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On S-Shaped Bifurcation Curves for a Class of Perturbed Semilinear Equations**

Benlong XU*　　Zhongliang WANG*

Abstract By making use of bifurcation analysis and continuation method, the authors discuss the exact number of positive solutions for a class of perturbed equations. The nonlinearities concerned are the so-called convex-concave functions and their behaviors may be asymptotic sublinear or asymptotic linear. Moreover, precise global bifurcation diagrams are obtained.

Keywords Semilinear elliptic equations, Exact multiplicity of solutions, S-Shaped bifurcation curve
2000 MR Subject Classification 35J65, 35J60, 35B32

1 Introduction

We study the global structure of all positive solutions of the equation

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in } B^n, \\ u > 0, & \text{in } B^n, \\ u = 0, & \text{on } \partial B^n \end{cases} \tag{1.1}$$

and the perturbation equation

$$\begin{cases} \Delta u + \lambda f(u+\epsilon) = 0, & \text{in } B^n, \\ u > 0, & \text{in } B^n, \\ u = 0, & \text{on } \partial B^n, \end{cases} \tag{1.2}$$

where B^n is the unit ball in R^n with $n \geq 1$, $\epsilon > 0$ is a constant, and $\lambda > 0$ is treated as a bifurcation parameter. By a positive solution, we mean a solution $u \in C_0^{2,\alpha}$ $(0 < \alpha \leq 1)$ such that $u > 0$ in B^n.

We assume that

(A1) $f \in C^2[0,\infty)$, $f(0) = 0$; and either

(F1) There exists $c \in (0,\infty)$ such that $f(u) > 0$ in $(0,c)$, $f(u) < 0$ in (c,∞), or

(F2) $f(u) > 0$ in $(0,\infty)$.

Our goal of this paper is to study the exact multiplicity of positive solutions to (1.2). Our results are for the spherical domain B^n, and we assume most of the time that the dimension n

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*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.
E-mail: bxu@shnu.edu.cn　likemath@163.com
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is any positive integer. It is known that determining the exact number of solutions of semilinear equations is usually a very hard and challenging task, even for the one-dimensional case (see [17]). Recently, through the work of P. Korman, T. Ouyang, Y. Li, J. Shi, J. Wei, etc., a systematic bifurcation analysis method has been established to study exact two or less number of solutions for certain equations defined on a unit ball (see [12–14, 18, 19, 21]). This method not only tells us the exact number of solutions (no more than 2), but also tells us the shape of the bifurcation diagram, such as monotone shape, $\subset$-shape and $\supset$-shape. Unfortunately, it seems at present that the bifurcation analysis method does not work well for the problem with more than two solutions. In other words, if the bifurcation curve has two or more turning points, such as S-shaped curve, the bifurcation analysis method will encounter some difficulties. For all that, S-shaped bifurcation was discussed by several mathematicians in the past thirty years. The early results on exact S-shaped bifurcation are for the one-dimensional case (see [3, 12, 22, 23]), by using the time-map method which does not work for $n \geq 2$. We note that P. Korman and Y. Li [12] used some bifurcation analysis combined with time-map technique. Recently, by using perturbation and continuation method, Y. Du and Y. Lou [9] get an exact S-shaped bifurcation result of perturbed Gelfand equation from combustion theory for $n = 1, 2$, and completely solve a long standing conjecture. There are some further results in [8]. By using the idea of [8] and [9], some exact S-shaped bifurcation results are given in [24] for a perturbed equation coming from chemical reaction. We note that the equations discussed in [8, 9, 24] are actual special cases of the problem (1.2) in this paper.

In this paper, we also need the following assumptions:

(A2) There exists $\alpha \in (0, \infty)$ ($\alpha \in (0, c)$, if $f(u)$ satisfies (F1)) such that $f''(u) > 0$ for $u < \alpha$, $f''(u) < 0$ for $u > \alpha$;

(A3) There exists $\beta \in (0, \infty)$ ($\beta \in (0, c)$, if $f(u)$ satisfies (F1)) such that $uf'(u) - f(u) > 0$ for $0 < u < \beta$, $uf'(u) - f(u) < 0$ for $u > \beta$;

Moreover, we define $\rho = \alpha - \frac{f(\alpha)}{f'(\alpha)}$, $K_f(u) = \frac{uf'(u)}{f(u)}$, and we assume that

(A4) $K_f(u)$ is nonincreasing in $[0, \rho]$, and $K_f(u) \leq K_f(\rho)$ for all $u \in [\rho, \alpha]$.

