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Some Remarks Concerning Hyperholomorphic B-Manifolds****

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(*Dedicated to the memory of Vladimir Vishnevskii*)

Abstract The authors consider a differentiable manifold with Π-structure which is an isomorphic representation of an associative, commutative and unitial algebra. For Riemannian metric tensor fields, the Φ-operators associated with r-regular Π-structure are introduced. With the help of Φ-operators, the hyperholomorphity condition of B-manifolds is established.

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1 Introduction

Let $\mathfrak{A}_m$ be an associative commutative unitial algebra (hypercomplex algebra) of order m over the field of real numbers $\mathbb{R}$. We consider the exact (monomorphic) representation $\Phi : \mathfrak{A}_m \to \operatorname{End} L_n$ of algebra $\mathfrak{A}_m$ in a linear space L_n over $\mathbb{R}$. Note that the algebra $\mathfrak{A}_m$ admits in its vector space, the so-called regular representation, given by linear operators $S_\alpha(x) = ax$, where a is a fixed element of $\mathfrak{A}_m$. It is not difficult to see that the regular representation is exact. For the regular representation, we have

$$(S_\alpha)^\beta_\alpha = C^\beta_{\sigma\alpha} a^\sigma, \quad \alpha, \beta, \sigma = 1, \cdots, m,$$

where $C^\beta_{\sigma\alpha}$ are structure constants of the algebra $\mathfrak{A}_m$. In particular, to the base units $e_\sigma \in \mathfrak{A}_m$, there correspond the matrices $S_\sigma = (C^\beta_{\sigma\alpha})$. It is known that for the linear operator (affinor) to belong to regular representation $\{S_\alpha\}$, the necessary and sufficient condition is that it commutes with all S_α (see [3]). With the aid of regular representation, we build the so-called r-regular representation of algebra $\mathfrak{A}_m$ in the linear space L_n $(n = mr)$, which is also exact, and the matrix of r-regular representation has the form

$$\underset{\alpha}{(\varphi)}{}^i_j = \delta^u_v (S_\alpha)^\alpha_\beta,$$

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where δ_v^u is the Kronecker symbol and $u, v = 1, \cdots, r,\ i, j = 1, \cdots, n$.

In this work, we consider only r-regular representations of algebra $\mathfrak{A}_m$.

Let $L_r(\mathfrak{A}_m)$ be an $\mathfrak{A}$-module or a module over algebra $\mathfrak{A}_m$ of order r (see [9, p. 65]), which is defined with the aid of the operators $\{\varphi\}$ or r-regular representation

$$\Phi : \mathfrak{A}_m \to \mathrm{End}\, L_n,$$

where $\{\varphi\} = \Phi(\mathfrak{A}_m) \subset \mathrm{End}\, L_n,\ n = mr$. Note that the $\mathfrak{A}$-module $L_r(\mathfrak{A}_m)$ arises after comparison

$$\xi^i = \xi^{(u-1)m+\alpha} = \xi^{u\alpha} \to \overset{*}{\xi}{}^u = \xi^{u\alpha} e_\alpha.$$

In fact, if $\eta^i = \varphi_j^i \xi^j$, where $\varphi_j^i \in \Phi(\mathfrak{A}_m)$, then $\eta^{u\alpha} = \delta_v^u C_{\sigma\beta}^\alpha \xi^{v\beta}$, or

$$\overset{*}{\eta}{}^u = \eta^{u\alpha} e_\alpha = C_{\sigma\beta}^\alpha \xi^{u\beta} e_\alpha = e_\sigma e_\beta \xi^{u\beta} = e_\sigma \overset{*}{\xi}{}^u.$$

The vector transformation law for quantities $\overset{*}{\xi}{}^u$ is verified after the definition of a fundamental group of module $L_r(\mathfrak{A}_m)$. The fundamental group of the module $L_r(\mathfrak{A}_m)$ is realized in L_n as the subgroup $G_\varphi \subset \mathrm{GL}(n, \mathbb{R})$, which preserves affinors of representation, i.e., $\forall\, p \in G_\varphi$ and $\forall\, \varphi \in \Phi(\mathfrak{A}_m)$,

$$\varphi p = p \varphi, \quad \det(p_{j'}^i) \neq 0.$$

Thus, any block of matrix P of order m commutes with all S_α. That is why

$$p_{j'}^i = \Delta_{u'}^{\sigma u} C_{\sigma\alpha'}^\alpha,$$

where $\Delta_{u'}^{\sigma u}$ are arbitrary coefficients subject only to the regularity condition $\det(p_{j'}^i) \neq 0$. It is easily seen that $\overset{*}{\xi}{}^u = S_{u'}^u \overset{*}{\xi}{}^{u'}$, where $S_{u'}^u = \Delta_{u'}^{\sigma u} e_\sigma$, i.e., the comparison $\xi^{u\alpha} \to \overset{*}{\xi}{}^u$ is defined correctly on the vector module $L_r(\mathfrak{A}_m)$ over algebra $\mathfrak{A}_m$.

