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Nontrivial Solutions of Superquadratic Hamiltonian Systems with Lagrangian Boundary Conditions and the L-index Theory***

Chong LI* Chungen LIU**

Abstract In this paper, the authors study the existence of nontrivial solutions for the Hamiltonian systems $\dot{z}(t) = J\nabla H(t, z(t))$ with Lagrangian boundary conditions, where $H(t,z) = \frac{1}{2}(\widehat{B}(t)z, z) + \widehat{H}(t,z)$, $\widehat{B}(t)$ is a semipositive symmetric continuous matrix and $\widehat{H}$ satisfies a superquadratic condition at infinity. We also obtain a result about the L-index.

Keywords L-index, Nontrivial solution, Hamiltonian systems, Lagrangian boundary conditions, Superquadratic condition
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1 Introduction and the Main Results

This paper deals with the existence of nontrivial solutions of superquadratic Hamiltonian systems with Lagrangian boundary conditions and the property of the L-index.

Consider the Hamiltonian systems

$$\begin{cases} \dot{z}(t) = J\nabla H(t, z(t)), & \forall\, z \in \mathbb{R}^{2n}, \forall\, t \in [0, T], \\ z(0) \in L,\ z(T) \in L, \end{cases} \tag{1.1}$$

where $L \in \Lambda(n)$, $T > 0$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the symplectic matrix, I_n is the unit matrix of order n, $H \in C^2([0,T] \times \mathbb{R}^{2n}, \mathbb{R})$ and $\nabla H(t,z)$ is the gradient of $H(t,z)$ respect to z.

We recall that $\Lambda(n)$ is the set of all linear Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0)$, here the standard symplectic form is defined by $\omega_0 = \sum\limits_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$. A Lagrangian subspace L of $\mathbb{R}^{2n}$ is an n dimensional subspace satisfying $\omega_0|_L = 0$. We denote the standard norm and inner product in $\mathbb{R}^{2n}$ by $|\cdot|$ and $(\cdot\,,\cdot)$, respectively.

After the work of P. Rabinowitz [24], many mathematicians have considered the problem of finding periodic solutions of the Hamiltonian systems. For example we can see the references of this paper [4–10, 15–18, 20, 23–25] and the references therein. In recent papers [13, 14], the second author of this paper introduced an index $i_L(\gamma)$ for symplectic paths γ starting from the identity with a Lagrangian subspace L by the algebraic methods and developed the various

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*School of Mathematical Sciences, Nankai University, Tianjin 300071, China.
E-mail: plumechong@yahoo.com.cn
**School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China.
E-mail: liucg@nankai.edu.cn
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properties of this index theory. We call this index the L-Maslov-type index. In a previous paper [19], Y. Long, D. Zhang and C. Zhu studied the multiplicity problem of the brake orbit on a convex domain, there they established two indices $\mu_1(\gamma)$ and $\mu_2(\gamma)$ for the fundamental solution γ of a linear Hamiltonian system by the methods of functional analysis which are special cases of the L-Maslov-type index $i_L(\gamma)$ for Lagrangian subspaces $L_0=\{0\}\oplus\mathbb{R}^n$ and $L_1=\mathbb{R}^n\oplus\{0\}$ up to a constant n.

Multiplicity results for solutions of various boundary value problems are known for dynamical systems on compact configuration manifolds, given by Lagrangians or Hamiltonians which have quadratic growth in the velocities or in the momenta. In a recent paper [1], A. Abbondandolo and A. Figalli extended these results to the classical setting of Tonelli Lagrangians (Lagrangians which are C^2-convex and superlinear in the velocities), or to Hamiltonians which are superlinear in the momenta and have a coercive action integrand. In this paper, we use the variational methods to prove the existence of nontrivial solutions of superquadratic Hamiltonian systems with Lagrangian boundary conditions and study the property of the L-index defined in [14].

This paper is divided into 3 sections. In Section 2, we use the L-index theory in [13, 14] to study the existence of a nontrivial solution of (1.1) for which the Hamiltonian function H is given by

$$H(t,z)=\frac{1}{2}(\widehat{B}(t)z,z)+\widehat{H}(t,z),$$

where $\widehat{B}(t)$ is a semipositive symmetric continuous matrix for all $t\in[0,T]$ and $\widehat{H}$ satisfies a superquadratic condition at infinity. Moreover, we obtain a property of the L-index of this nontrivial solution of (1.1).

We suppose $\widehat{H}$ and $\widehat{B}$ satisfy the following conditions:

(H1) There is a $\theta\in(0,\frac{1}{2})$ and $\overline{r}>0$ such that

$$0<\widehat{H}(t,z)\le\theta(z,\nabla\widehat{H}(t,z))\quad \text{for all } z\in\mathbb{R}^{2n},\ |z|\ge\overline{r},\ t\in[0,T],$$

(H2) $\widehat{H}(t,z)\ge 0$ for all $z\in\mathbb{R}^{2n}$, $t\in[0,T]$,

(H3) $\widehat{H}(t,z)=o(|z|^2)$ at $z=0$,

(H4) There are constants a, $b>0$ such that

$$|\nabla\widehat{H}(t,z)|\le a(z,\nabla\widehat{H}(t,z))+b\qquad \text{for all } z\in\mathbb{R}^{2n},\ t\in[0,T],$$

(H5) $\widehat{B}(t)$ is a semipositive symmetric continuous matrix for all $t\in[0,T]$.