Remark 1.1 From the assumptions above, it is easy to check that $0 < \rho < \alpha < \beta$. The assumption (A3) is equivalent to the statement that $\frac{f(u)}{u}$ is increasing in $(0, \beta)$ and decreasing in (β, ∞). If the assumption (A3) holds, then either $f'(\infty) \leq 0$ or $0 < f'(\infty) < \infty$, where $f'(\infty) = \lim\limits_{u \to \infty} \frac{f(u)}{u}$.

The following exact multiplicity results of the perturbation problem (1.2) are the main results of this paper and can be described by the diagrams in Figure 1.

Theorem 1.1 *Suppose that f satisfies* (A1)–(A4) *and* (F1) *or* (F2). *Suppose that either*

(i) $f'(0) > 0$, *or*

(ii) $f'(0) = 0$, $n \leq 2$, *or* $n \geq 3$ *and* $K_f(u) \leq \frac{n+2}{n-2}$.

Then for all small $\epsilon > 0$, the bifurcation diagram of (1.2) *is exactly S-shaped. More precisely,*

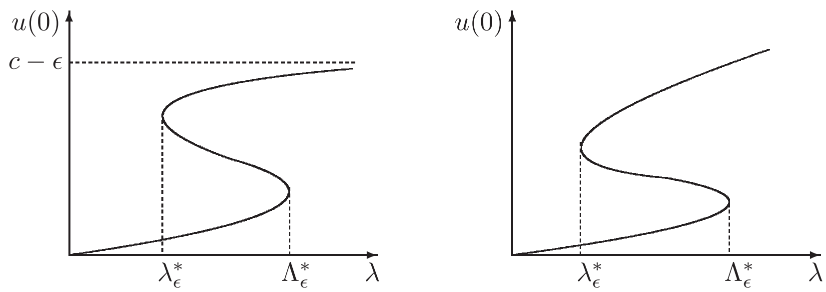
(1) *If f satisfies* (F1), *then there exists $0 < \lambda_\epsilon^* < \Lambda_\epsilon^* < \infty$ such that* (1.2) *has exactly one positive solution for $\lambda < \lambda_\epsilon^*$ or $\lambda > \Lambda_\epsilon^*$, has exactly two positive solutions for $\lambda = \lambda_\epsilon^*$ or $\lambda = \Lambda_\epsilon^*$ and has exactly three positive solutions for $\lambda_\epsilon^* < \lambda < \Lambda_\epsilon^*$ (see Figure* 1(a)).

(2) *If f satisfies* (F2), *then there exists $0 < \lambda_\epsilon^* < \Lambda_\epsilon^* < \infty$ and $\lambda_\infty > 0$, and there are three cases:*

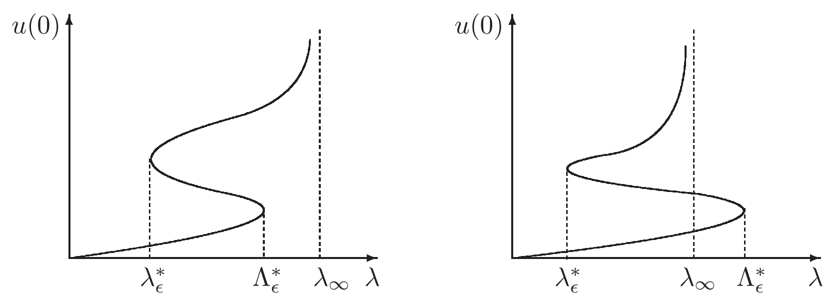
Case 1 $\lambda_\infty = \infty$, (1.2) *has exactly one positive solution for $\lambda < \lambda_\epsilon^*$ or $\lambda > \Lambda_\epsilon^*$, has exactly two positive solutions for $\lambda = \lambda_\epsilon^*$ or $\lambda = \Lambda_\epsilon^*$ and has exactly three positive solutions for $\lambda_\epsilon^* < \lambda < \Lambda_\epsilon^*$ (see Figure* 1(b)).

Case 2 $\Lambda_\epsilon^* < \lambda_\infty < \infty$, (1.2) *has exactly one positive solution for $\lambda < \lambda_\epsilon^*$ or $\Lambda_\epsilon^* < \lambda < \lambda_\infty$, has exactly two positive solutions for $\lambda = \lambda_\epsilon^*$ or $\lambda = \Lambda_\epsilon^*$ and has exactly three positive solutions for $\lambda_\epsilon^* < \lambda < \Lambda_\epsilon^*$ (see Figure* 1(c)).