Let $z = x^\alpha e_\alpha$ be a variable in algebra $\mathfrak{A}_m$ and $f^1(x), f^2(x), \cdots, f^m(x)$ be the set of functions of all x^α. Then $\omega = f^\alpha(x) e_\alpha$ is a function of z. We define the differentials

$$\mathrm{d}\omega = \mathrm{d} f^\alpha e_\alpha, \quad \mathrm{d}z = \mathrm{d}x^\alpha e_\alpha.$$

The function $\omega = \omega(z)$ is called hyperholomorphic, if there exists a function $\omega'(z)$ such that

$$\mathrm{d}\omega = \omega'(z) \mathrm{d}z.$$

The necessary and sufficient condition for hyperholomorphity of function $\omega = \omega(z)$ is the condition (see [3])

$$S_\alpha D = D S_{\alpha'}, \tag{1.1}$$

where

$$S_\sigma = (C_{\alpha\beta}^\gamma), \quad D = \Big(\frac{\partial f^\alpha}{\partial x^\beta}\Big).$$

Condition (1.1) will be called the Scheffers condition (see [8]). In particular, in the case of the algebra of complex numbers $\mathfrak{A}_2 = \mathbb{R}(\mathrm{i})$, where $\mathrm{i}^2 = -1$, the Scheffers condition coincides with Cauchy-Riemann conditions. If we consider the algebra of dual numbers $\mathfrak{A}_2 = \mathbb{R}(\varepsilon)$, where $\varepsilon^2 = 0$, then from (1.1) it follows that the condition of existence of derivative

$$\omega'(z) = \frac{\mathrm{d}\omega}{\mathrm{d}z}, \quad \omega = f^1(x^1, x^2) + \varepsilon f^2(x^1, x^2), \quad z = x^1 + \varepsilon x^2$$

has the form

$$\frac{\partial f^1}{\partial x^2}=0, \quad \frac{\partial f^2}{\partial x^2}=\frac{\partial f^1}{\partial x^1}.$$

Hence, we obtain that the dualholomorphic function $\omega=\omega(z)$ has the structure

$$\omega = F(x^1)+\varepsilon(x^2F'(x^1)+G(x^1)). \tag{1.2}$$

The dualholomorphic function in form (1.2) is called synectic. In particular, if $G(x^1)=0$ in (1.2), then the dualholomorphic function in form (1.2) is called the natural extension of real differentiable function $F(x^1)$ to the algebra $\mathbb{R}(\varepsilon)$.

The notion of hyperholomorphic function of several variables from algebra is introduced in a natural way (see [9]): the function $\omega=f^\alpha(x^1,\cdots,x^{rm})e_\alpha$ is hyperholomorphic with respect to $z^u=x^{(u-1)m+\alpha}e_\alpha,\ u,v=1,\cdots,r$, if and only if the Scheffers condition is valid for Jacobian matrix

$$\frac{D(f^1,\cdots,f^m)}{D(x^{(u-1)m+1},\cdots,x^{um})}, \quad u=1,\cdots,r.$$

Let M_n be a connected manifold of class C^∞. The field of endomorphisms $\Pi=\{\varphi\}$ is called an algebric hypercomplex Π-structure over M_n. By the structure, we mean affinors $\underset{\alpha}{\varphi},\alpha=1,\cdots,m$, which correspond to the base units $e_\alpha\in\mathfrak{A}_m$ under the isomorphism Φ. Then

$$\underset{\alpha}{\varphi}{}^i_m \underset{\beta}{\varphi}{}^m_j = C^\gamma_{\alpha\beta}\underset{\gamma}{\varphi}{}^i_j. \tag{1.3}$$

If Φ is the r-regular representation of algebra $\mathfrak{A}_m$, then the hypercomplex Π-structure is called an r-regular Π-structure over M_n $(n=mr)$. Note that if $\mathfrak{A}_2$ is a complex algebra, then the r-regularity of its representation over M_n at once follows from (1.3). Therefore, an almost complex structure over M_{2r} is an example of r-regular Π-structure. A. P. Shirokov proved in [2] that in the tangent bundle, the r-regular Π-structure arises in a natural way and is defined by algebra of dual numbers.

If the coordinate neighbourhood $U\subset M_n$ is endowed with an affine connection in which $\nabla\varphi=0,\ \forall\,\varphi\in\Pi$, then such a connection is called a Π-connection. A Π-structure is called integrable, if M_n admits a smooth atlas of local charts such that any affinor $\varphi\in\Pi$ in any of the charts of this atlas has constant components. A Π-structure is called almost integrable, if in a neighbourhood of any point of M_n, there exists at least one Π-connection without torsion. It is known that any integrable r-regular Π-structure is almost integrable and vice versa.

From the facts mentioned above, it follows that if on M_{rm} the r-regular Π-structure is given, then the tangent space $T_x(M_{rm})$ at any point $x\in M_n$ is transformed to the module $L_r(\mathfrak{A}_m)$ over algebra $\mathfrak{A}_m$. Moreover, if the r-regular Π-structure on M_n is integrable, then as proved in [3], the adapted charts on M_{rm} consist of charts that are connected by hyperholomorphic transition functions, i.e., M_n carries the structure of hyperholomorphic manifold of order r over algebra $\mathfrak{A}_m:\mathfrak{X}_r(\mathfrak{A})$.