We prove the following:

Theorem 1.1 *Suppose $H(t,z)\in C^2([0,T]\times\mathbb{R}^{2n},\mathbb{R})$ satisfies* (H1)–(H5)*, then* (1.1) *possesses at least one nontrivial solution z whose L-index pair $(i_L(z),\nu_L(z))$ satisfies*

$$i_L(z)\le i_L(\widehat{B})+\nu_L(\widehat{B})+1\le i_L(z)+\nu_L(z).$$

In Section 3, we study the general case

$$\begin{cases}\dot{z}(t)=J\nabla H(t,z(t)), & \forall\, z\in\mathbb{R}^{2n},\ \forall\, t\in[0,1],\\ z(0)\in L_1,\ z(1)\in L_2,\end{cases}\tag{1.2}$$

where L_1 and L_2 are any two linear Lagrangian subspaces of $\mathbb{R}^{2n}$ and $H(t,z)\in C^2([0,1]\times\mathbb{R}^{2n},\mathbb{R})$ satisfies (H1)–(H4). With these conditions, we have the following result:

Theorem 1.2 *Suppose* $H(t,z) \in C^2([0,1]\times\mathbb{R}^{2n},\mathbb{R})$ *satisfies* (H1)–(H4), *then* (1.2) *possesses at least one nontrivial solution.*

Remark 1.1 We observe that for $L_0 = \mathbb{R}^n \oplus \{0\} \subset \mathbb{R}^{2n}$ and any $L \in \Lambda(n)$, there exists an orthogonal symplectic matrix P such that $PL = L_0$ (see [14, 21]). By taking $P^{-1}\overline{z}(t) = z(t)$ in (1.1), we change the systems into the following one:

$$\begin{cases} \dot{\overline{z}}(t) = J\nabla\overline{H}(t,\overline{z}(t)), \qquad \forall\, \overline{z} \in \mathbb{R}^{2n},\ \forall\, t \in [0,T], \\ \overline{z}(0) \in L_0,\ \overline{z}(T) \in L_0, \end{cases} \tag{1.3}$$

where $\overline{H}$ depending on P satisfies the same conditions as in Theorem 1.1. So we only need to prove Theorem 1.1 for the special case $L = L_0$.

Remark 1.2 We use the same transformation as in Remark 1.1 for (1.2). Choose an orthogonal symplectic matrix $\check{P}$ such that $\check{P}L_1 = L_0$, then $\check{P}L_2 = L'$ with $L' \in \Lambda(n)$ (see [21]). By taking $\check{P}^{-1}\check{z}(t) = z(t)$ in (1.2), we change the systems into the following one:

$$\begin{cases} \dot{\check{z}}(t) = J\nabla\check{H}(t,\check{z}(t)), \quad \forall\, \check{z} \in \mathbb{R}^{2n},\ \forall\, t \in [0,1], \\ \check{z}(0) \in L_0,\ \check{z}(1) \in L', \end{cases} \tag{1.4}$$

where $\check{H}$ depending on $\check{P}$ satisfies the same conditions as in Theorem 1.2. So we only need to prove Theorem 1.2 for the special case (1.4).

The "open string problem" (1.1) is closely related with many dynamical systems problems. It is well-known that one can transform the famous Arnold chord conjecture (see [2]) into an autonomous Hamiltonian system with Lagrangian boundary conditions as in (1.1). The Arnold chord conjecture, which has been proved by K. Mohnke [22], says that on the contact manifold (S^{2n-1},ξ_0) with standard contact structure ξ_0, for any closed Legendrian submanifold $L \subset S^{2n-1}$, there always exists a Reeb chord intersecting the Legendrian submanifold L at least twice for any choice of contact form. On the other hand, one can take the brake orbit problem as a special case of the problem (1.1). For example, the following brake orbit problem

$$\begin{cases} \ddot{q}(t) + V'(q(t)) = 0, \\ \dot{q}(0) = \dot{q}\Big(\dfrac{T}{2}\Big) = 0, \\ q\Big(\dfrac{T}{2} + t\Big) = q\Big(\dfrac{T}{2} - t\Big), \\ q(T+t) = q(t) \end{cases}$$

can be transformed into the problem (1.3).

2 Proof of Theorem 1.1

Let L_0^2 be the subspace of $L^2([0,T],\mathbb{R}^{2n})$ which consists of all elements $z(t) = \sum\limits_{k\in\mathbb{Z}} \mathrm{e}^{\frac{k\pi}{T}Jt} z_k$, $z_k \in L_0$, satisfying $\|z\|_{L_0^2}^2 = \int_0^T |z|^2 \mathrm{d}t < +\infty$, where the norm $\|z\|_{L_0^2}$ is the usual L^2 norm. Correspondingly, the inner product of L_0^2 is the usual L^2 inner product

$$(u,v)_{L_0^2} = \int_0^T (u(t),v(t))\mathrm{d}t, \quad \forall\, u,\, v \in L_0^2.$$

Then L_0^2 becomes a Hilbert space.

Let $X := \{z \in W^{\frac{1}{2},2}([0,T],\mathbb{R}^{2n}) \mid z = \sum\limits_{k\in\mathbb{Z}} \mathrm{e}^{\frac{k\pi}{T}Jt} z_k,\ z_k \in L_0,\ \|z\|_X < +\infty\}$ be the Hilbert space with the inner product

$$(u,v)_X = \sum_{k\in\mathbb{Z}} (1+|k|)(u_k, v_k), \quad \forall\, u,\, v \in X.$$

In the following, we use $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\|\cdot\|$ to denote the inner product and norm in X, respectively. It is well-known that if $r \in [1,+\infty)$ and $z \in L^r([0,T],\mathbb{R}^{2n})$, then there exists a constant $c_r > 0$ such that $\|z\|_{L^r} \le c_r \|z\|$.

We define an operator $Ax = -J\dot{x}$ on L_0^2. Then the domain of A is $\mathrm{dom}\, A = W_{L_0}$, where $W_{L_0} := \left\{ z \in W^{1,2}([0,T],\mathbb{R}^{2n}) \,\middle|\, z = \sum\limits_{k\in\mathbb{Z}} \mathrm{e}^{\frac{k\pi}{T}Jt} z_k,\ z_k \in L_0,\ \|z\|_{W_{L_0}} < +\infty \right\}$ is a dense subspace of X. Under the inner product of the space L_0^2, the spectrum of the operator A is $\sigma(A) = \frac{\pi}{T}\mathbb{Z}$. It is a point spectrum, i.e., it contains only eigenvalues, and the multiplicity of every eigenvalue is n. We can prove that A is a self-adjoint operator, i.e., $(Au,v)_{L_0^2} = (u, A^*v)_{L_0^2} = (u, Av)_{L_0^2}$, $\forall u,\, v \in \mathrm{dom}\, A = W_{L_0}$.

