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Regularity Properties of the Degenerate Monge-Ampère Equations on Compact Kähler Manifolds

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Abstract The author establishes a result concerning the regularity properties of the degenerate complex Monge-Ampère equations on compact Kähler manifolds.

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1 Introduction

Let (X, ω) be an n-dimensional compact Kähler manifold, and $\alpha \in \mathcal{C}^{\infty}_{1,1}(X, \mathbb{R})$ be a closed $(1, 1)$ form on X, such that

(1) $\alpha \geq 0$ pointwise on X and $\int_X \alpha^n > 0$;

(2) $\frac{\omega^n}{\alpha^n} \in L^{\varepsilon_0}(X, \omega)$ for some $\varepsilon_0 > 0$.

A large class of such $(1, 1)$-forms can be obtained as follows: let $\pi : X \to Y$ be a generically finite map, and let ω_Y be a Kähler metric on Y. Then $\alpha := \pi^*\omega_Y$ verifies the conditions above.

We recall that, according to [3], a function $\phi : X \to [-\infty, \infty)$ is called quasi-plurisubharmonic (quasi-psh for short) if it is locally equal to the sum of a smooth function and a plurisubharmonic (psh) function. Then there exists a constant $C \in \mathbb{R}$ such that $\sqrt{-1}\partial\overline{\partial}\phi \geq -C\omega$ in the sense of currents on X. We say that a function ψ has logarithmic poles if for each open set $U \subset X$ there exists a family of holomorphic functions (f_j^U) such that $\psi \equiv \sum\limits_j |f_j^U|^2$ modulo $\mathcal{C}^{\infty}(U)$. It is an important class of quasi-psh functions.

In this setting, the aim of our note is to prove the next result.

Theorem 1.1 *Let (X, ω) be a compact Kähler manifold, and α be a smooth $(1, 1)$ form on X, having the properties* (1) *and* (2) *above. Consider the quasi-psh functions ψ_1, ψ_2, such that*

(i) $\int_X \exp(p(\psi_1 - \psi_2))\mathrm{d}V_\omega < \infty$ *for some* $p > 1$;

(ii) $\int_X \alpha^n = \int_X \exp(\psi_1 - \psi_2)\mathrm{d}V_\omega$.

Then for each $\gamma \in [0, 1)$, there exists $Y_\gamma \subset X$ such that the solution φ of the equation

$$(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \exp(\psi_1 - \psi_2)\omega^n \tag{1.1}$$

belongs to the Hölder class $\mathcal{C}^{1,\gamma}(X \setminus Y_\gamma)$.

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The estimates for the norm of the solution above can be obtained from the proof, but since they are not very enlighting, we have decided to skip them.

Before stating the next result, we would like to recall the following conjecture, formulated by J.-P. Demailly and J. Kollár [4].

Conjecture 1.1 (see [4]) *Let ψ be a psh function on the unit ball $B \subset \mathbb{C}^n$, such that $\int_B \exp(-\psi) \mathrm{d}\lambda < \infty$. Then there exists a positive real number $\delta > 0$ such that $\int_{(\mathbb{C}^n, 0)} \exp(-(1+\delta)\psi) \mathrm{d}\lambda < \infty$.*

A consequence of this conjecture would be the following statement.

Conjecture 1.2 *Let (X, ω) be a compact Kähler manifold, and ψ be a quasi-psh function such that $\int_X \omega^n = \int_X \exp(-\psi) \mathrm{d}V_\omega < \infty$. Then the Monge-Ampère equation*

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \exp(-\psi)\omega^n$$

has a unique continuous solution up to normalisation.

Now if the previous statement is correct, then by the Chern-Levine-Nirenberg inequalities we get $\psi \exp(-\psi) \in L^1(X)$. Let us assume for simplicity that $\psi \leq -1$ on X. Then the function $\psi - \log(-\psi)$ is equally quasi-psh, so inductively we would get $(-\psi)^k \exp(-\psi) \in L^1(X)$. Unfortunately, the L^1-integrability statements above seem to be more or less equivalent to the existence of the continuous solution in (1.1).