2 Φ_φ-Operator

Let $\mathfrak{A}_m$ be a Frobenius hypercomplex algebra and $\overset{*}{K}=(\overset{*}{K}{}^{u_1\cdots u_p}_{v_1\cdots v_q})$ be a hypercomplex tensor field on $\mathfrak{X}_r(\mathfrak{A}_m)$. Then the real model of such a tensor field is a tensor field $K=(K^{i_1\cdots i_p}_{j_1\cdots j_q})$ on M_{mr} of the same order that is independent of whether its vector or covector arguments are subject to the action of affinors $\underset{\alpha}{\varphi},\ \alpha = 1, \cdots, m$. Such tensor fields are said to be pure with respect to $\Pi = \{\underset{\alpha}{\varphi}\},\ \alpha = 1, \cdots, m$. They were studied by many authors (see [3, 5–7, 9]). Applied to $K \in \Im^p_q(M_n),\ p + q > 1$, the purity means that for any $X_1, X_2, \cdots, X_q \in \Im^1_0(M_n)$ and $\xi_1, \xi_2, \cdots, \xi_p \in \Im^0_1(M_n)$ the following conditions should hold:

$$\begin{aligned}
&K(\varphi X_1, X_2, \cdots, X_q, \xi_1, \xi_2, \cdots, \xi_p)\\
&= K(X_1, \varphi X_2, \cdots, X_q, \xi_1, \xi_2, \cdots, \xi_p) = \cdots = K(X_1, X_2, \cdots, \varphi X_q, \xi_1, \xi_2, \cdots, \xi_p)\\
&= K(X_1, X_2, \cdots, X_q, \varphi' \xi_1, \xi_2, \cdots, \xi_p) = K(X_1, X_2, \cdots, X_q, \xi_1, \varphi' \xi_2, \cdots, \xi_p)\\
&= \cdots = K(X_1, X_2, \cdots, X_q, \xi_1, \xi_2, \cdots, \varphi' \xi_p),
\end{aligned}$$

where φ' is the adjoint operator of φ. The vector (covector) field and scalar is considered to be pure by convention.

We denote by $\overset{*}{\Im}{}^p_q(M_n)$ the module of all pure tensor fields of type (p, q) on M_n with respect to the affinor field $\varphi \in \Im^1_1(M_n)$. We now fix a positive integer λ. If K and L are pure tensor fields of types (p_1, q_1) and (p_2, q_2) respectively, then the tensor product of K and L with contraction $K \overset{C}{\otimes} L = K^{i_1 \cdots m_\lambda \cdots i_{p_1}}_{j_1 \cdots j_{q_1}} L^{r_1 \cdots r_{p_2}}_{s_1 \cdots m_\lambda \cdots s_{q_2}}$ is also a pure tensor field. We shall prove only the case when $K \in \overset{*}{\Im}{}^1_1(M_n)$ and $L \in \overset{*}{\Im}{}^0_2(M_n)$. In fact, we have

$$(K \overset{C}{\otimes} L)(\varphi X, Y) = K(L(\varphi X, Y)) = K(L(X, \varphi Y)) = (K \overset{C}{\otimes} L)(X, \varphi Y).$$

We shall now make the direct sum $\overset{*}{\Im}(M_n) = \sum\limits_{p,q=0}^{\infty} \overset{*}{\Im}{}^p_q(M_n)$ into an algebra over the real number $\mathbb{R}$ by defining the pure product (denoted by $\overset{C}{\otimes}$) of $K \in \overset{*}{\Im}{}^{p_1}_{q_1}(M_n)$ and $L \in \overset{*}{\Im}{}^{p_2}_{q_2}(M_n)$ as follows:

$$\overset{C}{\otimes} : (K, L) \to K \overset{C}{\otimes} L = \begin{cases} K^{i_1 \cdots m_\lambda \cdots i_{p_1}}_{j_1 \cdots j_{q_1}} L^{r_1 \cdots r_{p_2}}_{s_1 \cdots m_\lambda \cdots s_{q_2}} & \text{for } \lambda \le p_1, q_2 \ (\lambda \text{ is a fixed positive integer}),\\ K^{i_1 \cdots i_{p_1}}_{j_1 \cdots m_\mu \cdots j_{q_1}} L^{r_1 \cdots m_\mu \cdots r_{p_2}}_{s_1 \cdots s_{q_2}} & \text{for } \mu \le p_2, q_1 \ (\mu \text{ is a fixed positive integer}),\\ 0 & \text{for } p_1 = 0,\ p_2 = 0,\\ 0 & \text{for } q_1 = 0,\ q_2 = 0. \end{cases}$$

Let $K \in \Im^1_0(M_n)$ and $L \in \Lambda_{q_2}(M_n)$ be a q_2-form. Then the pure product coincides with the interior product $i_X L$.

Definition 2.1 *A map* $\Phi_\varphi : \overset{*}{\Im}(M_n) \to \Im(M_n)$ $\left(\Im(M_n) = \sum\limits_{p,q=0}^{\infty} \Im^p_q(M_n)\right)$ *is a* Φ_φ*-operator on* M_n*, if*

(a) Φ_φ *is linear with respect to constant coefficients,*

(b) *for all* p, q, $\Phi_\varphi : \overset{*}{\Im}{}^p_q(M_n) \to \Im^p_{q+1}(M_n)$,

(c) *for all* K, $L \in \overset{*}{\Im}(M_r)$,

$$\Phi_\varphi(K \overset{C}{\otimes} L) = (\Phi_\varphi K) \overset{C}{\otimes} L + K \overset{C}{\otimes} \Phi_\varphi L,$$

(d) *for all* $X, Y \in \Im^1_0(M_n)$, $\Phi_{\varphi X} Y = -(L_Y \varphi)X$, *where* L_Y *is the Lie derivation with respect to* Y.