Indeed, by definition,

$$\begin{aligned}
(Au,v)_{L_0^2} &= \int_0^T (-J\dot{u}(t), v(t))\mathrm{d}t \\
&= (-Ju(t), v(t))|_0^T - \int_0^T (-Ju(t), \dot{v}(t))\mathrm{d}t \\
&= (-Ju(t), v(t))|_0^T + \int_0^T (u(t), -J\dot{v}(t))\mathrm{d}t \\
&= (-Ju(t), v(t))|_0^T + (u, Av)_{L_0^2}.
\end{aligned}$$

Since $(-Ju(t), v(t))|_0^T = \omega_0(u(T), v(T)) - \omega_0(u(0), v(0)) = 0$, so $(Au,v)_{L_0^2} = (u,Av)_{L_0^2}$, i.e., A is a self-adjoint operator.

Define the linear operator B on L_0^2 by $(Bu,v)_{L_0^2} = \int_0^T (\widehat{B}(t)u, v)\mathrm{d}t$, $\forall\, u,\, v \in L_0^2$. Then B is a compact self-adjoint linear operator. Furthermore, we define the self-adjoint linear operator $A - \widehat{B}$ on X by $\langle (A-\widehat{B})u, v\rangle := ((A-B)u, v)_{L_0^2}$, $\forall\, u,\, v \in X$. Similar to the operator A, the spectrum of the operator $A - B$ is a point spectrum and the multiplicity of each eigenvalue is finite. Similarly to [16, Lemma 4.1.5], but with the inner product of the space L_0^2, we can prove that $0 \in \{0\} \cup \sigma(A-B)$ is isolated. Following the ideas of [16], we can list the set $\{0\}\cup\sigma(A-B)$ by

$$\cdots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \text{with } |\lambda_q| \to +\infty \text{ as } q \to \infty,$$

where a nonzero eigenvalue λ_j of $A - B$ should be repeated k times if its multiplicity is k, and the eigenvector of $A - B$ belonging to λ_j by e_j. For $m > 0$, set

$$\begin{aligned}
&X_m^+ = \mathrm{Span}\{e_1, \cdots, e_m\}, \quad X_m^- = \mathrm{Span}\{e_{-1}, \cdots, e_{-m}\}, \\
&X^0 = \ker(A - \widehat{B}), \quad X_m = X_m^+ \oplus X^0 \oplus X_m^-.
\end{aligned}$$

Let $P^m : X \to X_m$ be the corresponding orthogonal projection for $m \in \mathbb{N}$. Then $\Gamma = \{P^m; m \in \mathbb{N}\}$ is a Galerkin approximation scheme with respect to $A - \widehat{B}$.

For $d > 0$, we denote by $M_d^*(Q)$, $* = +, 0, -$, the eigenspaces corresponding to the eigenvalues λ of the linear operator $Q : X \to X$ belonging to $[d, +\infty)$, $(-d, d)$ and $(-\infty, -d]$, respectively. And denote by $M^*(Q)$, $* = +, 0, -$, the eigenspaces corresponding to the eigenvalues λ of Q belonging to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$, respectively. For any adjoint operator Q, we denote $Q^\sharp = (Q|_{ImQ})^{-1}$, and we also denote $P^m Q P^m = (P^m Q P^m)|_{X_m}$. From [10] and [13], we have

Theorem 2.1 *For any $B(t) \in C([0,1], \mathscr{L}_s(\mathbb{R}^{2n}))$, where $\mathscr{L}_s(\mathbb{R}^{2n})$ denotes the set of symmetric matrices, with the L- index pair $(i_L(B), \nu_L(B))$ and any constant $0 < d \le \frac{1}{4}\|(A-B)^\sharp\|^{-1}$, there exists $m_0 > 0$ such that for $m \ge m_0$, we have*

$$\dim M_d^+(P^m(A-B)P^m) = m + i_L(\widehat{B}) - i_L(B) + \nu_L(\widehat{B}) - \nu_L(B),$$
$$\dim M_d^-(P^m(A-B)P^m) = m - i_L(\widehat{B}) + i_L(B),$$
$$\dim M_d^0(P^m(A-B)P^m) = \nu_L(B).$$

Proof We follow the ideas of [10].

Case 1 $\nu_L(B) = 0$. We have $\dim\ker(A-B) = 0$.

Since B and $\widehat{B}$ are compact, there exists an $m^* > 0$ such that for $m \ge m^*$,

$$\|(I-P^m)(\widehat{B}-B)\| + \|(\widehat{B}-B)(I-P^m)\| \le \frac{1}{2}\|(A-B)^{-1}\|^{-1}.$$

Since $P^m(A-B)P^m = (A-B)P^m + (P^m - I)(\widehat{B}-B)P^m$, for $m \ge m^*$, then

$$\|P^m(A-B)P^m z\| \ge \frac{1}{2}\|(A-B)^{-1}\|^{-1}\|z\| \quad \text{for all } z \in X_m.$$

Hence we have

$$M_d^*(P^m(A-B)P^m) = M^*(P^m(A-B)P^m) \quad \text{for } * = +, 0, -.$$

Notice that

$$\begin{aligned} A - B &= P^m(A-B)P^m + (I-P^m)(A-\widehat{B}) + (I-P^m)(\widehat{B}-B) + P^m(\widehat{B}-B)(I-P^m) \\ &= A - (\widehat{B} + P^m(B-\widehat{B})P^m) + (I-P^m)(\widehat{B}-B) + P^m(\widehat{B}-B)(I-P^m). \end{aligned}$$

By Theorem 5.1, Theorem 5.2 and Definition 5.1 in [10], we have

$$\begin{aligned} I(B, \widehat{B}) &= I(\widehat{B} + P^m(B-\widehat{B})P^m, \widehat{B}) \\ &= \dim M^+(P^m(A-B)P^m) - \dim M^+(P^m(A-\widehat{B})P^m) - \nu_L(\widehat{B}). \end{aligned}$$

Hence

$$\begin{aligned} \dim M^+(P^m(A-B)P^m) &= I(B, \widehat{B}) + m + \nu_L(\widehat{B}) \\ &= m + i_L(\widehat{B}) - i_L(B) + \nu_L(\widehat{B}) - \nu_L(B). \end{aligned}$$

Similarly, $\dim M^-(P^m(A-B)P^m) = m - i_L(\widehat{B}) + i_L(B)$.