Anyway, an immediate consequence of the theorem above is the next statement, which addresses a question proposed in [7].

Corollary 1.1 *Let (X, ω) be a compact Kähler manifold, and α be a smooth $(1,1)$ form on X, with the properties (1)–(2) above. Consider the quasi–psh functions ψ_1, ψ_2, which are assumed to have logarithmic poles and satisfy the following conditions $\int_X \alpha^n = \int_X \exp(\psi_1 - \psi_2) \mathrm{d}V_\omega < \infty$. Then there exists an analytic set $Y \subset X$ such that the solution φ of the equation*

$$(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \exp(\psi_1 - \psi_2)\omega^n \tag{1.2}$$

is smooth on $X \setminus Y$.

In [7], the authors proved the corollary under the assumptions that the manifold X is projective, and the cohomology class $\{\alpha\}$ lies in the real Neron-Severi group of X. We remark that the same result was obtained by Tian-Zhang [13] (see also [9], as well as [6]).

Let us outline the main steps in the proof of Theorem 1.1. First of all, if the functions $(\psi_j)_{j=1,2}$ are regular enough and α is a genuine Kähler metric, then the question was completely solved by S.-T. Yau [15]. Therefore, the natural idea under the hypothesis of the theorem is to regularise the functions $(\psi_j)_{j=1,2}$, then to use the result of Yau in order to solve the equations with the regularised right-hand side member, and finally to take the limit. As it is well-known, to carry out this programme, we have to provide uniform a priori estimates for the solutions.

Now, if α is a Kähler metric, the $\mathcal{C}^0$-estimates we need were obtained by S. Kolodziej [8] by using simple, tricky and elegant considerations in the pluripotential theory. In [7, 13], the authors proved that the methods of S. Kolodziej can be extended to cover the case where α is only semi-positive on X and strictly positive at some point of X. Therefore, the solution of the equation (1.1) is known to be continuous.

In order to achieve further regularity, we would like to use the $\mathcal{C}^2$-estimates in the theory of Monge-Ampère equations, but a new difficulty occurs: since the $(1,1)$-form α may be zero at some points of X, the quantity $\mathrm{tr}_\alpha \partial\overline{\partial} f$ (clearly needed in the estimates) will be unbounded, even for smooth functions f. This difficulty already appeared in [14], where he considered the case where X is projective, and $\alpha = c_1(L)$, for some big line bundle L. He solved the problem by modifying the $(1,1)$-form α within its cohomology class in order to get a strongly positive representative. Remark that this is possible by the well-known fact (the Kodaira's lemma): a big line bundle can be decomposed as a sum of effective and ample $\mathbb{Q}$-line bundles. Then he observed that the singularities which come into the picture via the effective part of the decomposition do not affect in a significant way the usual $\mathcal{C}^2$-estimates (recently, the same circle of ideas were used in [7, 13]).

In our case, the $(1,1)$-form α does not correspond to a line bundle, and X is not necessarily projective. But recall that a result of J.-P. Demailly and ourself [5] shows that if α is a semi-positive $(1,1)$-form such that $\int_X \alpha^n > 0$, then there exists a function τ with at worst logarithmic poles, such that the current $T := \alpha + \sqrt{-1}\partial\overline{\partial}\tau$ dominates a small multiple of the Kähler metric ω. Thus, we "trade" the smoothness of α for the strong positivity of T, as in the case of line bundles. The only new phenomenon is that the poles of τ may not be of divisorial type. The existence of this current is crucial for the regularity analysis, since it is the right substitute for the Kodaira lemma quoted above.

For the rest of the proof, we follow the classical approach in the Monge-Ampère theory and we show that Y.-T. Siu's version of the second order estimates in [11] can be adapted in our context to give the result.