(e) *for all* $\omega \in \Im^0_1(M_n)$ *and* $X, Y \in \Im^1_0(M_n)$, $\Phi_{\varphi X}(i_Y \omega) = (\varphi X)(i_Y \omega) - X(i_{\varphi Y} \omega)$.

Remark 2.1 It follows that Φ_φ possesses also the following property:

$$\Phi_{\varphi X}(\omega(Y_1,\cdots,Y_q))=(\varphi X)(\omega(Y_1,\cdots,Y_q))-X(\omega(\varphi Y_1,\cdots,Y_q)).$$

Proof We shall prove the formula for the case $q=2$. By Definition 2.1(d) and the purity of ω, we have

$$\begin{aligned}\Phi_{\varphi X}(\omega(Y,Z))&=\Phi_{\varphi X}((i_Y\omega)Z)=\Phi_{\varphi X}(i_Z(i_Y\omega))=(\varphi X)(i_Z(i_Y\omega))-X(i_{\varphi Z}(i_Y\omega))\\&=(\varphi X)(i_Y\omega)(Z)-X(i_Y\omega)(\varphi Z)=(\varphi X)(\omega(Y,Z))-X(\omega(\varphi Y,Z)).\end{aligned}$$

Let $K\in\overset{*}{\Im}{}^1_q(M_n)$. Using the condition (c) of Definition 2.1, we have, for any operator Φ_φ,

$$\Phi_{\varphi X}(K(Y_1,\cdots,Y_q))=(\Phi_{\varphi X}K)(Y_1,\cdots,Y_q)+\sum_{\lambda=1}^{q}K(Y_1,\cdots,\Phi_{\varphi X}Y_\lambda,\cdots,Y_q).$$

Then Definition 2.1(d) implies

$$\begin{aligned}(\Phi_\varphi K)(X;Y_1,\cdots,Y_q)&=(\Phi_{\varphi X}K)(Y_1,\cdots,Y_q)\\&=-(L_{K(Y_1,\cdots,Y_q)}\varphi)X+\sum_{\lambda=1}^{q}K(Y_1,\cdots,(L_{Y_\lambda}\varphi)X,\cdots,Y_q).\end{aligned}$$

Using (e) by similar devices for $\omega\in\overset{*}{\Im}{}^0_q(M_n)$, we have

$$\begin{aligned}(\Phi_\varphi\omega)(X;Y_1,\cdots,Y_q)&=(L_{\varphi X}\omega-L_X(\omega\circ\varphi))(Y_1,Y_2,\cdots,Y_q)\\&\quad+\sum_{\lambda=2}^{q}\omega(Y_1,Y_2,\cdots,\varphi(L_XY_\lambda),\cdots,Y_q)\\&\quad-\sum_{\lambda=2}^{q}\omega(\varphi Y_1,Y_2,\cdots,L_XY_\lambda,\cdots,Y_q).\end{aligned}\tag{2.1}$$

The following theorem is true.

Theorem 2.1 *Let on M_{rm} be given the integrable r-regular hypercomplex Π-structure. For hypercomplex tensor field $\overset{*}{t}$ of type $(1,q)$ (or of type $(0,q)$) on $\mathfrak{X}_r(\mathfrak{A})$ to be $\mathfrak{A}$-holomorphic tensor field, it is necessary and sufficient that*

$$\Phi_{\underset{\alpha}{\varphi}}t=0,\quad \alpha=1,\cdots,m,\ t\in\overset{*}{\Im}(M_{rm}).$$

Proof For simplicity, let $\overset{*}{t}\in\Im^0_q(\mathfrak{X}_r(\mathfrak{A}))$. By setting $X=\partial_k,\ Y_\lambda=\partial_{j_\lambda},\ \lambda=1,\cdots,q$ in the equation of (2.1), we see that the components $(\Phi_{\underset{\alpha}{\varphi}}t)_{kj_1\cdots j_q}$ of $\Phi_{\underset{\alpha}{\varphi}}t$ with respect to local coordinate system $x^1,\cdots,x^{rm}$ may be expressed as follows:

$$(\Phi_{\underset{\alpha}{\varphi}}t)_{kj_1\cdots j_q}=\underset{\alpha}{\varphi}{}^m_k\partial_m t_{j_1\cdots j_q}-\partial_k(t(\underset{\alpha}{\varphi}))_{j_1\cdots j_q}+\sum_{\lambda=1}^{q}(\partial_{j_\lambda}\underset{\alpha}{\varphi}{}^m_k)t_{j_1\cdots m\cdots j_q}.$$

By virtue of [3],

$$t_{j_1\cdots j_q}=\Im_{v_1\cdots v_q\sigma}C^{\sigma}_{\beta_1\lambda_1}C^{\lambda_1}_{\beta_2\lambda_2}\cdots C^{\lambda_{q-2}}_{\beta_{q-1}\beta_q}.$$