Case 3 $\lambda_\infty < \Lambda_\epsilon^*$, (1.2) *has exactly one positive solution for $\lambda < \lambda_\epsilon^*$ or $\lambda = \Lambda_\epsilon^*$, has exactly two positive solutions for $\lambda = \lambda_\epsilon^*$ or $\lambda_\infty < \lambda < \Lambda_\epsilon^*$ and has exactly three positive solutions for $\lambda_\epsilon^* < \lambda < \lambda_\infty$ (see Figure* 1(d)).



(a) f satisfying (F1) (b) f satisfying (F2) with $\lambda_\infty = \infty$



(c) f satisfying (F2) with $\Lambda_\epsilon^* < \lambda_\infty < \infty$ (d) f satisfying (F2) with $\lambda_\infty < \Lambda_\epsilon^*$

Figure 1 Bifurcation Diagram of (1.2)

Furthermore, all positive solutions of (1.2) *lie on a single smooth solution curve in the space $R^+ \times C^2(\overline{B^n})$. If f satisfies* (F1), *we denote the upper branch by*

$$\{(\lambda, u^*) : \lambda_\epsilon^* < \lambda < \infty\},$$

or if f satisfies (F2), *then we denote the upper branch by*

$$\{(\lambda, u^*) : \lambda_\epsilon^* < \lambda < \lambda_\infty\},$$

and we denote the middle and lower branches by

$$\{(\lambda, \widetilde{u}) : \lambda_\epsilon^* < \lambda < \Lambda_\epsilon^*\},$$

and

$$\{(\lambda, u_*) : 0 < \lambda < \Lambda_\epsilon^*\},$$

respectively. Then $\lambda \mapsto u^*(\lambda, r)$ *and* $\lambda \mapsto u_*(\lambda, r)$ *are strictly increasing for any fixed* $r < 1$, $\lambda \mapsto \widetilde{u}(\lambda, r)$ *is strictly decreasing, and*

$$\lim_{\lambda \to 0^+} u_*(\lambda, r) = 0, \quad \forall\, r < 1.$$

Moreover $\lim_{\lambda\to\infty} u^*(\lambda, 0) = c - \epsilon$ *if* f *satisfies* (F1)*, and* $\lim_{\lambda\to\lambda_\infty} u^*(\lambda, 0) = \infty$ *if* f *satisfies* (F2).

The rest of this paper is organized as follows. In Section 2, we will recall some preliminaries of bifurcation approach and analyze the limiting equation (1.1). We give the proof of the main theorem in Section 3. For clarity, the two long technical proofs for the positivity of the solutions of the linearized equations are put to Sections 4 and 5.

2 Some Preliminaries and Exact Multiplicity Results of Problem (1.1)

We briefly review the basic setting for bifurcation approach to the set of positive solutions of equation (1.1). The following bifurcation theorems (Lemmas 2.1–2.3) are well-known (see [1, 5, 7, 20]).

Lemma 2.1 (see [5]) *Let* X *and* Y *be Banach spaces. Let* $(\overline{\lambda}, \overline{x}) \in R \times X$ *and let* F *be a continuously differentiable mapping of an open neighborhood of* $(\overline{\lambda}, \overline{x})$ *into* Y*. Let the null-space* $N(F_x(\overline{\lambda}, \overline{x})) = \mathrm{span}\{x_0\}$ *be one-dimensional and* $\mathrm{codim} R(F_x(\overline{\lambda}, \overline{x})) = 1$*. And* $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$*. If* Z *is a complement of span*$\{x_0\}$ *in* X*, then the solutions of* $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ *near* $(\overline{\lambda}, \overline{x})$ *form a curve* $(\lambda(s), x(x)) = (\overline{\lambda} + \tau(s), \overline{x} + s x_0 + z(s))$*, where* $s \mapsto (\tau(s), z(s)) \in R \times Z$ *is a continuously differentiable function near* $s = 0$ *and* $\tau(0) = \tau'(0) = 0$*,* $z(0) = z'(0) = 0$.