Case 2 $\nu_L(B) > 0$. Let $\gamma \in \mathcal{P}(2n) = \{\gamma \in C([0,1], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\}$ be the fundamental solution of the linear Hamiltonian systems

$$\dot{z} = JB(t)z.$$

Let γ_s, $0 \le s \le 1$, be the perturbed path defined in [14]. Define

$$B_s(t) = -J\dot{\gamma}_s(t)\gamma_s(t)^{-1}, \quad \forall\, t \in [0,1].$$

Let B_s be the compact operator defined as above corresponding to $B_s(t)$. For $s \neq 0$, there hold $B_0(t) = B(t)$, $\nu_L(B_s) = 0$ for $s \in (0,1]$, and $\|B_s - B\| \to 0$ as $s \to 0$. If $s \in (0,1]$, we have

$$i_L(\gamma_s) - i_L(\gamma_{-s}) = \nu_L(\gamma) = \nu_L(B), \quad i_L(\gamma_{-s}) = i_L(B) = i_L(\gamma).$$

Choose $0 < s < 1$ such that $\|B - B_{\pm s}\| \le \frac{d}{2}$, let $\overline{s} = \pm s$. By Case 1, there exists an $m_1 \ge 0$ such that for $m \ge m_1$,

$$\begin{aligned} \dim M^+(P^m(A - B_{\overline{s}})P^m) &= m + i_L(\widehat{B}) - i_L(\gamma_{\overline{s}}) + \nu_L(\widehat{B}), \\ \dim M^-(P^m(A - B_{\overline{s}})P^m) &= m - i_L(\widehat{B}) + i_L(\gamma_{\overline{s}}), \\ \dim M^0(P^m(A - B_{\overline{s}})P^m) &= 0. \end{aligned}$$

In [13], there exists an $m^* \ge m_1$ such that for $m \ge m^*$,

$$\dim M_d^0(P^m(A - B)P^m) \le \nu_L(B).$$

For otherwise, there exists $y \in M_d^0(P^m(A - B)P^m) \cap R_m$, $\|y\| = 1$, where

$$X_m = P^m \ker(A - B) \oplus R_m, \quad \dim P^m \ker(A - B) = \nu_L(B).$$

Then $\|P^m(A - B)P^m y\| \le d\|y\|$ contradicts to $\|y\| \le \frac{1}{2d}\|P^m(A - B)P^m y\|$.

Since $P^m(A - B_{\overline{s}})P^m = P^m(A - B)P^m + P^m(B - B_{\overline{s}})P^m$, for $m \ge m^*$, then

$$\begin{aligned} \dim M_d^+(P^m(A - B)P^m) &\le \dim M^+(P^m(A - B_s)P^m) \\ &= m + i_L(\widehat{B}) - i_L(B) + \nu_L(\widehat{B}) - \nu_L(B), \\ \dim M_d^+(P^m(A - B)P^m) &\ge \dim M^+(P^m(A - B_{-s})P^m) - \dim M_d^0(P^m(A - B)P^m) \\ &= m + i_L(\widehat{B}) - i_L(B) + \nu_L(\widehat{B}) - \dim M_d^0(P^m(A - B)P^m). \end{aligned}$$

We have $\dim M_d^0(P^m(A - B)P^m) = \nu_L(B)$ and

$$\dim M_d^+(P^m(A - B)P^m) = m + i_L(\widehat{B}) - i_L(B) + \nu_L(\widehat{B}) - \nu_L(B).$$

Similarly, we have $\dim M_d^-(P^m(A - B)P^m) = m - i_L(\widehat{B}) + i_L(B)$.

The proof of Theorem 2.1 is complete.

We need to truncate the function $\widehat{H}$ at infinite. That is to replace $\widehat{H}$ by a modified function $\widehat{H}_k$ which grows at a prescribed rate near ∞. The truncated function was defined by P. Rabinowitz in [23]. Let $k > 0$ and select $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 1$ for $s \le k$, $\chi(s) = 0$ for $s \ge k+1$, and $\chi'(s) < 0$ for $s \in (k, k+1)$. Set

$$\widehat{H}_k(t,z) = \chi(|z|)\widehat{H}(t,z) + (1 - \chi(|z|))r_k|z|^4,$$

where $r_k = \max\{\frac{\widehat{H}(t,z)}{|z|^4} \mid k \le |z| \le k+1,\, t \in [0,T]\}$.

It is known that $\widehat{H}_k$ still satisfies (H1)–(H4) with θ and (H4) being replaced by $\widehat{\theta} = \max\{\theta, \frac{1}{4}\}$ and $|\nabla \widehat{H}_k(t,z)| \le (z, \nabla \widehat{H}_k(t,z)) + b'$, $b' > 0$, respectively.

Define a function φ on X by

$$\varphi(z) = \frac{1}{2}\langle (A - \widehat{B})z, z\rangle - \int_0^T \widehat{H}_k(t, z(t))\mathrm{d}t.$$

Suppose W is a real Banach space, $g \in C^1(W, \mathbb{R})$. g is said satisfying the (PS) condition, if for any sequence $\{x_q\} \subset W$ satisfying $g(x_q)$ is bounded and $g'(x_q) \to 0$ as $q \to \infty$, there exists a convergent subsequence $\{x_{q_j}\}$ of $\{x_q\}$ (see [23]). Let $\varphi_m = \varphi|_{X_m}$ be the restriction of φ on X_m. When H satisfies (H1)–(H5), similarly to [3, Proposition A], we have the following two lemmas.

Lemma 2.1 *For all $m \in \mathbb{N}$, φ_m satisfies the* (PS) *condition on X_m.*

Lemma 2.2 *φ satisfies the* (PS)* *condition on X with respect to $\{z_m\}$, i.e., for any sequence $\{z_m\} \subset X$ satisfying $z_m \in X_m$, $\varphi_m(z_m)$ is bounded and $\|\varphi_m'(z_m)\|_{(X_m)'} \to 0$ in $(X_m)'$ as $m \to +\infty$, where $(X_m)'$ is the dual space of X_m, there exists a convergent subsequence $\{z_{m_j}\}$ of $\{z_m\}$ in X.*

In order to prove Theorem 1.1, we need the following definition and the saddle-point theorem.

Definition 2.1 (see [11]) *Let E be a C^2-Riemannian manifold and D be a closed subset of E. A family $\phi(\alpha)$ of subsets of E is said to be a homological family of dimensional q with boundary D if for some nontrivial class $\alpha \in H_q(E, D)$. The family $\phi(\alpha)$ is defined by*

$$\phi(\alpha) = \{G \subset E : \alpha \text{ is in the image of } i_* : H_q(G, D) \to H_q(E, D)\},$$

where i_* is the homomorphism induced by the immersion $i : G \to E$.

