G}_{\mathcal{F}} \xrightarrow{\beta} \mathcal{E} \xrightarrow{\gamma} \mathcal{T} \longrightarrow 0,$$

induces the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{F} \arrow[r, equal] \arrow[d] & \mathcal{F} \arrow[d] & & \\
0 \arrow[r] & \mathcal{G}_{\mathcal{F}} \arrow[r, "\beta"] \arrow[d, "\pi"'] & \mathcal{E} \arrow[r, "\gamma"] \arrow[d, "\pi'"'] & \mathcal{T} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r, "\beta'"] \arrow[d] & \mathcal{E}' \arrow[r, "\gamma'"] \arrow[d] & \mathcal{T} \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\mathrm{Ext}^1(\mathcal{T}, \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty) = 0$, the lower short exact sequence splits. Consequently, $\mathcal{E}'$ has $(\oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}) \oplus \mathcal{T}$ as a subsheaf. Considering the pullback of $\iota'$ and $\pi'$, where $\iota' : (\oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}) \oplus \mathcal{T} \to \mathcal{E}'$ is the embedding, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E}'' \arrow[r] \arrow[d, "\iota''"'] & (\oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}) \oplus \mathcal{T} \arrow[r] \arrow[d, "\iota'"'] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}' \arrow[r] & 0.
\end{tikzcd}$$

The fact that $\iota'$ is a monomorphism implies that so is $\iota''$. On the other hand, $\mathcal{E}''$ has a subsheaf isomorphic to $\mathcal{T}$ by Lemma 3.2. Thus we have the inclusion $\sigma : \mathcal{T} \to \mathcal{E}$. It is obvious that $\gamma\sigma \neq 0$ and $\gamma$ is a split monomorphism, a contradiction.

Now we begin to prove that there is only one indecomposable torsionfree divisible sheaf.

**Lemma 3.5** *Let $\mathcal{L}$ be a line bundle, $\mathcal{G}$ be a torsionfree divisible sheaf. Then* $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) \neq 0$.

**Proof** It is sufficient to prove that $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) = 0$ implies $\mathcal{G} = 0$.

Suppose $\mathcal{G} \neq 0$ and $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) = 0$. Since $\mathcal{G}$ is torsionfree, all the coherent subsheaves of $\mathcal{G}$ are bundles. Note that each bundle has a line bundle filtration. Then $\mathcal{G}$ has a subsheaf $\mathcal{L}'$ which is a line bundle with the inclusion $\iota : \mathcal{L}' \to \mathcal{G}$. Now we consider three possibilities.

(1) $\mu(\mathcal{L}') < \mu(\mathcal{L})$. Then $\mathrm{Hom}(\mathcal{L}', \mathcal{L}) \neq 0$, and there is a short exact sequence

$$0 \longrightarrow \mathcal{L}' \xrightarrow{\alpha} \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Note that $\mathcal{E} \in \mathcal{H}_0$ since $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{L}) - \mathrm{rk}(\mathcal{L}') = 0$. The pushout of $\iota$ and $\alpha$ induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}' \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \mathcal{L} \arrow[r] \arrow[d, "\iota'"'] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r, "\alpha'"] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{G}$ is a divisible sheaf, we get $\mathrm{Ext}^1(\mathcal{E}, \mathcal{G}) = 0$. Then there exists a homomorphism $\pi : \mathcal{F} \to \mathcal{G}$ such that $\pi\alpha' = \mathrm{id}_{\mathcal{G}}$. Let $f = \pi\iota' : \mathcal{L} \to \mathcal{G}$. We have $\iota = \pi\alpha'\iota = \pi\iota'\alpha = f\alpha$, which implies $f \neq 0$. This contradicts the fact that $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) = 0$.

(2) $\mu(\mathcal{L}) < \mu(\mathcal{L}')$. Then $\mathrm{Hom}(\mathcal{L}, \mathcal{L}') \neq 0$, and there is a monomorphism $\iota'' : \mathcal{L} \to \mathcal{L}'$. Thus, $0 \neq \iota\iota'' : \mathcal{L} \to \mathcal{G}$ implies that $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) \neq 0$. It is a contradiction.