Lemma 2.2 (Bifurcation from the Trivial Solutions) *If* $f(0) = 0$ *and* $f'(0) > 0$*,* $\lambda_0 = \frac{\lambda_1}{f'(0)}$*, then all positive solutions of* (1.1) *near* $(\lambda_0, 0)$ *have a form of* $(\lambda(s), sw + z(s))$ *for* $s \in (0, \delta)$ *and some* $\delta > 0$*, and* w *is a positive solution of*

$$\begin{cases} \Delta w + \lambda_1 w = 0, & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{2.1}$$

and $\lambda(0) = \lambda_0$*,* $z(0) = z'(0) = 0$.

Lemma 2.3 (Bifurcation from Infinity) *Let* $f'(\infty) = \lim_{u\to\infty} \frac{f(u)}{u} \in (0, \infty)$ *and* $\lambda_\infty = \frac{\lambda_1}{f'(\infty)}$*. Then all positive solutions of* (1.1) *near* (λ_∞, ∞) *has a form of* $(\lambda(s), sw + z(s))$ *for* $s \in (\delta, \infty)$ *and some* $\delta > 0$*, where* w *is a positive solution of* (2.1)*,* $\lim_{s\to\infty} \lambda(s) = \lambda_\infty$*, and* $\|z(s)\|_{C^{2,\alpha}(\overline{B^n})} = o(s)$ *as* $s \to \infty$.

The next remarkable results regarding (1.1) are due to B. Gidas, W.-M. Ni and L. Nirenberg [10], and C-S. Lin and W-M. Ni [16].

Lemma 2.4 (1) *If* f *is locally Lipschitz continuous in* $[0, \infty)$*, then all positive solutions of* (1.1) *are radially symmetric, that is,* $u(x) = u(r)$*,* $r = |x|$*, and satisfy*

$$\begin{cases} u'' + \frac{n-1}{r} u' + \lambda f(u) = 0, & r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \tag{2.2}$$

Moreover, $u'(r) < 0$ *for all* $r \in (0, 1]$*, and hence* $u(0) = \max_{0 \le r \le 1} u(r)$*.*

(2) *If* u *is a positive solution to* (1.1)*, and* w *is a solution of the linearized problem* (*if it exists*)

$$\begin{cases} \Delta w + \lambda f'(u) w = 0, & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{2.3}$$

then w *is also radially symmetric and satisfies*

$$\begin{cases} w'' + \frac{n-1}{r} w' + \lambda f'(u) w = 0, & r \in (0, 1), \\ w'(0) = w(1) = 0. \end{cases} \tag{2.4}$$

The next lemma plays a crucial rule in this paper.

Lemma 2.5 (1) *For any* $d > 0$*, there is at most one* $\lambda_d > 0$ *such that* (1.1) *has a positive solution* $u(\cdot)$ *with* $\lambda = \lambda_d$ *and* $u(0) = d$*.*

(2) *Let* $T = \{d > 0 : (1.1)$ *has a positive solution with* $u(0) = d\}$*. Then* T *is open;* $\lambda(d) = \lambda_d$ *is a well-defined continuous function from* T *to* R^+*.*

Lemma 2.5 is well-known (see for example [4, 11, 18, 19]). A simple proof of the first part of the lemma can be found in [9].

Because of Lemma 2.5, we call $R^+ \times R^+ = \{(\lambda, d) : \lambda > 0,\ d > 0\}$ the phase space, and $\{(\lambda(d), d) : d \in T\}$ the bifurcation diagram. A solution (λ, u) of (1.1) is called a degenerate solution if (2.3) has a nontrivial solution. If (λ^*, u^*) is degenerate, then the solution set of (1.1) near (λ^*, u^*) could be extremely complicated. However, if one can show that any nontrivial solution w of (2.3) does not change sign in B^n, then it is easy to verify that the conditions of Lemma 2.1 are satisfied, and hence, by this theorem, near the degenerate solution (λ^*, u^*), the solutions of (1.1) form a smooth curve which is expressed in the form

$$(\lambda(s), u(s)) = (\lambda^* + \tau(s), u^* + sw + z(s)), \tag{2.5}$$

where $s \mapsto (\tau(s), z(s)) \in R \times Z$ is a smooth function near $s = 0$ with $\tau(0) = \tau'(0) = 0,\ z(0) = z'(0) = 0$, where Z is the complement of span$\{w\}$ in X, and w is the positive solution of (2.3), which is unique if normalized.