Theorem 2.2 (see [11]) *For above E, D and α, let $\phi(\alpha)$ be a homological family of dimension q with boundary D. Suppose that $f \in C^2(E, \mathbb{R})$ satisfies the* (PS) *condition. Define*

$$c = \inf_{G\in\phi(\alpha)} \sup_{x\in G} f(x).$$

Suppose that $\sup\limits_{x\in D} f(x) < c$ and f' is Fredholm on

$$\mathscr{K}_c(f) \equiv \{x \in E : f'(x) = 0,\ f(x) = c\}.$$

Then there exists an $x \in \mathscr{K}_c(f)$ such that the Morse index $m^-(x)$ and the nullity $m^0(x)$ of the functional f at x satisfy

$$m^-(x) \le q \le m^-(x) + m^0(x).$$

It is clear that a critical point of φ is a solution of (1.3). For a critical point $z = z(t)$, let $B(t) = H''(t, z(t))$, define the linearized systems at $z(t)$ by

$$\begin{cases} \dot{y}(t) = JH''(t, z(t))y(t), & \forall\, y \in \mathbb{R}^{2n},\ \forall\, t \in [0, T], \\ y(0) \in L_0,\ y(T) \in L_0. \end{cases}$$

Then the L_0-index pair of z is defined by $(i_{L_0}(z), \nu_{L_0}(z)) = (i_{L_0}(B), \nu_{L_0}(B))$.

Proof of Theorem 1.1 We carry out the proof in 3 steps.

Step 1 The critical points of φ_m.

Set $S_m = X_m^- \oplus X^0$. Then $\dim S_m = m + \dim X^0 = m + \dim\ker(A-\widehat{B}) = m + \nu_{L_0}(\widehat{B})$, $\dim X_m^+ = m$.

In the following, we prove that $\varphi_m(z)$ satisfies:

(I) $\varphi_m(z) \geq \beta > 0,\ \forall\, z \in Y_m = X_m^+ \cap \partial B_\rho(0)$,

(II) $\varphi_m(z) \leq 0 < \beta,\ \forall\, z \in \partial Q_m$, where $Q_m = \{re \mid r \in [0, r_1]\} \oplus (B_{r_2}(0) \cap S_m)$, $e \in X_m^+ \cap \partial B_1(0)$, $r_1 > \rho$, $r_2 > 0$.

First we prove (I). By (H3), for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\widehat{H}_k(t,z) \leq \varepsilon|z|^2$ if $|z| \leq \delta$. Since $\widehat{H}_k(t,z)|z|^{-4}$ is uniformly bounded as $|z| \to +\infty$, there is an $M_1 = M_1(\varepsilon, k)$ such that $\widehat{H}_k(t,z) \leq M_1|z|^4$ for $|z| \geq \delta$. Hence

$$\widehat{H}_k(t,z) \leq \varepsilon|z|^2 + M_1|z|^4 \quad \text{for all } z \in \mathbb{R}^{2n}. \tag{2.1}$$

For $z \in Y_m$, we have

$$\int_0^T \widehat{H}_k(t,z)\mathrm{d}t \leq \varepsilon\|z\|_{L^2}^2 + M_1\|z\|_{L^4}^4 \leq (\varepsilon c_2^2 + M_1 c_4^4\|z\|^2)\|z\|^2. \tag{2.2}$$

Since for all $z \in Y_m$, similarly to [16, Lemma 4.1.5], under the inner product of the space X, we can prove that $0 \in \{0\} \cup \sigma(A-\widehat{B})$ is isolated. Then the operator $A-\widehat{B}$ has a minimum positive eigenvalue μ on X, and the eigenvector of $A-\widehat{B}$ belonging to μ by $\overline{e}$. Thus we have $\langle (A-\widehat{B})z, z\rangle \geq \mu\|z\|^2$. Hence by (2.2),

$$\begin{aligned}\varphi_m(z) &= \frac{1}{2}\langle (A-\widehat{B})z, z\rangle - \int_0^T \widehat{H}_k(t,z(t))\mathrm{d}t \\ &\geq \frac{\mu}{2}\|z\|^2 - \int_0^T \widehat{H}_k(t,z(t))\mathrm{d}t \\ &\geq \frac{\mu}{2}\|z\|^2 - (\varepsilon c_2^2 + M_1 c_4^4\|z\|^2)\|z\|^2 \\ &= \frac{\mu}{2}\rho^2 - (\varepsilon c_2^2 + M_1 c_4^4\rho^2)\rho^2.\end{aligned} \tag{2.3}$$

Choose $\varepsilon = \frac{\mu}{5c_2^2}$ and ρ independent of k and m so that $5M_1c_4^4\rho^2 = \mu$. Then for $z \in Y_m$,

$$\varphi_m(z) \geq \frac{\mu}{10}\rho^2 \triangleq \beta > 0.$$

β is independent of k and m. Hence (I) holds.

Next prove (II). Let $e \in X_m^+ \cap \partial B_1$ and $z = z^- + z^0 \in S_m$. Without loss of generality, choose $e = \frac{\overline{e}}{\|\overline{e}\|}$. Then similar to (I), the operator $A-\widehat{B}$ has a maximal negative eigenvalue τ on X:

$$\begin{aligned}\varphi_m(z+re) &= \frac{1}{2}\langle (A-\widehat{B})z^-, z^-\rangle + \frac{1}{2}r^2\langle (A-\widehat{B})e, e\rangle - \int_0^T \widehat{H}_k(t, z+re)\mathrm{d}t \\ &\leq \frac{\tau}{2}\|z^-\|^2 + \frac{\mu}{2}r^2 - \int_0^T \widehat{H}_k(t, z+re)\mathrm{d}t,\end{aligned} \tag{2.4}$$

From (H1), We have

$$\widehat{H}_k(t,z) \geq b_1|z|^{\frac{1}{\theta}} - b_2, \tag{2.5}$$

where $b_1 > 0$, b_2 are two constants independent of k and m. Then by (2.5),

$$\begin{aligned}\int_0^T \widehat{H}_k(t, z+re)\mathrm{d}t &\ge b_1 \int_0^T |z+re|^{\frac{1}{\theta}}\mathrm{d}t - Tb_2\\ &\ge b_3\Big(\int_0^T |z+re|^2\mathrm{d}t\Big)^{\frac{1}{2\theta}} - b_4\\ &\ge b_5(\|z^0\|^{\frac{1}{\theta}} + r^{\frac{1}{\theta}}) - b_4,\end{aligned} \tag{2.6}$$

where b_3, b_4 are constants and $b_5 > 0$ independent of k and m. Thus by (2.6),

$$\varphi_m(z+re) \le \frac{\tau}{2}\|z^-\|^2 + \frac{\mu}{2}r^2 - b_5(\|z^0\|^{\frac{1}{\theta}} + r^{\frac{1}{\theta}}) + b_4 \tag{2.7}$$

for $\tau < 0$ and $\mu > 0$ independent of k and m. Thus we can choose large enough r_1 and r_2 independent of k and m such that

$$\varphi_m(z+re) \le 0, \quad \text{on } \partial Q_m.$$

Then (II) holds.