(3) $\mu(\mathcal{L}') = \mu(\mathcal{L})$. Let $\mathcal{L}''$ be a line bundle with $\mu(\mathcal{L}'') < \mu(\mathcal{L})$. Similarly to (2), we can regard $\mathcal{L}''$ as a subsheaf of $\mathcal{G}$. Then the result is true by an analogue to case (1).

**Lemma 3.6** *Let $\mathcal{G}$ be an indecomposable torsionfree divisible sheaf, $\mathcal{Q}$ be a torsionfree divisible sheaf which has no subsheaf isomorphic to $\mathcal{G}$. Then* $\mathrm{Hom}(\mathcal{G}, \mathcal{Q}) = 0$.

**Proof** Suppose $\mathrm{Hom}(\mathcal{G}, \mathcal{Q}) \neq 0$. Then, for each $0 \neq f \in \mathrm{Hom}(\mathcal{G}, \mathcal{Q})$, there are two short exact sequences

$$0 \longrightarrow \mathrm{Ker}f \longrightarrow \mathcal{G} \longrightarrow \mathrm{Im}f \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow \mathrm{Im}f \longrightarrow \mathcal{Q} \longrightarrow \mathrm{Coker}f \longrightarrow 0 \tag{3.2}$$

We consider two possibilities.

(1) $\mathrm{Ker}f = 0$. Then $\mathcal{G} \cong \mathrm{Im}f$. Thus, $\mathcal{Q}$ has a subsheaf $\mathrm{Im}f$ isomorphic to $\mathcal{G}$. It is a contradiction.

(2) $\mathrm{Ker}f \neq 0$. Then $\mathrm{Im}f$ is divisible according to (3.1), and is torsionfree according to (3.2). Let $\mathcal{S}$ be any simple sheaf. Applying $\mathrm{Hom}(\mathcal{S}, -)$ to (3.1), we obtain the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(\mathcal{S}, \mathrm{Ker}f) \longrightarrow \mathrm{Hom}(\mathcal{S}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{S}, \mathrm{Im}f) \\ &\longrightarrow \mathrm{Ext}^1(\mathcal{S}, \mathrm{Ker}f) \longrightarrow \mathrm{Ext}^1(\mathcal{S}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1(\mathcal{S}, \mathrm{Im}f) \longrightarrow 0. \end{aligned}$$

Then $\mathrm{Ker}f$ is torsionfree divisible since $\mathcal{G}$ and $\mathrm{Im}f$ are torsionfree divisible. Thus, (3.1) splits, and then $\mathrm{Ker}f$ is a direct summand of $\mathcal{G}$. Hence $\mathrm{Ker}f \cong \mathcal{G}$ since $\mathcal{G}$ is indecomposable. This means $\mathrm{Im}f = 0$, which contradicts $f \neq 0$.

**Lemma 3.7** *Let $\mathcal{L}$ be a line bundle, and $\mathcal{G}$ be an indecomposable torsionfree divisible sheaf. Then $\mathcal{G}/\mathcal{L}$ is a torsion sheaf.*

**Proof** By Lemma 3.5, there exists a short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{G}/\mathcal{L} \longrightarrow 0.$$

Set $\mathcal{G}_1 = \mathcal{G}/\mathcal{L}$. Suppose that $\mathcal{G}_1$ is not a torsion sheaf, i.e., $t\mathcal{G}_1 \neq \mathcal{G}_1$. Then there is a short exact sequence

$$0 \longrightarrow t\mathcal{G}_1 \xrightarrow{\alpha} \mathcal{G}_1 \xrightarrow{\pi'} \mathcal{G}_1/t\mathcal{G}_1 \longrightarrow 0.$$

Thus, the composition $\pi'\pi : \mathcal{G} \to \mathcal{G}_1/t\mathcal{G}_1$ is an epimorphism. Note that the class of divisible sheaves is closed under quotients. We have that $\mathcal{G}_1/t\mathcal{G}_1$ is a torsionfree divisible sheaf. By Lemma 3.6, $\mathcal{G}_1/t\mathcal{G}_1$ has a subsheaf isomorphic to $\mathcal{G}$. Thus, we can regard $\mathcal{G}$ as a direct summand of $\mathcal{G}_1/t\mathcal{G}_1$. This induces an epimorphism $\beta : \mathcal{G}_1/t\mathcal{G}_1 \to \mathcal{G}$ which gives a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\beta\pi'\pi \longrightarrow \mathcal{G} \xrightarrow{\beta\pi'\pi} \mathcal{G} \longrightarrow 0.$$