From the expression (2.5), we see that the solution curve makes a turn to the left at (λ^*, u^*) if $\tau''(0) < 0$, and it turns to the right if $\tau''(0) > 0$. Substituting the expression (2.5) to (1.1), differentiating (1.1) twice, and evaluating at $s = 0$, we have

$$\begin{aligned} &\Delta u_{ss} + \lambda f'(u) u_{ss} + 2\lambda' f'(u) u_s + \lambda f''(u) u_s^2 + \lambda'' f(u) = 0, \\ &\Delta u_{ss} + \lambda^* f'(u) u_{ss} + \lambda^* f''(u) w^2 + \tau''(0) f(u) = 0. \end{aligned} \tag{2.6}$$

Multiplying (2.6) by w, (2.3) by u_{ss}, subtracting and integrating, we obtain

$$\tau''(0) = -\lambda^* \frac{\int_{B^n} f''(u_0) w^3 \mathrm{d}x}{\int_{B^n} f(u_0) w \mathrm{d}x}. \tag{2.7}$$

The following lemma only needs the assumptions (A1) and (A2).

Lemma 2.6 *Suppose that f satisfies* (A1) *and* (A2), $\Omega = B^n$, *and* (λ^*, u^*) *is a degenerate solution of* (1.1), $u^* \not\equiv 0$, *with w being the corresponding solution of linearized problem* (2.3). *Suppose that $w > 0$ in B^n. Then*

(1) *All solutions of* (1.1) *near (λ^*, u^*) have a form of $(\lambda^* + \tau(s), u^* + sw + z(s))$, with* $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

(2) *The solution curve is C^2 near (λ^*, u^*), and $\tau''(0) > 0$.*

Part 1 of the lemma is just the consequence of Lemma 2.1 and the proof of part 2 can be found in [18] and the result there is more general, so we omit the proof.

Lemma 2.7 *Suppose that f satisfies* (A1)–(A4) *and* (F1) *or* (F2). *If u is a degenerate positive solution of* (1.1) *and w is the corresponding solution of* (2.3), *then w does not change sign in B^n*

The proof of Lemma 2.7 is now becoming standard but rather long and technical. We put the proof, which has some new technique in it, in Section 4 for interested readers.

For f satisfying (A1)–(A3) and (F1) or (F2), we define two numbers λ_0 and λ_∞ as

$$\lambda_0 = \begin{cases} \dfrac{\lambda_1}{f'(0)}, & \text{if } f'(0) > 0, \\ \infty, & \text{if } f'(0) = 0, \end{cases} \tag{2.8}$$

$$\lambda_\infty = \begin{cases} \infty, & \text{if } f'(\infty) \le 0, \\ \dfrac{\lambda_1}{f'(\infty)}, & \text{if } 0 < f'(\infty) < \infty. \end{cases} \tag{2.9}$$

Note that $\lambda_\infty = \infty$ if f satisfies (F1). By Lemmas 2.2 and 2.3, λ_0 and λ_∞ are the bifurcation points of (1.1) when they are finite.

Now we state the exact multiplicity and bifurcation results for the equation (1.1).

Theorem 2.1 *Suppose that $f'(0) > 0$ and f satisfies* (A1)–(A4) *and* (F1) *or* (F2). *Then there exist $0 < \lambda^* < \infty$, $0 < \lambda_0 < \infty$ and $0 < \lambda_\infty \le \infty$ with $\lambda^* < \min\{\lambda_0, \lambda_\infty\}$ such that* (1.1) *has no positive solution for $\lambda < \lambda^*$ or $\lambda \ge \max\{\lambda_0, \lambda_\infty\}$, exactly one positive solution for $\lambda = \lambda^*$ or $\min\{\lambda_0, \lambda_\infty\} \le \lambda < \max\{\lambda_0, \lambda_\infty\}$, exactly two positive solutions for $\lambda^* < \lambda < \min\{\lambda_0, \lambda_\infty\}$. Furthermore, all positive solutions of* (1.1) *lie on a single smooth solution curve in the space $R^+ \times C^2(\overline{B^n})$, which for $\lambda > \lambda^*$ and near λ^* has two branches denoted by u^λ (the upper branch) and u_λ (the lower branch). u^λ continues to the right up to (∞, c) if f satisfies* (F1) *and to (λ_∞, ∞) if f satisfies* (F2); *u_λ continues to the right down to $(\lambda_0, 0)$, where a bifurcation from the trivial solution occurs (see Figure 2).*