Because Q_m is deformation retract of X_m, then $H_q(Q_m, \partial Q_m) \cong H_q(X_m, \partial Q_m)$, where $q = \dim S_m + 1 = m + \nu_{L_0}(\widehat{B}) + 1 = \dim Q_m$, and ∂Q_m is the boundary of Q_m in $S_m \oplus \{\mathbb{R}e\}$. But $H_q(Q_m, \partial Q_m) \cong H_{q-1}(S^{q-1}) \cong \mathbb{R}$. Denote by $i : Q_m \to X_m$ the inclusion map. Let $\alpha = [Q_m] \in H_q(Q_m, D)$ be a generator. Then $i_*\alpha$ is nontrivial in $H_q(X_m, \partial Q_m)$, and $\phi(i_*\alpha)$ defined by Definition 2.1 is a homological family of dimension q with boundary $D := \partial Q_m$ and $Q_m \in \phi(i_*\alpha)$. ∂Q_m and Y_m are homologically link (see [5]). By Lemma 2.1, φ_m satisfies the (PS) condition. Define $c_m = \inf\limits_{G\in\phi(i_*\alpha)} \sup\limits_{z\in G} \varphi_m(z)$. We have

$$\sup_{z\in\partial Q_m} \varphi_m(z) \le 0 < \beta \le c_m \le \sup_{z\in Q_m} \varphi_m(z) \le \frac{\mu}{2}r_1^2. \tag{2.8}$$

Since X_m is finite dimensional, φ_m' is Fredholm. By Theorem 2.2, φ_m has a critical point z_m with critical value c_m, and the Morse index $m^-(z_m)$ and nullity $m^0(z_m)$ of z_m satisfy

$$m^-(z_m) \le m + \nu_{L_0}(\widehat{B}) + 1 \le m^-(z_m) + m^0(z_m). \tag{2.9}$$

Since $\{c_m\}$ is bounded, passing to a subsequence, suppose $c_m \to c \in [\beta, \frac{\mu}{2}r_1^2]$. By the (PS)* condition of Lemma 2.2, passing to a subsequence, there exists an $z \in X$ such that

$$z_m \to z, \quad \varphi(z) = c, \quad \varphi'(z) = 0.$$

Step 2 The solution of (1.3).

Because the critical value c has an upper bound $\frac{\mu}{2}r_1^2$ independent of k, then

$$\begin{aligned}\frac{\mu}{2}r_1^2 \ge c &= \varphi(z) - \frac{1}{2}\langle\varphi'(z), z\rangle\\ &\ge \Big(\frac{1}{2} - \widehat{\theta}\Big)\int_0^T (z, \nabla\widehat{H}_k(t,z))\mathrm{d}t.\end{aligned} \tag{2.10}$$

Then by (2.10), $\int_0^T (z, \nabla\widehat{H}_k(t,z))\mathrm{d}t$ has an upper bound independent of k,

$$\int_0^T (z, \nabla\widehat{H}_k(t,z))\mathrm{d}t \le \overline{M} \quad \text{for constant } \overline{M} \text{ independent of } k. \tag{2.11}$$

By $\widehat{H}_k(t,z) \le \widehat{\theta}(z, \nabla \widehat{H}_k(t,z))$, one has

$$\int_0^T \widehat{H}_k(t,z)\mathrm{d}t \le \widehat{\theta}\,\overline{M}. \tag{2.12}$$

Thus by (2.5) and (2.12),

$$\int_0^T (b_1|z|^{\frac{1}{\theta}} - b_2)\mathrm{d}t \le \int_0^T \widehat{H}_k(t,z)\mathrm{d}t \le \widehat{\theta}\,\overline{M},$$

i.e.,

$$\widehat{\theta}\,\overline{M} \ge b_1 \int_0^T |z|^{\frac{1}{\theta}}\mathrm{d}t - b_2 T \ge b_1' \Big(\int_0^T |z|^2 \mathrm{d}t\Big)^{\frac{1}{2\theta}} - b_2 T. \tag{2.13}$$

Thus by (2.13), $\|z\|_{L^2}^{\frac{1}{\theta}} \le M_2$, where M_2 is independent of k, i.e.,

$$\|z\|_{L^2} \le M_3, \quad \text{where } M_3 \text{ is independent of } k. \tag{2.14}$$

Since

$$\|z\|_{L^1} \le C\|z\|_{L^2} \le M_3', \quad \text{where } C>0 \text{ is independent of } k. \tag{2.15}$$