If $\mathrm{Ker}\beta\pi'\pi \neq 0$, then $\mathrm{Ker}\beta\pi'\pi$ is torsionfree divisible since $\mathcal{G}$ is torsionfree divisible. Thus, $\mathrm{Ker}\beta\pi'\pi$ is a direct summand of $\mathcal{G}$, and then $\mathrm{Ker}\beta\pi'\pi \cong \mathcal{G}$. This is impossible. Hence $\mathrm{Ker}\beta\pi'\pi = 0$ and then $\beta\pi'\pi$ is an isomorphism. This implies that $\pi$ is a monomorphism and then $\pi$ is an isomorphism. This is a contradiction. Therefore, we conclude that $\mathcal{G}_1$ is a torsion sheaf.

**Theorem 3.2** *There exists a unique indecomposable torsionfree divisible sheaf. Its endomorphism ring is a division ring.*

**Proof** Let $\mathcal{L}$ be a line bundle. By Theorem 3.1, there exists a short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{G}_{\mathcal{L}} \xrightarrow{\pi} \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{L}}} \mathcal{S}_\infty \longrightarrow 0,$$

where $\mathcal{G}_{\mathcal{L}}$ is a torsionfree divisible sheaf. According to the structure of coh$\mathbb{E}$, we have that $\dim \mathrm{Ext}^1(\mathcal{S},\mathcal{L})_{\mathrm{End}\mathcal{S}} = \dim_k \mathrm{Ext}^1(\mathcal{S},\mathcal{L}) = \dim_k \mathrm{Hom}(\mathcal{L},\mathcal{S}) = 1$ for each simple sheaf $\mathcal{S}$, i.e., $e_{\mathcal{S}\mathcal{L}} = 1$. Thus, the above exact sequence is just

$$0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{G}_{\mathcal{L}} \xrightarrow{\pi} \oplus_{\mathcal{S}}\mathcal{S}_\infty \longrightarrow 0.$$

We claim that $\mathcal{G}_{\mathcal{L}}$ is the unique indecomposable torsionfree divisible sheaf. Denote $\mathcal{G}_{\mathcal{L}}$ by $\mathcal{G}$.

First, we prove that $\mathcal{G}$ is indecomposable. Let $\mathcal{G}_1$ be a direct summand of $\mathcal{G}$. Then there exist homomorphisms $\iota : \mathcal{G}_1 \to \mathcal{G}$ and $\sigma : \mathcal{G} \to \mathcal{G}_1$ such that $\sigma\iota = \mathrm{id}_{\mathcal{G}_1}$. If $\pi\iota = 0$, then there exists a homomorphism $f : \mathcal{G}_1 \to \mathcal{L}$ such that $\iota = \alpha f$ and $\sigma\alpha f = \sigma\iota = \mathrm{id}_{\mathcal{G}_1}$. This means $\mathcal{L} \cong \mathcal{G}_1$ and then $\mathcal{L}$ is a direct summand of $\mathcal{G}$. This contradicts the fact that $\mathcal{L}$ is not divisible. Therefore $\pi\iota \neq 0$. If $\mathrm{Ker}\pi\iota = 0$, then we can regard $\mathcal{G}_1$ as a subsheaf of $\oplus_{\mathcal{S}}\mathcal{S}_\infty$. This contradicts the fact that $\mathcal{G}_1$ is a torsionfree divisible. If $\mathrm{Ker}\pi\iota \neq 0$, then we have $\pi\iota\alpha_1 = 0$, where $\alpha_1 : \mathrm{Ker}\pi\iota \to \mathcal{G}_1$ is the inclusion. Thus, there exists a monomorphism $\iota' : \mathrm{Ker}\pi\iota \to \mathcal{L}$ such that $\iota\alpha_1 = \alpha\iota'$. This induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\pi\iota \arrow[r, "\iota'"] \arrow[d, "\alpha_1"'] & \mathcal{L} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{L}/\mathrm{Ker}\pi\iota \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_1 \arrow[r, "\iota"] & \mathcal{G} \arrow[r, "\pi'"] & \mathcal{G}/\mathcal{G}_1 \arrow[r] & 0.
\end{tikzcd}$$