2 Regularization of Currents and $\mathcal{C}^0$-Estimates

As the title of this paragraph tries to suggest, we will collect here some facts about the regularization of quasi-psh functions. We also recall some results concerning the $\mathcal{C}^0$-estimates for the Monge-Ampère operators which will be used later. The convention all over this paper is that we will use the same letter "C" to denote a generic constant, which may change from one line to another, but it is independent of the pertinent parameters involved.

Let (X, ω) be an n-dimensional compact Kähler manifold, and ψ be a quasi-psh function. By definition, there exists a constant $C > 0$ such that $C\omega + \sqrt{-1}\partial\overline{\partial}\psi \geq 0$ on X. We recall the next result due to J.-P. Demailly, on the regularization of ψ. In fact, the statement in [3] is much more precise, but all we need is the following particular case.

Theorem 2.1 (see [3]) *There exist a family of smooth functions $(\psi_\varepsilon) \subset \mathcal{C}^\infty(X)$ and a constant $C > 0$ such that*

(i) $\psi_\varepsilon \to \psi$ *in* $L^1(X)$ *as* $\varepsilon \to 0$, *and* $\psi_\varepsilon \geq \psi - 1$ *for all* $0 < \varepsilon \ll 1$;

(ii) $C\omega + \sqrt{-1}\partial\overline{\partial}\psi_\varepsilon \geq 0$.

The functions ψ_ε are obtained by means of the flow of the Chern connection on the tangent bundle T_X. This can be seen as a global version of the familiar local convolution by smoothing kernels. We apply the previous regularization theorem to ψ_1 and ψ_2. Thanks to the fact that $\exp(\psi_1 - \psi_2)$ is in L^p, for some $p > 1$, by the above considerations we infer that $\exp(\psi_{1;\varepsilon} - \psi_{2;\varepsilon}) \to \exp(\psi_1 - \psi_2)$ in L^p.

We recall now the theorem of S.-T. Yau [15], which will be used (in direct or indirect manner) several times in this note.

Theorem 2.2 (see [15]) *Let (X, ω) be a compact Kähler manifold, and $\mathrm{d}V$ be a smooth volume element, such that $\int_X \mathrm{d}V = \int_X \omega^n$. Then there exists a smooth function φ, unique up* *to normalisation, such that* $\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0$ *and such that*

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \mathrm{d}V.$$

By hypothesis, α is a semi-positive $(1,1)$-form of positive top self-intersection. Thus for each $\varepsilon > 0$ we have $\alpha + \varepsilon\omega > 0$, and by the previous result, there exists a function $\varphi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\alpha + \varepsilon\omega + \sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon > 0$ on X and it is a solution of the equation

$$(\alpha + \varepsilon\omega + \sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon)^n = (1+\delta_\varepsilon)\exp(\psi_{1;\varepsilon} - \psi_{2;\varepsilon})\omega^n \tag{2.1}$$

on X. We assume that $\int_X \varphi_\varepsilon \mathrm{d}V_\omega = 0$; in the previous expression, the real numbers δ_ε are normalisation constants, and we have $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.

We want to prove that some subsequence of the family (φ_ε) converges to the solution of the equation (1.2), and that this limit has the regularity properties stated in the theorem.

The first step in this direction is provided by the next result, due to S. Kolodziej [8].

Theorem 2.3 (see [8]) *Let (X, α) be a compact Kähler manifold and $(f_j) \subset \mathcal{C}^\infty(X)$ be a sequence of functions on X, such that the following requirements are satisfied:*

(a) $\sup_j \|\exp(f_j)\|_{L^p(X)} < \infty$ *for some* $p > 1$*;*

(b) $\int_X \exp(f_j)\mathrm{d}V_\alpha = \int_X \alpha^n$ *for* $j \geq 1$.