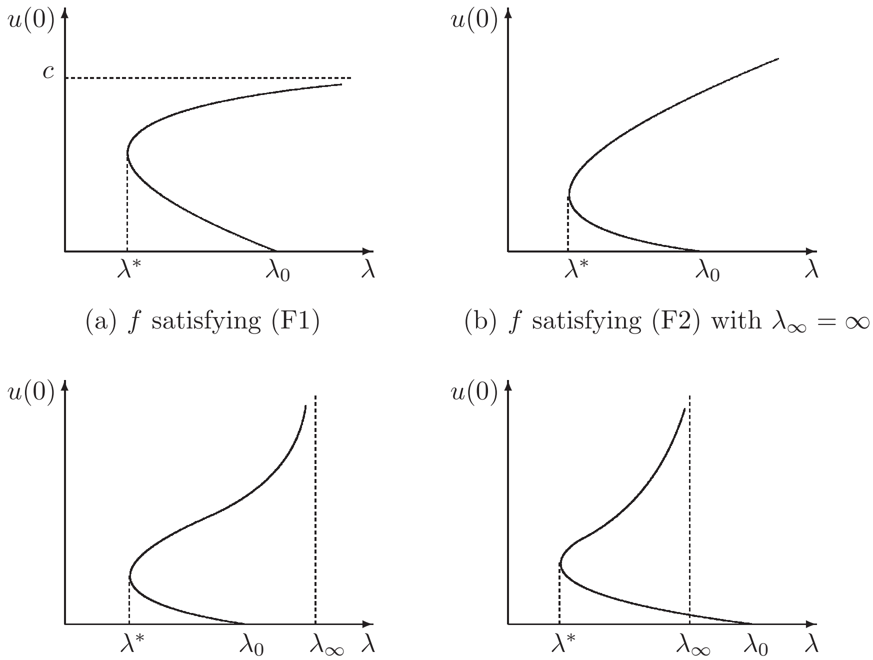
Proof The proof will be a standard bifurcation analysis combined with some techniques used in [8, 9].

Step 1 Claim The problem (1.1) has no positive solution when $\lambda > 0$ is small.

Indeed under our assumptions (A2) and (A3), there is a constant $a > 0$ such that $f(u) \le au$ for all $u > 0$. Then for any solution (λ, u) of (1.1)

$$\lambda_1 \int_{B^n} u^2 \mathrm{d}x \le \int_{B^n} |\nabla u|^2 \mathrm{d}x = \int_{B^n} u\lambda f(u) \mathrm{d}x \le \lambda a \int_{B^n} u^2 \mathrm{d}x,$$

and the claim follows.



(a) f satisfying (F1)

(b) f satisfying (F2) with $\lambda_\infty = \infty$

(c) f satisfying (F2) with $\lambda_0 < \lambda_\infty < \infty$

(d) f satisfying (F2) with $\lambda_\infty < \lambda_0$

Figure 2 Bifurcation Diagram of (1.2) with $f'(0) > 0$

Step 2 Let $(\lambda(s), u(s))$ be the bifurcation curve from the trivial solution $(\lambda_0, 0)$ described in Lemma 2.2. We claim that the bifurcation curve goes to the left near $(\lambda_0, 0)$, that is, $\lambda(s) < \lambda_0$ when $s > 0$ is small.

By (A3), $\frac{f(u)}{u}$ is strictly increasing for $u \in (0, \beta]$, and so $\frac{f(u)}{u} > f'(0)$ for $0 < u \leq \beta$. Since

$$\begin{cases} \Delta u(s) + \lambda(s) f(u(s)) = 0, & \text{in } B^n, \\ u(s) = 0, & \text{on } \partial B^n, \end{cases} \tag{2.10}$$

by the standard regularity theory of elliptic equation, $u(s) \in C^{2,\alpha}(\overline{B^n})$, and $\|u(s)\|_{C^{2,\alpha}(\overline{B^n})} \leq \beta$ for $s > 0$ small. It follows that $\frac{f(u(s))}{u(s)} > f'(0)$ for $s > 0$ small.