Thus by (2.14) and (2.15), $\|z\|_{L^1}$ has an upper bound independent of k. We use Young's inequality. For any $w \in W^{1,2}([0,T],\mathbb{R}^{2n})$, $w(\tau) - w(t) = \int_t^\tau \dot{w}(s)\mathrm{d}s$. Integrating with respect to t shows that

$$Tw(\tau) - \int_0^T w(t)\mathrm{d}t = \int_0^T\int_t^\tau \dot{w}(s)\mathrm{d}s\mathrm{d}t,$$

i.e.,

$$\begin{aligned} |Tw(\tau)| &= \Big|\int_0^T w(t)\mathrm{d}t + \int_0^T\int_t^\tau \dot{w}(s)\mathrm{d}s\mathrm{d}t\Big| \\ &\le \int_0^T |w(t)|\mathrm{d}t + \int_0^T\int_t^\tau |\dot{w}(s)|\mathrm{d}s\mathrm{d}t \\ &\le \|w\|_{L^1} + T\int_0^T |\dot{w}(s)|\mathrm{d}s \\ &= \|w\|_{L^1} + T\|\dot{w}\|_{L^1}, \end{aligned}$$

i.e.,

$$|w(\tau)| \le \frac{\|w\|_{L^1}}{T} + \|\dot{w}\|_{L^1},$$

i.e.,

$$\|w\|_{L^\infty} \le \frac{\|w\|_{L^1}}{T} + \|\dot{w}\|_{L^1}. \tag{2.16}$$

By (H5), we know that $\widehat{B}(t)$ is bounded for all $t \in [0,T]$, therefore

$$\begin{aligned} \|\dot{z}\|_{L^1} &= \|J\widehat{B}(t)z + J\nabla \widehat{H}_k(t,z)\|_{L^1} \\ &\le \widehat{\delta}\|z\|_{L^1} + \int_0^T |\nabla \widehat{H}_k(t,z)|\mathrm{d}t \\ &\le \widehat{\delta}M_3' + \int_0^T (z, \nabla \widehat{H}_k(t,z))\mathrm{d}t + b'T \\ &\le \widehat{\delta}M_3' + \overline{M}T + b'T \le M_4. \end{aligned} \tag{2.17}$$

Thus $\|\dot{z}\|_{L^1}$ has an upper bound independent of k. Then from (2.15)–(2.17), $\|z\|_{L^\infty} \le k_0$, where k_0 is independent of k. We choose $k > k_0$, therefore $\widehat{H}_k(t,z) = \widehat{H}(t,z)$. Consequently, z is a nontrivial solution of (1.3).

Step 3 Let $B(t) = H_k''(t, z(t))$, $d = \frac{1}{4}\|(A-B)^\sharp\|^{-1}$. Since

$$\|\varphi''(x) - (A-B)\| \to 0, \quad \text{as } \|x - z\| \to 0,$$

there exists an $r_3 > 0$ such that

$$\|\varphi''(x) - (A-B)\| < \frac{1}{4}d, \quad \forall\, x \in V_{r_3}(z) = \{x \in X \mid \|x - z\| \le r_3\}.$$

Then for m large enough, there holds

$$\|\varphi_m''(x) - P^m(A-B)P^m\| < \frac{1}{2}d, \quad \forall\, x \in V_{r_3}(z) \cap X_m. \tag{2.18}$$

For $x \in V_{r_3}(z) \cap X_m$, $\forall\, u \in M_d^-(P^m(A-B)P^m) \setminus \{0\}$, from (2.18) we have

$$\begin{aligned}\langle \varphi_m''(x)u, u\rangle &\le \langle P^m(A-B)P^m u, u\rangle + \|\varphi_m''(x) - P^m(A-B)P^m\| \cdot \|u\|^2 \\ &\le -\frac{1}{2}d\|u\|^2 < 0.\end{aligned} \tag{2.19}$$

Thus by (2.19),

$$\dim M^-(\varphi_m''(x)) \ge \dim M_d^-(P^m(A-B)P^m), \quad \forall\, x \in V_{r_3}(z) \cap X_m. \tag{2.20}$$

Similarly, we have

$$\dim M^+(\varphi_m''(x)) \ge \dim M_d^+(P^m(A-B)P^m), \quad \forall\, x \in V_{r_3}(z) \cap X_m. \tag{2.21}$$

By Theorem 2.1 and (2.9), (2.20), (2.21), for large m we have

$$\begin{aligned} m + \nu_{L_0}(\widehat{B}) + 1 &\ge m^-(z_m) \\ &\ge \dim M_d^-(P^m(A-B)P^m) \\ &= m - i_{L_0}(\widehat{B}) + i_{L_0}(B). \end{aligned} \tag{2.22}$$

We also have

$$\begin{aligned} m + \nu_{L_0}(\widehat{B}) + 1 &\le m^-(z_m) + m^0(z_m) \\ &\le \dim M_d^-(P^m(A-B)P^m) \oplus \dim M_d^0(P^m(A-B)P^m) \\ &= m - i_{L_0}(\widehat{B}) + i_{L_0}(B) + \nu_{L_0}(B). \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), we have

$$i_{L_0}(z) \le i_{L_0}(\widehat{B}) + \nu_{L_0}(\widehat{B}) + 1 \le i_{L_0}(z) + \nu_{L_0}(z).$$

The proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 We know that there exists an orthogonal symplectic matrix $\widetilde{P}$ such that $L' = \widetilde{P}L_0$ (see [14, 21]) and $\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R}) \cong \mathrm{U}(n,\mathbb{C})$ (see [16]), where $\mathrm{Sp}(2n,\mathbb{R})$ is the symplectic group, $\mathrm{O}(2n,\mathbb{R})$ is the orthogonal group and $\mathrm{U}(n,\mathbb{C})$ is the unitary group. By this isomorphism, we know that $\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R})$ is a Lie group and its Lie algebra is $\mathrm{sp}(2n,\mathbb{R}) \cap \mathrm{o}(2n,\mathbb{R}) = \{N \in \mathrm{gl}(2n,\mathbb{R}) \mid N^TJ + JN = 0,\ N^T + N = 0\}$. Then there exists a matrix M, satisfying $M^TJ + JM = 0$ and $M^T + M = 0$, such that $\widetilde{P} = \exp(M)$.

Let $z(t) = \exp(tM)u(t)$, where $u(0) \in L_0$ and $u(1) \in L_0$, then $z(0) \in L_0$ and $z(1) \in L'$. By $\dot{z}(t) = J\nabla H(t, z(t))$, then we have

$$\dot{z}(t) = \exp(tM)Mu(t) + \exp(tM)\dot{u}(t). \tag{3.1}$$

$$H(t, z(t)) = H(t, \exp(tM)u(t)) \triangleq \widetilde{H}(t, u(t)). \tag{3.2}$$

Thus from (3.2),

$$\nabla\widetilde{H}(t, u(t)) = (\exp(tM))^T\nabla H(t, \exp(tM)u(t)). \tag{3.3}$$

Besides by $M^T = -M$ and $M^TJ = -JM$, we have $JM = MJ$. Furthermore, we obtain

$$(\exp(tM))^{-1} = (\exp(tM))^T \quad \text{and} \quad (\exp(tM))^TJ\exp(tM) = J. \tag{3.4}$$