In the adapted charts, we have ($j_\lambda = v_\lambda \beta_\lambda,\ k = \omega\gamma,\ \lambda = 1, \cdots, q$)

$$\begin{aligned}(\Phi_{\underset{\alpha}{\varphi}} t)_{kj_1\cdots j_q} &= \underset{\alpha}{\varphi}{}_k^m \partial_m t_{j_1\cdots j_q} - \partial_k (t(\underset{\alpha}{\varphi}))_{j_1\cdots j_q} \\ &= (C^\mu_{\alpha\gamma}\partial_{\omega\mu}\Im_{v_1\cdots v_q\sigma} - C^\lambda_{\alpha\sigma}\partial_{\omega\gamma}\Im_{v_1\cdots v_s\lambda}) C^\sigma_{\beta_1\lambda_1} C^{\lambda_1}_{\beta_2\lambda_2}\cdots C^{\lambda_{q-2}}_{\beta_{q-1}\beta_q} = 0\end{aligned}$$

or

$$C^\mu_{\alpha\gamma}\partial_{\omega\mu}\Im_{v_1\cdots v_q\sigma} = C^\mu_{\alpha\sigma}\partial_{\omega\gamma}\Im_{v_1\cdots v_q\mu},$$

which is the Scheffers condition of $\mathfrak{A}$-holomorphity of $\overset{*}{t}_{v_1\cdots v_q} = \Im_{v_1\cdots v_q\sigma} e^\sigma$ ($e^\sigma = q^{\sigma\alpha} e_\alpha$, $q^{\sigma\alpha}$ is a Frobenius metric) with respect to local coordinates $z^u = x^{u\alpha} e_\alpha$ from $\mathfrak{X}_r(\mathfrak{A})$. This completes the proof.

3 Hyperholomorphic B-Manifold

Let M_{rm} be a Riemannian manifold with metric g, which is not necessarily positive definite. A pure metric with respect to the hypercomplex structure is a Riemannian metric g such that

$$g(\underset{\alpha}{\varphi}X, Y) = g(X, \underset{\alpha}{\varphi}Y), \quad \alpha = 1, \cdots, m \tag{3.1}$$

for any $X, Y \in \Im^1_0(M_{rm})$. Such Riemannian metrics were studied in [9], where they were said to be B-metrics, since the metric tensor g with respect to the Π-structure is B-tensor according to the terminology accepted in [4]. If (M_{rm}, Π) is an almost hypercomplex manifold with B-metric, we say that (M_{rm}, Π, g) is an almost hypercomplex B-manifold. If $\Pi = \{\underset{\alpha}{\varphi}\}$ is integrable, we say that (M_{rm}, Π, g) is a hypercomplex B-manifold.

In a B-manifold, a B-metric is called hyperholomorphic, if

$$(\Phi_{\underset{\alpha}{\varphi}} g)(X, Y, Z) = 0, \quad \alpha = 1, \cdots, m.$$

If (M_{rm}, Π, g) is a B-manifold with hyperholomorphic B-metric g, we say that (M_{rm}, Π, g) is a hyperholomorphic B-manifold. Since in dimension m, such a manifold is flat (see [9, p. 113]), we assume in the sequel that $\dim M \geq 2m$, i.e., $r \geq 2$.

Theorem 3.1 *An almost B-manifold is a hyperholomorphic B-manifold, if and only if the almost hypercomplex structure is parallel with respect to the Levi-Civita connection* ∇.

Proof By virtue of (3.1) and $\nabla g = 0$, we have

$$g(Z, (\nabla_Y \underset{\alpha}{\varphi})X) = g((\nabla_Y \underset{\alpha}{\varphi})Z, X). \tag{3.2}$$

Using (3.1) and $[X, Y] = \nabla_X Y - \nabla_Y X$, we have transform $\Phi_{\underset{\alpha}{\varphi}} g$ as follows:

$$(\Phi_{\underset{\alpha}{\varphi}} g)(X; Z_1, Z_2) = -g((\nabla_X \underset{\alpha}{\varphi})Z_1, Z_2) + g((\nabla_{Z_1}\underset{\alpha}{\varphi})X, Z_2) + g(Z_1, (\nabla_{Z_2}\underset{\alpha}{\varphi})X). \tag{3.3}$$

From this, we have

$$(\Phi_{\underset{\alpha}{\varphi}} g)(Z_2; Z_1, X) = g((\nabla_{Z_2}\underset{\alpha}{\varphi})Z_1, X) + g((\nabla_{Z_1}\underset{\alpha}{\varphi})Z_2, X) + g(Z_1, (\nabla_X \underset{\alpha}{\varphi})Z_2). \tag{3.4}$$

If we add (3.3) to (3.4), we find

$$(\Phi_{\underset{\alpha}{\varphi}} g)(X; Z_1, Z_2) + (\Phi_{\underset{\alpha}{\varphi}} g)(Z_2; Z_1, X) = 2g(X, (\nabla_{Z_1} \underset{\alpha}{\varphi}) Z_2). \tag{3.5}$$

Putting $\Phi_{\underset{\alpha}{\varphi}} g = 0$ in (3.5), we find $\nabla \underset{\alpha}{\varphi} = 0$. Conversely, if $\nabla \underset{\alpha}{\varphi} = 0$, then the condition $\Phi_{\underset{\alpha}{\varphi}} g = 0$ follows from (3.3) or (3.4).