Now let φ be the normalized positive eigenfunction corresponding to $\lambda_1 = \lambda_0 f'(0)$. Then

$$\begin{cases} \Delta \varphi + \lambda_0 f'(0) \varphi = 0, & \text{in } B^n, \\ \varphi = 0, & \text{on } \partial B^n. \end{cases} \tag{2.11}$$

By integration to (2.10) and (2.11), we get

$$(\lambda(s) - \lambda_0) f'(0) \int_{B^n} u(s) \varphi \mathrm{d}x + \lambda(s) \int_{B^n} \Big[\frac{f(u(s))}{u(s)} - f'(0) \Big] u(s) \varphi \mathrm{d}x = 0. \tag{2.12}$$

For $s > 0$ small, the second integral is positive, and hence $\lambda(s) < \lambda_0$.

Step 3 Claim If f satisfies (F1), (1.1) has a positive solution for large λ.

Choose $\psi \in C_0^\infty(B^n)$, $\psi \geq 0$, $\psi \not\equiv 0$, and $\|\psi\|_\infty$ small such that the unique solution $\underline{v}$ of the problem

$$\begin{cases} \Delta v + \psi = 0, & \text{in } B^n, \\ v = 0, & \text{on } \partial B^n, \end{cases}$$

satisfies $\|\underline{v}\|_\infty < c$, by standard prior estimate. Since $\underline{v} > 0$ in B^n and ψ has a compact support in B^n, it is easy to see that $\underline{v}$ is a lower solution of (1.1) for large λ. It is also evident that $\overline{v} \equiv c$ is an upper solution of (1.1). It follows from the lower and supper solution method that (1.1) has a positive solution for large λ.

Step 4 Claim (i) If f satisfies (F2) with $f'(\infty) = 0$, then (1.1) has positive solutions for large λ; (ii) If f satisfies (F2) with $f'(\infty) > 0$, then (1.1) has no positive solutions for large λ.

Claim (i) is well-known (see, for example, [1]).

If f satisfies (F2) with $f'(\infty) > 0$, since we also have $f'(0) > 0$, there exists a constant $b > 0$, such that $f(u) \geq bu$ for all $u \geq 0$. Let φ be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$. Then for any solution (λ, u) of (1.1)

$$\lambda_1 \int_{B^n} u\varphi \mathrm{d}x = \int_{B^n} (-\Delta u)\varphi \mathrm{d}x = \int_{B^n} \lambda f(u)\varphi \mathrm{d}x \geq \lambda b \int_{B^n} u\varphi \mathrm{d}x,$$

and the claim (ii) follows.

Step 5 Claim If f satisfies (F2) with $f'(\infty) > 0$, then the bifurcation curve $(\lambda(s), u(s))$ from infinity are on the left of (λ_∞, ∞) when $|\lambda - \lambda_\infty|$ is small.

Let $(\lambda(s), u(s))$ be the bifurcation curve obtained by Lemma 2.3. Similarly to (2.12), we get

$$(\lambda(s) - \lambda_\infty) f'(\infty) \int_{B^n} u(s)\varphi \mathrm{d}x + \lambda(s) \int_{B^n} [f(u(s)) - f'(\infty)u(s)]\varphi \mathrm{d}x = 0. \tag{2.13}$$

Let $g(u) = f(u) - uf'(u)$, and M be a constant such that $M > \max\{\alpha, \beta\}$. By (A2) and (A3), $g(u) > 0$, $f''(u) < 0$ for $u \geq M$, and hence $g'(u) = -uf''(u) > 0$, $\left(\frac{f(u)}{u}\right)' = -\frac{g(u)}{u^2} < 0$ for $u \geq M$. It follows that $g(u) > g(M)$. On the other hand, since $f''(u) < 0$ for $u \geq M$, we have $f'(\infty) = \lim_{u \to \infty} f'(u) < f'(u)$ for $u \geq M$. Hence for $u \geq M$

$$f(u) - f'(\infty)u \geq f(u) - uf'(u) \geq g(M) > 0. \tag{2.14}$$

Since $u(s)(x) \to \infty$ $(s \to \infty)$ almost everywhere in B^n, by the Fatou's Lemma we have

$$\begin{aligned} \liminf_{s \to \infty} \int_{B^n} [f(u(s)) - f'(\infty)u(s)]\varphi \mathrm{d}x &\geq \int_{B^n} \liminf_{s \to \infty} [f(u(s)) - f'(\infty)u(s)]\varphi \mathrm{d}x \\ &\geq g(M) \int_{B^n} \varphi \mathrm{d}x > 0. \end{aligned} \tag{2.15}$$

It follows from (2.13) that $\lambda(s) < \lambda_\infty$ when s is large.