Then there exists a constant $C \in \mathbb{R}$ such that for each solution φ_j of the equation

$$(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_j)^n = \exp(f_j)\alpha^n$$

such that $\int_X \varphi_j \mathrm{d}V_\alpha = 0$, we have $\sup_X |\varphi_j| \leq C$. Moreover, if $\exp(f_j) \to \exp(f_\infty)$ in L^p, then there exists a continuous function φ_∞ such that $\varphi_j \to \varphi_\infty$ and it is the solution of the equation

$$(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\infty)^n = \exp(f_\infty)\alpha^n. \tag{2.2}$$

Remark at this point that if the number p above is large enough (compared to the dimension of X), then the above result follows from S.-T. Yau's original proof of the $\mathcal{C}^0$ estimates (by an obvious modification). But the arguments provided by S. Kolodziej seem to be more flexible, since they were adapted in [7, 13] by Eyssidieux, Guedj and Zeriahi, respectively Tian and Zhang, to get the next statement.

Proposition 2.1 (see [7, 13]) *The above statement holds true, if $\alpha \geq 0$ pointwise on X, and $\int_X \alpha^n > 0$.*

As a consequence of this proposition the family of solutions of the equation (1.2) admits an a priori L^∞ bound; this is the part of the argument where the integrability condition (2) on α is needed (to insure the L^p integrability hypothesis; remark that in the equation (1.1), we have ω^n instead of α^n in the right-hand side).

A general conclusion of the results collected here is the next statement.

Corollary 2.1 *There exists a constant $C > 0$ depending on p and the geometry of (X, ω) such that $|\varphi_\varepsilon|_{L^\infty} \leq C$ for each $\varepsilon > 0$. In addition, we can extract a continuous limit φ of the family (φ_ε).*

Remark 2.1 Quite recently, S. Kolodziej proved that the solution of the equation (2.2) belongs to the Hölder space $\mathcal{C}^\gamma(X)$ (for some γ depending on p) if α is the inverse image of a Kaehler metric by a generically finite map. At this moment, it is unclear whether his methods can be used to prove an analogous regularity result in the hypothesis of the proposition above.

3 End of the Proof

In order to achieve further regularity, we would like to use the $\mathcal{C}^2$-estimates in the theory of Monge-Ampère equations. But we cannot do it directly, because the eigenvalues of α may vanish at some points of X. We overcome this difficulty by using the next result of J.-P. Demailly and ourself (see [5]).

Theorem 3.1 (see [5]) *Let (X, ω) be a compact Kähler manifold, and let α be a semi-positive, closed $(1,1)$-form on X, such that $\int_X \alpha^n > 0$. Then there exist $\varepsilon_0 > 0$ and $\tau \in L^1(X)$ which have at worst logarithmic poles, such that*

$$\alpha + \sqrt{-1}\partial\overline{\partial}\tau \geq \varepsilon_0 \omega$$

as currents on X.

Thus, even if the eigenvalues of the $(1,1)$-form α may be zero on an open set of X, we can modify it by the Hessian of a function τ, such that it dominates a small multiple of the Kähler metric. Of course, now we have to deal with the poles of τ. But along the following lines we will show that a careful reading of the computations performed by Y.-T. Siu [11] will give the result.

Before that, remark that in general we cannot expect the poles of τ to be divisorial (as in the case of line bundles), so we have to proceed to an intermediate step. There exists a modification (composition of blow-up maps with smooth centers) $\pi : \widehat{X} \to X$ such that

$$\pi^*\alpha = \widehat{\omega} + [E] - \sqrt{-1}\partial\overline{\partial}\eta, \tag{3.1}$$

where $\widehat{\omega}$ is a Kähler metric on $\widehat{X}$, E is an effective $\mathbb{Q}$-divisor on $\widetilde{X}$, and η is a quasi-psh function on $\widehat{X}$. Indeed, we first use a sequence of blow up maps to get rid of the poles of τ. Thus on a model of X the absolutely continuous part of the inverse image of $\alpha + \sqrt{-1}\partial\overline{\partial}\tau$ dominates a small multiple of the inverse image of a Kähler metric, and now just recall the way one constructs a metric on the blow-up of a manifold; we refer to [5] for a complete description of this process.

We are going to use the equality (3.1) in order to study the regularity of $\varphi := \lim_{\varepsilon} \varphi_\varepsilon$. We will use along the next lines the following notation: if f is a function on X, we denote by $\widehat{f}$ the function $f \circ \pi$.