4 Examples of Hyperholomorphic B-Manifolds

A Kahler-Norden manifold (see [1]) can be defined as a triple (M_{2n}, φ, g), $n \geq 2$, which consists of a manifold M_{2n}, endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla \varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian: $g(\varphi X, Y) = g(X, \varphi Y)$. Thus, the Kahler-Norden manifold is a complex holomorphic B-manifold.

Let $T(M_n)$ be a tangent bundle of a Riemannian manifold (M_n, g). It is well-known that there exists a tensor field of type $(1,1)$ which has components of the form

$$\gamma = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}$$

with respect to the induced coordinates (x^i, x^{n+i}) in $T(M_n)$, E being unit matrix in M_n and γ satisfying $\gamma^2 = 0$. Thus $T(M_n)$ has a natural integrable n-regular dual Π-structure $\Pi = \{I, \gamma\}$, where I denotes the identity transformation. The complete lift ${}^C g$ of g is a B-metric with respect to γ. Thus $(T(M_n), \gamma, {}^C g)$ is a B-manifold. Moreover, we easily see that ${}^C \nabla \gamma = 0$, where ${}^C \nabla$ is the complete lift of the Levi-Civita connection ∇ in M_n. Thus $(T(M_n), \Pi, {}^C g)$ is a dualholomorphic B-manifold.

By similar devices, we can prove that $(T(M_n), \gamma, {}^C g)$ is also a dualholomorphic B-manifold, where ${}^C g = {}^S g + {}^V a$ (${}^V a$ is a vertical lift of a symmetric tensor field $a \in T_2^0(M_n)$) is a synectic lift of g (see [2]).

Let, now, $T^2(M_n)$ be a tangent bundle of order 2 over M_n. It is also well-known that there exists an affinor field $\gamma \in \Im_1^1(T^2(M_n))$ which has components of the form

$$\widehat{\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ E & 0 & 0 \\ 0 & E & 0 \end{pmatrix}, \quad \widehat{\gamma}^3 = 0$$

with respect to the induced coordinates (x^i, x^{n+i}, x^{2n+i}) in $T^2(M_n)$, i.e., $T^2(M_n)$ has a natural integrable n-regular plural Π-structure $\Pi = \{I, \widehat{\gamma}, \widehat{\gamma}^2\}$. The second lift of g, i.e., ${}^{CC} g = {}^{II} g$ (see [10, p. 332]), is a B-metric with respect to $\widehat{\gamma}$ and ${}^{CC} \nabla {}^C g = 0$ where ${}^{CC} \nabla$ denotes the second lift of the Levi-Civita connection ∇, which is necessarily the Levi-Civita connection determined by ${}^{CC} g$. Thus, $(T^2(M_n), \widehat{\Pi}, {}^{CC} g)$ is a pluralholomorphic B-manifold.

A locally decomposable Riemannian manifold M_{2k} is a paraholomorphic B-manifold (see [7]).

5 Curvature Tensors in a Hyperholomorphic B-Manifold

Let R be the Riemannian curvature tensor formed by g. If a torsion free connection ∇ preserving the structure ($\nabla \underset{\alpha}{\varphi} = 0$) satisfies the condition $\nabla_{\underset{\alpha}{\varphi} X} Y = \underset{\alpha}{\varphi}(\nabla_X Y)$, then ∇ is called a hyperholomorphic connection (see [9, p. 185]). The purity of the curvature tensor field of a connection ∇ is a necessary and sufficient condition for its holomorphy (see [3, 9]). Since the Levi-Civita connection of hyperholomorphic B-manifold is hyperholomorphic (see [3, 9]), we see that in a hyperholomorphic B-manifold, the Riemannian curvature tensor R of B-metric g is pure.

Since the Riemannian curvature tensor R is pure, we can apply the Φ-operator to R. By similar devices (see the proof of Theorem 3.1), we can prove that

$$(\Phi_{\underset{\alpha}{\varphi}} R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{\underset{\alpha}{\varphi} X} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(\underset{\alpha}{\varphi} Y_1, Y_2, Y_3, Y_4). \tag{5.1}$$

Applying the Ricci's identity to $\underset{\alpha}{\varphi}$, we get

$$\underset{\alpha}{\varphi}(R(X, Y)Z) = R(X, Y)\underset{\alpha}{\varphi} Z \tag{5.2}$$

by virtue of $\nabla \underset{\alpha}{\varphi} = 0$. Using (5.2) and applying the second Bianchi identity to (5.1), we get

$$\begin{aligned}(\Phi_{\underset{\alpha}{\varphi}} R)(X, Y_1, Y_2, Y_3, Y_4) &= g((\nabla_{\underset{\alpha}{\varphi} X} R)(Y_1, Y_2, Y_3) - (\nabla_X R)(\underset{\alpha}{\varphi} Y_1, Y_2, Y_3), Y_4) \\ &= g((\nabla_{\underset{\alpha}{\varphi} X} R)(Y_1, Y_2, Y_3) - \underset{\alpha}{\varphi}((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4) \\ &= g(-(\nabla_{Y_1} R)(Y_2, \underset{\alpha}{\varphi} X, Y_3) - (\nabla_{Y_2} R)(\underset{\alpha}{\varphi} X, Y_1, Y_3) \\ &\quad - \underset{\alpha}{\varphi}((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4).\end{aligned} \tag{5.3}$$