It is obvious that $\mathrm{Ker}\pi\iota$ is a line bundle and that $\mathrm{rk}(\mathcal{L}/\mathrm{Ker}\pi\iota) = \mathrm{rk}\mathcal{L} - \mathrm{rk}(\mathrm{Ker}\pi\iota) = 0$. This means that $\mathcal{L}/\mathrm{Ker}\pi\iota$ is torsion. Note that $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}/\mathcal{G}_1$. We know that $\mathcal{G}/\mathcal{G}_1$ is torsionfree and that $\mathrm{Hom}(\mathcal{L}/\mathrm{Ker}\pi\iota, \mathcal{G}/\mathcal{G}_1) = 0$. This implies that $\pi'\alpha = 0$. So there exists a homomorphism $\beta : \oplus_{\mathcal{S}}\mathcal{S}_\infty \to \mathcal{G}/\mathcal{G}_1$ such that $\pi' = \beta\pi$. Thus $\pi' = 0$ since $\mathrm{Hom}(\oplus_{\mathcal{S}}\mathcal{S}_\infty, \mathcal{G}/\mathcal{G}_1) = 0$. This means $\mathcal{G} \cong \mathcal{G}_1$. Therefore, $\mathcal{G}$ is indecomposable.

Next, we show that the endomorphism ring End$\mathcal{G}$ of $\mathcal{G}$ is a division ring. For a homomorphism $0 \neq \varphi : \mathcal{G} \to \mathcal{G}$, we claim that $\varphi(\mathcal{L}) \neq 0$. In fact, otherwise, $\varphi(\mathcal{L}) = 0$ implies $\varphi\alpha = 0$. This induces a homomorphism $\gamma : \oplus_{\mathcal{S}}\mathcal{S}_\infty \to \mathcal{G}$ such that $\varphi = \gamma\pi$. This contradicts the fact that $\mathrm{Hom}(\oplus_{\mathcal{S}}\mathcal{S}_\infty, \mathcal{G}) = 0$. Thus $\varphi(\mathcal{L}) \neq 0$. Thus, $\varphi(\mathcal{L})$, as a subsheaf of $\mathcal{G}$, is torsionfree. Note that a nonzero homomorphism from a line bundle to a torsionfree sheaf is a monomorphism. We have that $\varphi|_{\mathcal{L}} : \mathcal{L} \to \varphi(\mathcal{L})$ is an isomorphism. Following Lemma 3.4, the fact that $\varphi$ is an extension of $\varphi|_{\mathcal{L}}$ implies that $\varphi$ is an isomorphism. Consequently, End$\mathcal{G}$ is a division ring.

Finally, we show the uniqueness. Let $\mathcal{G}, \mathcal{G}'$ be two indecomposable torsionfree divisible sheaves. For any line bundle $\mathcal{L}$, both $\mathcal{G}/\mathcal{L}$ and $\mathcal{G}'/\mathcal{L}$ are torsion sheaves by Lemma 3.7. Thus, by Lemma 3.4, $\mathrm{id}_{\mathcal{L}}$ induces the isomorphism $\varphi : \mathcal{G} \to \mathcal{G}'$.

As we know, the rational function sheaf $\mathcal{K}$ on $\mathbb{E}$ is the constant sheaf having section the function field of $\mathbb{E}$, and $\mathcal{K}$ is an indecomposable injective sheaf. In addition, by [15], $\mathcal{K}$ is torsionfree. Thus, the uniqueness in Theorem 3.2 implies that $\mathcal{K}$ coincides with $\mathcal{G}_{\mathcal{L}}$ constructed in the proof of Theorem 3.2. In other words, the proof of Theorem 3.2 in fact gives a construction of $\mathcal{K}$.

## 4 The Relationship Between $\mathcal{K}$ and Coherent Sheaves

In this section, we use the construction to study the relationship between $\mathcal{K}$ and coherent sheaves on $\mathbb{E}$.