On $\widehat{X}$, the equality (2.1) reads as

$$(\pi^*(\alpha + \varepsilon\omega) + \sqrt{-1}\partial\overline{\partial}\widehat{\varphi}_\varepsilon)^n = (1 + \delta_\varepsilon)\exp(\widehat{\psi}_{1;\varepsilon} - \widetilde{\psi}_{2;\varepsilon})\|J(\pi)\|^2\widehat{\omega}^n, \tag{3.2}$$

where $\|J(\pi)\|^2 := \omega^n/\widehat{\omega}^n$. We denote $\Phi_\varepsilon := \widehat{\varphi}_\varepsilon - \eta$, and $\widehat{\omega}_\varepsilon := \widehat{\omega} + \varepsilon\pi^*\omega$. Remark that the geometry of $(\widehat{X}, \widehat{\omega}_\varepsilon)$ is bounded independently of ε. By using the relations (3.1) and (3.2) we get

$$(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\overline{\partial}\Phi_\varepsilon)^n = (1 + \delta'_\varepsilon)\exp(\widehat{\psi}_{1;\varepsilon} - \widehat{\psi}_{2;\varepsilon})\|J(\pi)\|^2\widehat{\omega}_\varepsilon^n \tag{3.3}$$

pointwise on $\widehat{X} \setminus E$ (the symbols δ'_ε are functions which tend to zero in $\mathcal{C}^\infty$ norm).

The inequality we start with is borrowed from Y.-T. Siu [11, p. 99]. Considering in general a compact Kähler manifold $(\widehat{X}, \widehat{\omega})$, we denote by Δ one half of the Laplace-Beltrami operator associated to the metric $\widehat{\omega}$, and for each function Φ such that $\widehat{\omega} + \sqrt{-1}\partial\overline{\partial}\Phi > 0$ we denote by Δ_Φ the Laplacian of the metric $\widehat{\omega} + \sqrt{-1}\partial\overline{\partial}\Phi$ on $\widehat{X}$.

Lemma 3.1 (see [11]) *Let $\Phi, f \subset \mathcal{C}^\infty(\widehat{X})$ be smooth functions on an open subset U of $\widehat{X}$, such that*

$$(\widehat{\omega} + \sqrt{-1}\partial\overline{\partial}\Phi)^n = \exp(f)\widehat{\omega}^n$$

pointwise on U. Then there exists a constant C depending on the geometry of $(\widehat{X}, \widehat{\omega})$ only, such that the following inequality holds true (pointwise on the open set U):

$$\Delta_\Phi\big(\log(n + \Delta\Phi)\big) \geq \frac{1}{n + \Delta\Phi}\big(\Delta f - C\big) - C\sum_{j=1}^n \frac{1}{1 + \Phi_{,j\overline{j}}}. \tag{3.4}$$

In our case, the geometry of the family of Kähler manifolds $(\widehat{X}, \widehat{\omega}_\varepsilon)$ is uniformly bounded; therefore the constant C above can be assumed to be independent of ε. Also, the Hessian of $\widehat{\psi}_{1;\varepsilon}$ is bounded from below independently of ε; thus we have $\Delta_\varepsilon(\widehat{\psi}_{1;\varepsilon} + \log\|J(\pi)\|^2) \geq -C$ uniformly with respect to ε. Thus, the inequality (3.4) implies

$$\Delta_{\Phi_\varepsilon} \log(n + \Delta_\varepsilon\Phi_\varepsilon) \geq -\frac{1}{n + \Delta\Phi_\varepsilon}\big(\Delta_\varepsilon\widehat{\psi}_{2;\varepsilon} + C\big) - C\sum_{j=1}^n \frac{1}{1 + \Phi_{\varepsilon,j\overline{j}}}. \tag{3.5}$$

We want next to move the term containing $\widehat{\psi}_{2;\varepsilon}$ inside the $\Delta_{\Phi_\varepsilon}$; for this, we need the following simple observation.