On the other hand, using $\nabla \underset{\alpha}{\varphi} = 0$, we find

$$\begin{aligned}(\nabla_{Y_2} R)(\underset{\alpha}{\varphi} X, Y_1, Y_3) &= \nabla_{Y_2}(R(\underset{\alpha}{\varphi} X, Y_1, Y_3)) - R(\nabla_{Y_2}(\underset{\alpha}{\varphi} X), Y_1, Y_3) \\ &\quad - R(\underset{\alpha}{\varphi} X, \nabla_{Y_2} Y_1, Y_3) - R(\underset{\alpha}{\varphi} X, Y_1, \nabla_{Y_2} Y_3) \\ &= (\nabla_{Y_2} \underset{\alpha}{\varphi})(R(X, Y_1, Y_3)) + \underset{\alpha}{\varphi}(\nabla_{Y_2} R(X, Y_1, Y_3)) \\ &\quad - R((\nabla_{Y_2} \underset{\alpha}{\varphi})X + \underset{\alpha}{\varphi}(\nabla_{Y_2} X), Y_1, Y_3) \\ &\quad - R(\underset{\alpha}{\varphi} X, \nabla_{Y_2} Y_1, Y_3) - R(\underset{\alpha}{\varphi} X, Y_1, \nabla_{Y_2} Y_3) \\ &= \underset{\alpha}{\varphi}(\nabla_{Y_2} R(X, Y_1, Y_3)) - \underset{\alpha}{\varphi}(R(\nabla_{Y_2} X, Y_1, Y_3)) \\ &\quad - \underset{\alpha}{\varphi}(R(X, \nabla_{Y_2} Y_1, Y_3)) - \underset{\alpha}{\varphi}(R(X, Y_1, \nabla_{Y_2} Y_3)) \\ &= \underset{\alpha}{\varphi}((\nabla_{Y_2} R)(X, Y_1, Y_3)).\end{aligned} \tag{5.4}$$

Similarly,

$$(\nabla_{Y_1} R)(Y_2, \underset{\alpha}{\varphi} X, Y_3) = \underset{\alpha}{\varphi}((\nabla_{Y_1} R)(Y_2, X, Y_3)). \tag{5.5}$$

Substituting (5.4) and (5.5) in (5.3) and using again the second Bianchi identity, we obtain

$$\begin{aligned}(\Phi_{\underset{\alpha}{\varphi}} R)(X, Y_1, Y_2, Y_3, Y_4) &= g(-\underset{\alpha}{\varphi}((\nabla_{Y_1} R)(Y_2, X, Y_3)) - \underset{\alpha}{\varphi}((\nabla_{Y_2} R)(X, Y_1, Y_3)) \\ &\quad - \underset{\alpha}{\varphi}((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4) \\ &= -g(\underset{\alpha}{\varphi}(\sigma\{(\nabla_X R)(Y_1, Y_2, Y_3)\}), Y_4) = 0,\end{aligned}$$

where σ denotes the cyclic sum with respect to X, Y_1 and Y_2. Therefore, we have

Theorem 5.1 *In a hyperholomorphic B-manifold, the Riemannian curvature tensor field is a pluralholomorphic tensor field.*

Theorem 5.2 *A necessary and sufficient condition for an exact* 1*-form* $\mathrm{d}f,\ f \in \Im_0^0(M_{2m})$ *to be hyperholomorphic, i.e.,* $\Phi_{\underset{\alpha}{\varphi}}(\mathrm{d}f) = 0$*, is that an associated* 1*-form* $\mathrm{d}f \circ \underset{\alpha}{\varphi}$ *be closed, i.e.,* $\mathrm{d}(\mathrm{d}f \circ \underset{\alpha}{\varphi}) = 0$*.*

Proof Using

$$(\mathrm{d}\omega)(X,Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\}, \quad X, Y \in \Im_0^1(M_{2n}),\ \omega \in \Im_1^0(M_{2n})$$

for $(\omega \circ \underset{\alpha}{\varphi})(X) = \omega(\underset{\alpha}{\varphi}(X))$, we have

$$\begin{aligned}(\mathrm{d}\omega)(Y,\underset{\alpha}{\varphi}X) &= \frac{1}{2}\{Y(\omega(\underset{\alpha}{\varphi}X)) - (\underset{\alpha}{\varphi}X)(\omega(Y)) - \omega([Y,\underset{\alpha}{\varphi}X])\}\\ &= \frac{1}{2}\{Y(\omega(\underset{\alpha}{\varphi}X)) - (\underset{\alpha}{\varphi}X)(\omega(Y)) + \omega([\underset{\alpha}{\varphi}X,Y])\}\\ &= \frac{1}{2}\{Y(\omega(\underset{\alpha}{\varphi}X)) - (\underset{\alpha}{\varphi}X)(\omega(Y)) + \omega([\underset{\alpha}{\varphi}X,Y] - \underset{\alpha}{\varphi}[X,Y]) + \omega(\underset{\alpha}{\varphi}[X,Y])\}.\end{aligned} \tag{5.6}$$