Then (3.3) is

$$\nabla H(t, \exp(tM)u(t)) = \exp(tM)\nabla\widetilde{H}(t, u(t)). \tag{3.5}$$

By (3.1)–(3.5), we can change (1.4) into the following one:

$$\exp(tM)Mu(t) + \exp(tM)\dot{u}(t) = J\exp(tM)\nabla\widetilde{H}(t, u(t)),$$

i.e.,

$$\exp(tM)\dot{u}(t) = -\exp(tM)Mu(t) + J\exp(tM)\nabla\widetilde{H}(t, u(t)),$$

i.e.,

$$\begin{aligned}\dot{u}(t) &= -(\exp(tM))^{-1}\exp(tM)Mu(t) + (\exp(tM))^{-1}J\exp(tM)\nabla\widetilde{H}(t, u(t))\\ &= JJMu(t) + J\nabla\widetilde{H}(t, u(t))\\ &= J(JMu(t) + \nabla\widetilde{H}(t, u(t))).\end{aligned} \tag{3.6}$$

Let $\widetilde{B} = JM$. Next, we prove that in (3.6), $\widetilde{H}(t, u(t))$ and $\widetilde{B}$ satisfy (H1)–(H5).

Indeed, for (H1), there is a $\theta' \in (0, \frac{1}{2})$ and $\overline{r}' > 0$ such that

$$0 < H(t, z(t)) \le \theta'(z, \nabla H(t, z(t))) \quad \text{for all } z \in \mathbb{R}^{2n},\ |z| \ge \overline{r}',\ t \in [0, 1]. \tag{3.7}$$

By (3.2), (3.4) and (3.5), then (3.7) is

$$0 < \widetilde{H}(t, u(t)) \le \theta'(\exp(tM)u(t), \exp(tM)\nabla\widetilde{H}(t, u(t))), \tag{3.8}$$

and

$$|z|^2 = (\exp(tM)u, \exp(tM)u) = (\exp(tM)^T \exp(tM)u, u) = |u|^2, \tag{3.9}$$

$$\begin{aligned}(\exp(tM)u(t), \exp(tM)\nabla\widetilde{H}(t, u(t))) &= (\exp(tM)^T \exp(tM)u(t), \nabla\widetilde{H}(t, u(t))) \\ &= (u(t), \nabla\widetilde{H}(t, u(t))).\end{aligned} \tag{3.10}$$

Then by (3.9) and (3.10), (3.8) is

$$0 < \widetilde{H}(t, u(t)) \le \theta'(u(t), \nabla\widetilde{H}(t, u(t))) \quad \text{for all } u \in \mathbb{R}^{2n},\ |u| \ge \overline{r}',\ t \in [0, 1]. \tag{3.11}$$

For (H2),

$$\widetilde{H}(t, u(t)) = H(t, \exp(tM)u(t)) = H(t, z(t)) \ge 0 \quad \text{for all } u \in \mathbb{R}^{2n},\ t \in [0, 1]. \tag{3.12}$$

For (H3),

$$\frac{\widetilde{H}(t, u)}{|u|^2} = \frac{H(t, z)}{|\exp(tM)^{-1}z|^2}. \tag{3.13}$$

Besides by (3.4),

$$\begin{aligned}|\exp(tM)^{-1}z|^2 &= (\exp(tM)^{-1}z, \exp(tM)^{-1}z) = (\exp(tM)\exp(tM)^{-1}z, z) \\ &= (z, z) = |z|^2.\end{aligned} \tag{3.14}$$

Thus by $H(t, z) = o(|z|^2)$ at $z = 0$ and (3.14), (3.13) is

$$\frac{\widetilde{H}(t, u)}{|u|^2} = \frac{H(t, z)}{|z|^2} \to 0, \quad \text{at } u = 0,$$

i.e.,

$$\widetilde{H}(t, u) = o(|u|^2), \quad \text{at } u = 0. \tag{3.15}$$

For (H4), there are constants a', $b'' > 0$ such that

$$|\nabla H(t, z)| \le a'(z, \nabla H(t, z)) + b'' \quad \text{for all } z \in \mathbb{R}^{2n},\ t \in [0, 1]. \tag{3.16}$$

By (3.2), (3.4) and (3.5), then (3.16) is

$$|\exp(tM)\nabla\widetilde{H}(t, u(t))| \le a'(\exp(tM)u(t), \exp(tM)\nabla\widetilde{H}(t, u(t))) + b'', \tag{3.17}$$

$$\begin{aligned}|\exp(tM)\nabla\widetilde{H}(t, u(t))|^2 &= (\exp(tM)\nabla\widetilde{H}(t, u(t)), \exp(tM)\nabla\widetilde{H}(t, u(t))) \\ &= (\exp(tM)^T \exp(tM)\nabla\widetilde{H}(t, u(t)), \nabla\widetilde{H}(t, u(t))) \\ &= (\nabla\widetilde{H}(t, u(t)), \nabla\widetilde{H}(t, u(t))) = |\nabla\widetilde{H}(t, u(t))|^2.\end{aligned} \tag{3.18}$$

Then by (3.10) and (3.18), (3.17) is

$$|\nabla\widetilde{H}(t, u(t))| \le a'(u(t), \nabla\widetilde{H}(t, u(t))) + b'' \quad \text{for all } u \in \mathbb{R}^{2n}, t \in [0, 1]. \tag{3.19}$$

For (H5), the choice of matrix M is not unique, for any choice M satisfying $\widetilde{P} = \exp(M)$, $M^T = -M$ and $JM = MJ$. Without loss of generality, we fix an M, let $\widetilde{M} = M + 2k\pi J$, then $\widetilde{M}$ satisfies the above conditions. By taking $k < 0$ and $|k|$ large enough, we can see that $J\widetilde{M}$ is a semipositive symmetric continuous matrix.

From (3.11), (3.12), (3.15) and (3.19), we obtain that $\widetilde{H}(t,u(t)) \in C^2([0,1]\times\mathbb{R}^{2n},\mathbb{R})$ satisfies (H1)–(H4).

In all, in (3.6), $\widetilde{H}(t,u(t))$ and $\widetilde{B}$ satisfy the same conditions as in Theorem 1.1, then by Theorem 1.1, Theorem 1.2 holds.

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