**Lemma 4.1** *$\mathcal{K}$ is a direct summand of any torsionfree divisible sheaf.*

**Proof** Let $\mathcal{G}$ be a torsionfree divisible sheaf, and $\mathcal{L}$ be a line bundle. By Lemma 3.5, there are monomorphisms $\alpha : \mathcal{L} \to \mathcal{G}$ and $\iota : \mathcal{L} \to \mathcal{K}$. Considering the pushout of $\alpha$ and $\iota$, we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \mathcal{G} \arrow[r] \arrow[d, "\iota'"'] & \mathcal{G}/\mathcal{L} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "\alpha'"] & \mathcal{G}' \arrow[r] & \mathcal{G}/\mathcal{L} \arrow[r] & 0. \end{tikzcd}$$

Then $\iota'$ is a monomorphism. Since $\mathcal{G}$ is divisible, i.e., it is injective, there exists a homomorphism $\pi : \mathcal{G}' \to \mathcal{G}$ such that $\pi\iota' = \mathrm{id}_{\mathcal{G}}$. Set $\beta = \pi\alpha'$. We have $\beta\iota = \pi\alpha'\iota = \pi\iota'\alpha = \alpha$, and then $\beta \neq 0$. By Lemma 3.6, there is a monomorphism $\beta' : \mathcal{K} \to \mathcal{G}$. Since $\mathcal{K}$ is injective, we obtain that $\mathcal{K}$ is a direct summand of $\mathcal{G}$.

**Theorem 4.1** *Any torsionfree divisible sheaf is a direct sum of copies of $\mathcal{K}$.*

**Proof** Let $\mathcal{G}$ be a torsionfree divisible sheaf. By transfinite induction, we shall construct a torsionfree divisible subsheaf $\mathcal{G}_\lambda$ with $\mathcal{G}_\lambda \cong \mathcal{G}_{\lambda-1} \oplus \mathcal{K}$ for any ordinal $\lambda$, and $\mathcal{G}_\lambda = \bigcup\limits_{\mu<\lambda} \mathcal{G}_\mu$ for any limit ordinal $\lambda$, such that for any $\lambda$, $\mathcal{G} \cong \mathcal{G}_\lambda \oplus \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is a torsionfree divisible sheaf. The construction will stop when $\mathcal{H}_\lambda = 0$.

Let $\mathcal{G}_0 = 0$. Assume that $\mathcal{G}_\lambda$ has been defined for an ordinal $\lambda$, with $\mathcal{G} \cong \mathcal{G}_\lambda \oplus \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is a nonzero torsionfree divisible sheaf. By Lemma 4.1, $\mathcal{K}$ is a direct summand of $\mathcal{H}_\lambda$. This means that there exists a torsionfree divisible sheaf $\mathcal{H}_{\lambda+1}$ such that $\mathcal{H}_\lambda \cong \mathcal{K} \oplus \mathcal{H}_{\lambda+1}$. Let $\mathcal{G}_{\lambda+1} = \mathcal{G}_\lambda \oplus \mathcal{K}$. Then $\mathcal{G}_{\lambda+1}$ is also a torsionfree divisible sheaf and $\mathcal{G} \cong \mathcal{G}_{\lambda+1} \oplus \mathcal{H}_{\lambda+1}$. Assume that $\mathcal{G}_\mu$ has been defined for all $\mu < \lambda$. Let $\mathcal{G}_\lambda = \bigcup\limits_{\mu<\lambda} \mathcal{G}_\mu$. We claim that $\mathcal{G}_\lambda$ is also a torsionfree divisible subsheaf of $\mathcal{G}$. In fact, suppose that there is a simple sheaf $\mathcal{S}$ satisfying $\mathrm{Hom}(\mathcal{S},\mathcal{G}_\lambda)\neq 0$; then there exists $\mu<\lambda$ such that $\mathrm{Hom}(\mathcal{S},\mathcal{G}_\mu)\neq 0$. This is a contradiction. Suppose that there is a simple sheaf $\mathcal{T}$ satisfying $\mathrm{Ext}^1(\mathcal{T},\mathcal{G}_\lambda)\neq 0$. Since $\mathrm{Qcoh}\mathbb{E}$ is a hereditary Abelian category, we have $\mathrm{Ext}^1(\mathcal{T},\mathcal{G}_\lambda)\cong \mathrm{Hom}(\mathcal{T},\mathcal{G}_\lambda[1])$ in the derived category of $\mathrm{Qcoh}\mathbb{E}$, where [1] is the translation functor (see [10, Theorem 2.1]). Then there exists $\nu<\lambda$ with $\mathrm{Hom}(\mathcal{T},\mathcal{G}_\nu[1])\neq 0$ and then $\mathrm{Ext}^1(\mathcal{T},\mathcal{G}_\nu)\neq 0$. This is also a contradiction. Thus, $\mathcal{G}_\lambda$ is also a torsionfree divisible subsheaf of $\mathcal{G}$. Then there exists a torsionfree divisible sheaf $\mathcal{H}_\lambda$ such that $\mathcal{G}\cong\mathcal{G}_\lambda\oplus\mathcal{H}_\lambda$. By the construction, we know that any $\mathcal{G}_\lambda$ is the direct sum of copies of $\mathcal{K}$. The proof is completed.