Lemma 3.2 *There exists a constant $C > 0$ such that*

$$\Delta_{\Phi_\varepsilon}\widehat{\psi}_{2;\varepsilon} \geq \frac{\Delta_\varepsilon\widehat{\psi}_{2;\varepsilon}}{n + \Delta_\varepsilon\Phi_\varepsilon} - C\sum_{j=1}^n \frac{1}{1 + \Phi_{\varepsilon,j\overline{j}}}. \tag{3.6}$$

Proof We will prove the lemma by a local computation. By using an appropriate coordinate system (z^j) at a point $x \in \widehat{X}$, the quantities under consideration are

(1) $\Delta_{\Phi_\varepsilon}\widehat{\psi}_{2;\varepsilon} = \sum_j \frac{\widehat{\psi}_{2;\varepsilon,j\overline{j}}}{1+\Phi_{\varepsilon,j\overline{j}}}$;

(2) $\Delta_\varepsilon\widehat{\psi}_{2;\varepsilon} = \sum_j \psi_{2;\varepsilon,j\overline{j}}$.

We use again at this point the fact that the geometry of $(\widehat{X}, \widehat{\omega}_\varepsilon)$ is bounded, and thus there exists a constant $C > 0$ independent of ε, such that $\sqrt{-1}\partial\overline{\partial}\widehat{\psi}_{2;\varepsilon} \geq -C\widehat{\omega}_\varepsilon$ on $\widehat{X}$ (we use the statement (ii) of the regularization theorem quoted above). Therefore we have $\widehat{\psi}_{2;\varepsilon,j\overline{j}} \geq -C$ for all $j = 1, \cdots, n$. At the point x we have

$$\Delta_{\Phi_\varepsilon}\widehat{\psi}_{2;\varepsilon} = \sum_j \frac{\widehat{\psi}_{2;\varepsilon,j\overline{j}}}{1 + \Phi_{\varepsilon,j\overline{j}}} = \sum_j \frac{\widehat{\psi}_{2;\varepsilon,j\overline{j}} + C}{1 + \Phi_{\varepsilon,j\overline{j}}} - C\sum_j \frac{1}{1 + \Phi_{\varepsilon,j\overline{j}}}$$

and this quantity is greater than

$$\frac{\Delta_\varepsilon\widehat{\psi}_{2;\varepsilon}}{n + \Delta_\varepsilon(\Phi_\varepsilon)} - C\sum_{j=1}^n \frac{1}{1 + \Phi_{\varepsilon,j\overline{j}}}$$

and thus the lemma is proved.

By the inequalities (3.5) and (3.6), we get

$$\Delta_{\Phi_\varepsilon}(\widehat{\psi}_{2;\varepsilon} + \log(n + \Delta_\varepsilon(\Phi_\varepsilon))) \geq -C\sum_{j=1}^n \Big(1 + \frac{1}{1 + \Phi_{\varepsilon,j\overline{j}}}\Big).$$

Noting also the previous inequality and the fact that

$$\Delta_{\Phi_\varepsilon}(\Phi_\varepsilon) = n - \sum_{j=1}^{n} \frac{1}{1+\Phi_{\varepsilon,j\bar{j}}},$$

we infer

$$\Delta_{\Phi_\varepsilon}(-2C\Phi_\varepsilon + \widehat{\psi}_{2;\varepsilon} + \log(n + \Delta_\varepsilon(\Phi_\varepsilon))) \geq C\sum_{j=1}^{n} \frac{1}{1+\Phi_{\varepsilon,j\bar{j}}} - C. \tag{3.7}$$

Recall now that $\Phi_\varepsilon = \widehat{\varphi}_\varepsilon - \eta$, where the function η has at most logarithmic poles along an analytic set E. Thus for each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that the next inequality holds true uniformly on $\widehat{X} \setminus E$:

$$-2C\Phi_\varepsilon + \widehat{\psi}_{2;\varepsilon} + \log(n + \Delta_\varepsilon \Phi_\varepsilon) \leq C_\varepsilon$$

(this is so because η and $\Delta_\varepsilon(-\eta)$ are bounded from above).