From (2.1), we have

$$\begin{aligned}(\Phi_{\underset{\alpha}{\varphi}}\omega)(X,Y) &= (\underset{\alpha}{\varphi}X)(\omega(Y)) - X(\omega(\underset{\alpha}{\varphi}Y)) + \omega((L_Y\underset{\alpha}{\varphi})(X))\\ &= (\underset{\alpha}{\varphi}X)(\omega(Y)) - X(\omega(\underset{\alpha}{\varphi}Y)) - \omega([\underset{\alpha}{\varphi}X,Y] - \underset{\alpha}{\varphi}[X,Y]).\end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.6), we obtain

$$\begin{aligned}(\mathrm{d}\omega)(Y,\underset{\alpha}{\varphi}X) &= \frac{1}{2}\{-(\Phi_{\underset{\alpha}{\varphi}}\omega)(X,Y) + Y(\omega(\underset{\alpha}{\varphi}X)) - X(\omega(\underset{\alpha}{\varphi}Y)) + \omega(\underset{\alpha}{\varphi}[X,Y])\}\\ &= -\frac{1}{2}\{(\Phi_{\underset{\alpha}{\varphi}}\omega)(X,Y) + Y((\omega\circ\underset{\alpha}{\varphi})(X)) - X((\omega\circ\underset{\alpha}{\varphi})(Y)) - (\omega\circ\underset{\alpha}{\varphi})([Y,X])\}\\ &= -\frac{1}{2}(\Phi_{\underset{\alpha}{\varphi}}\omega)(X,Y) + (\mathrm{d}(\omega\circ\underset{\alpha}{\varphi})(Y,X)).\end{aligned}$$

From this we see that the equation $\Phi_{\underset{\alpha}{\varphi}}\omega = 0$ is equivalent to

$$(\mathrm{d}(\omega\circ\underset{\alpha}{\varphi}))(Y,X) = (\mathrm{d}\omega)(Y,\underset{\alpha}{\varphi}X). \tag{5.8}$$

For $\omega = \mathrm{d}f$, equation (5.8) turns into the following simple form

$$(\mathrm{d}(\mathrm{d}f\circ\underset{\alpha}{\varphi}))(Y,X) = (\mathrm{d}^2 f)(Y,\underset{\alpha}{\varphi}X) = 0, \quad \text{i.e.,} \quad \mathrm{d}(\mathrm{d}f\circ\underset{\alpha}{\varphi}) = 0. \tag{5.9}$$

Thus Theorem 5.2 is proved.

If there exists a function g in a hyperholomorphic B-manifold such that $\mathrm{d}f \circ \underset{\alpha}{\varphi} = \mathrm{d}\underset{\alpha}{g}$ for a function f, then we call f a hyperholomorphic function and g an associated function. If such a function f is defined locally, then we call it a locally hyperholomorphic function.

We notice that equation (5.9) is equivalent to $\mathrm{d}f \circ \underset{\alpha}{\varphi} = \mathrm{d}\underset{\alpha}{g}$ only locally. Hence, the condition for f to be locally hyperholomorphic $(\underset{\alpha}{\varphi}{}_i^m \partial_m f = \partial_i \underset{\alpha}{g})$ is also given by

$$(\Phi_{\underset{\alpha}{\varphi}} \mathrm{d}f)_{ij} = \underset{\alpha}{\varphi}{}_i^m \partial_m \partial_j f - \partial_i (\underset{\alpha}{\varphi}{}_j^m \partial_m f) + (\partial_j \underset{\alpha}{\varphi}{}_i^m) \partial_m f = 0.$$

Let $(M_{2m}, \underset{\alpha}{\varphi}, g)$ be a hyperholomorphic B-manifold with B-metric g. Then from Theorem 5.1 and (5.1), we find that in pluralholomorphic B-manifolds the covariant derivative of the curvature tensor field ∇R is also pure. Now, the covariant derivative of the Ricci tensor $R_{ji} = R^s_{sji} = g^{ts} R_{tjis}$ is pure in all its indices and hence

$$\underset{\alpha}{\varphi}{}_t^s \nabla_s R_{ji} = \underset{\alpha}{\varphi}{}_j^s \nabla_t R_{si}.$$

Contracting this equation with contravariant B-metric g^{ji}, we find

$$\underset{\alpha}{\varphi}{}_t^s \nabla_s R = g^{ji} \underset{\alpha}{\varphi}{}_j^s \nabla_t R_{si} = \nabla_t (\underset{\alpha}{G}{}^{si} R_{si}) = \nabla_t \overset{*}{\underset{\alpha}{R}}, \tag{5.10}$$

where $R = g^{ij} R_{ij}$ is the scalar curvature of B-metric g and $\overset{*}{\underset{\alpha}{R}} = g^{ji} \underset{\alpha}{\varphi}{}_j^s R_{si}$.

From (5.10), we have

Theorem 5.3 *In a hyperholomorphic B-manifold, the scalar curvature R is a locally hyperholomorphic function.*

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