By Theorem 3.1, for each bundle $\mathcal{F}$, there exists a short exact sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\mathcal{G}_\mathcal{F}\longrightarrow\oplus_\mathcal{S}\oplus_{e_{\mathcal{S}\mathcal{F}}}\mathcal{S}_\infty\longrightarrow 0.$$

**Theorem 4.2** *Let $\mathcal{F}$ be a bundle with $\mathrm{rk}\mathcal{F}=n$. Then $\mathcal{G}_\mathcal{F}\cong\oplus_n\mathcal{K}$.*

**Proof** In view of the structure of $\mathrm{coh}\mathbb{E}$, $\mathcal{F}$ has a line bundle filtration. That is, there exists a chain

$$0=\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots\subseteq\mathcal{F}_{n-1}\subseteq\mathcal{F}_n=\mathcal{F}$$

of subsheaves of $\mathcal{F}$, such that each filtration quotient $\mathcal{F}_{i+1}/\mathcal{F}_i$ is isomorphic to a line bundle, denoted by $\mathcal{L}_{i+1}$. Then there are short exact sequences

$$0\longrightarrow\mathcal{F}_i\xrightarrow{\alpha_i}\mathcal{F}_{i+1}\xrightarrow{\beta_i}\mathcal{L}_{i+1}\longrightarrow 0,\quad 0\le i\le n-1.$$

According to Lemma 3.4, it is sufficient to construct the following exact sequence

$$0\longrightarrow\mathcal{F}_j\xrightarrow{\iota_j}\oplus_j\mathcal{K}\xrightarrow{\pi_j}\mathcal{D}_j\longrightarrow 0,\quad 0\le j\le n,$$

with $\mathcal{D}_j$ torsion by using induction on $j$. By the construction of $\mathcal{K}$, the assertion holds obviously in case $i=1$. Assume that there is a short exact sequence

$$0\longrightarrow\mathcal{F}_i\xrightarrow{\iota_i}\oplus_i\mathcal{K}\xrightarrow{\pi_i}\mathcal{D}_i\longrightarrow 0$$

with $\mathcal{D}_i$ torsion. Considering the pushout of $\iota_i$ and $\alpha_i$, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{F}_i \arrow[r, "\alpha_i"] \arrow[d, "\iota_i"'] & \mathcal{F}_{i+1} \arrow[r] \arrow[d, "\sigma_{i+1}"'] & \mathcal{L}_{i+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \oplus_i\mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_{i+1} \arrow[r] \arrow[d] & \mathcal{L}_{i+1} \arrow[r] & 0. \\
 & \mathcal{D}_i \arrow[r, equal] \arrow[d] & \mathcal{D}_i \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then $\mathcal{E}_{i+1}/\mathcal{F}_{i+1} \cong \mathcal{D}_i$, and $\mathcal{E}_{i+1}/\mathcal{F}_{i+1}$ is torsion. In addition, the short exact sequence

$$0 \longrightarrow \oplus_i \mathcal{K} \longrightarrow \mathcal{E}_{i+1} \longrightarrow \mathcal{L}_{i+1} \longrightarrow 0$$

splits since $\oplus_i \mathcal{K}$ is an injective sheaf. This induces a short exact sequence

$$0 \longrightarrow \mathcal{L}_{i+1} \xrightarrow{\gamma_{i+1}} \mathcal{E}_{i+1} \longrightarrow \oplus_i \mathcal{K} \longrightarrow 0.$$