We are now in good position to apply the maximum principle: consider $x_\varepsilon \in \widehat{X} \setminus E$ the point where the maximum of the function considered above is achieved; at x_ε the relation (3.7) gives

$$\sum_{j=1}^{n} \frac{1}{1+\Phi_{\varepsilon,j\bar{j}}} \leq C. \tag{3.8}$$

On the other hand, the Monge-Ampère equation (3.2) implies

$$\prod_{j=1}^{n}(1+\Phi_{\varepsilon,j\bar{j}}) \leq C\exp(-\widehat{\psi}_{2;\varepsilon}).$$

Thus, for each $j = 1, \cdots, n$, we get

$$(1+\Phi_{\varepsilon,j\bar{j}})\exp(\widehat{\psi}_{2;\varepsilon}) \leq C$$

and therefore at the point x_ε we obtain

$$(n + \Delta_\varepsilon \Phi_\varepsilon)\exp(\widehat{\psi}_{2;\varepsilon}) \leq C.$$

Observe that so far, we did not use the uniform L^∞ bound for the functions φ_ε. We do it now and infer that at x_ε the next relation holds:

$$(n + \Delta_\varepsilon \Phi_\varepsilon)\exp(\widehat{\psi}_{2;\varepsilon} - 2C\Phi_\varepsilon) \leq C. \tag{3.9}$$

Since x_ε is the maximum point of the previous function, we see that the inequality (3.9) holds true at any point of $X_1 \setminus E$.

In conclusion, we have a uniform constant $C > 0$ such that

$$n + \Delta_\varepsilon(\widehat{\varphi}_\varepsilon - \eta) \leq C\exp(-\widehat{\psi}_{2;\varepsilon} + 2C(\widehat{\varphi}_\varepsilon - \eta)). \tag{3.10}$$

Now we have $\psi_{2;\varepsilon} \geq \psi_2$ by the regularization theorem, and moreover, we claim that for each $p > 0$ there exists $Y_p \subset \widehat{X}$ such that $\exp(-\widehat{\psi}_2 - 2C\eta) \in L^p_{\mathrm{loc}}(\widehat{X} \setminus Y_p)$. Indeed, it is enough to consider the analytic set Y_p where the Lelong numbers of the quasi-psh function $\widehat{\psi}_2 + 2C\eta$ are larger than $\frac{1}{p}$ (for the analyticity of the level sets, see [10]) and the local integrability of the function in the complement of this set is a consequence of a result of H. Skoda [12].

We quote now the next regularity result (see e.g. [1]).

Theorem 3.2 (Kondrakov) *Let $B \subset \mathbb{C}^n$ be the unit ball. Then the inclusion*

$$L_2^p(U) \mapsto \mathcal{C}^{1,\gamma}(U)$$

is compact, provided that $q(1-\gamma) > n$.

Thus Theorem 1.1 is proved. As for Corollary 1.1, it is an easy consequence of the previous considerations and the classical Schauder theory, as it is usually applied in the context of the Monge-Ampère operators (see e.g. [11, 15]).

Remark 3.1 Let us consider the following geometric context: Y is a compact complex Kähler space (eventually singular), X is a compact Kähler (smooth) manifold, $\pi : X \to Y$ is a generically finite map, $\alpha = \pi^*\omega_Y$ is the inverse image of a Kähler metric on Y and finally the functions ψ_j have logarithmic poles. Then it is very likely that all the results proved here are an obvious consequence of the original proof of the Calabi conjecture. Indeed, a strong indication in this direction is the next observation, due to Cascini-LaNave [2]: the holomorphic bisectional curvature of $\pi^*(\omega_Y)$ is bounded. On the dark side, the matter is not completely clear, since for example the components of the Ricci tensor of $\pi^*(\omega_Y)$ are not bounded (because of the additional contraction with the singular metric).

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