Again by the construction of $\mathcal{K}$, there is a short exact sequence

$$0 \longrightarrow \mathcal{L}_{i+1} \xrightarrow{\tau_{i+1}} \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{L}_{i+1} \longrightarrow 0,$$

where $\mathcal{K}/\mathcal{L}_{i+1}$ is torsion. Considering the pushout of $\tau_{i+1}$ and $\gamma_{i+1}$, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{L}_{i+1} \arrow[r, "\gamma_{i+1}"] \arrow[d, "\tau_{i+1}"'] & \mathcal{E}_{i+1} \arrow[r] \arrow[d, "\sigma'_{i+1}"'] & \oplus_i \mathcal{K} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}'_{i+1} \arrow[r] \arrow[d] & \oplus_i \mathcal{K} \arrow[r] & 0. \\
 & \mathcal{K}/\mathcal{L}_{i+1} \arrow[r, equal] \arrow[d] & \mathcal{K}/\mathcal{L}_{i+1} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then $\mathcal{E}'_{i+1}/\mathcal{E}_{i+1} \cong \mathcal{K}/\mathcal{L}_{i+1}$, and $\mathcal{E}'_{i+1}/\mathcal{E}_{i+1}$ is torsion. In addition, the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}'_{i+1} \longrightarrow \oplus_i \mathcal{K} \longrightarrow 0$$

splits, and $\mathcal{E}'_{i+1} \cong \oplus_{i+1} \mathcal{K}$. Set $\iota_{i+1} = \sigma'_{i+1}\sigma_{i+1}$. We have the exact sequence

$$0 \longrightarrow \mathcal{F}_{i+1} \xrightarrow{\iota_{i+1}} \oplus_{i+1} \mathcal{K} \xrightarrow{\pi_{i+1}} \oplus_{i+1} \mathcal{K}/\mathcal{F}_{i+1} \longrightarrow 0.$$

Since there is a short exact sequence

$$0 \longrightarrow \mathcal{E}_{i+1}/\mathcal{F}_{i+1} \longrightarrow \mathcal{E}'_{i+1}/\mathcal{F}_{i+1} \longrightarrow \mathcal{E}'_{i+1}/\mathcal{E}_{i+1} \longrightarrow 0$$

and the class of torsion sheaves is closed under extensions, we see that $\mathcal{E}'_{i+1}/\mathcal{F}_{i+1}$ is torsion, i.e., $\oplus_{i+1} \mathcal{K}/\mathcal{F}_{i+1}$ is torsion. This finishes the proof.

**Remark 4.1** The proof of Theorem 4.2 can be simplified by taking into account the quotient category $\mathrm{Qcoh}\mathbb{E}/\mathrm{Qcoh}_0\mathbb{E}$, where $\mathrm{Qcoh}_0\mathbb{E}$ is the full subcategory of $\mathrm{Qcoh}\mathbb{E}$ consisting of all torsion sheaves. In fact, by [12], $\mathrm{Qcoh}\mathbb{E}/\mathrm{Qcoh}_0\mathbb{E} \cong \mathrm{Mod}(K)$, where $K$ is the function field of $\mathbb{E}$, and then the bundles of rank $n$ become an $n$-dimensional vector space in the quotient category. The proof we present above is more constructible.

By Theorem 4.2, $\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \neq 0$ for each bundle $\mathcal{F}$. The following consequence is obvious, which is formulated in [9, Theorem 5.2] where the context is that of tubular weighted projective line.

**Theorem 4.3** *We have* $\mathcal{H}_0 = {}^{\perp}\mathcal{K} \cap \mathrm{coh}(\mathbb{E})$, *where* ${}^{\perp}\mathcal{K} = \{\mathcal{F} \in \mathrm{Qcoh}\mathbb{E} \mid \mathrm{Hom}(\mathcal{F}, \mathcal{K}) = 0\}$ *is the left perpendicular category of* $\mathcal{K}$.

**Acknowledgement** The authors would like to thank Professor H. Lenzing for his lectures and useful discussions in Tsinghua University in 2006. And they also thank Professor J. Xiao for his encouragement and helpful suggestions.

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