Step 6 By Step 2, the Implicit Function Theorem ensures that we can continue the bifurcation curve originating from $(\lambda_0, 0)$ in the direction of decreasing λ, and we denote the solution by u_λ. Step 1 tells us that the process of continuation towards smaller values of λ for the positive solution curve must stop at some $0 < \lambda^* < \lambda_0$, and there are only three possibilities for λ^*:

(a) $\|u_{\lambda_n}\|_\infty$ goes to infinity for some $\lambda_n \to \lambda^* + 0$;

(b) $\|u_{\lambda_n}\|_\infty$ goes to 0 for some $\lambda_n \to \lambda^* + 0$;

(c) $(\lambda^*, u_{\lambda^*})$ is a degenerate solution.

However, (b) cannot occur by Lemma 2.5. If (a) occurs, then again by Lemma 2.5, all the positive solutions (λ, u) of (1.1) are on the left of $(\lambda_0, 0)$, this is impossible for f satisfying (F1) or (F2) with $f'(\infty) = 0$, by Steps 3 and 4. (Note that (a) can also be easily ruled out by Maximal Principle for f satisfying (F1).) If (a) occurs and f satisfies (F2) with $f'(\infty) > 0$, then denoting $w_n = \frac{u_n}{\|u_n\|_\infty}$, we have

$$\Delta w_n + \lambda_n \frac{f(u_n)}{u_n} w_n = 0.$$

By Sobolev Imbedding Theorems and standard regularity of elliptic equation, it is easy to show that there exists $w \in C^{2,\alpha}(B^n)$, $w > 0$ in B^n, such that

$$\Delta w + \lambda^* f'(\infty) w = 0,$$

which implies that $\lambda^* = \frac{\lambda_1}{f'(\infty)} = \lambda_\infty$. However, by Step 5, all positive solutions (λ, u) are on the left side of (λ_∞, ∞), a contradiction. So in a word, (a) can not occur either. Hence $(\lambda^*, u_{\lambda^*})$ is a degenerate solution.

Step 7 Let $u^* = u_{\lambda^*}$. Assume that w is the solution of corresponding linearized problem

$$\begin{cases} \Delta w + \lambda^* f'(u^*) w = 0, & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n. \end{cases}$$

Then w can be chosen positive in B^n by Lemma 2.7. It follows from Lemma 2.6 that $\tau''(0) > 0$, so the solution curve "turn right" at (λ^*, u^*). Denote the lower branch still by u_λ (with $s < 0$) and upper branch by u^λ (with $s > 0$) respectively for $\lambda > \lambda^*$.

As long as (λ, u^λ) are non-degenerate, the Implicit Function Theorem ensures that we can continue to extend the upper branch u^λ in the direction of increasing λ. We still denote the extensions by u^λ. This process of continuation towards larger values of λ will not encounter any other degenerate solution. This is because, if, say (λ, u^λ) becomes degenerate at $\lambda = \lambda^\#$, then Lemma 2.6 tells us that all the solutions near $(\lambda^\#, u^{\lambda^\#})$ must lie to the right-hand side of it, which is a contradiction.

By Lemma 2.5, we see that the real functions $\lambda \mapsto u^\lambda(0)$ and $\lambda \mapsto u_\lambda(0)$ are strictly increasing and decreasing, respectively, and $u^\lambda(0) > u^*(0) > u_\lambda(0)$.

Step 8 Detailed analysis of the upper branch u^λ.

By the discussion in Step 7, the upper branch is unbounded and there are only two possibilities:

(I) u^λ stops at some finite $\lambda^\# > \lambda^*$ such that $\lim\limits_{\lambda \to \lambda^\# - 0} u^\lambda(0) = \infty$;

(II) u^λ extends to all $\lambda > \lambda^*$.

If (I) occurs, then by Lemma 2.5, all the solutions (λ, u) are contained in the bifurcation curve u_λ and u^λ, and hence λ is bounded. This is impossible for the cases that f satisfies (F1) or (F2) with $f'(\infty) = 0$, by Step 3 and Step 4(i).

Hence, if f satisfies (F1) or (F2) with $f'(\infty) = 0$, then (II) must occur. Since $u^\lambda(0)$ is increasing in λ, we have

$$\lim_{\lambda\to\infty} u^\lambda = \zeta \in (\lambda^*, \infty].$$

We claim that $\zeta = c$ for f satisfying (F1) and $\zeta = \infty$ for f satisfying (F2) with $f'(\infty) = 0$. In fact we will show a little more than that. A similar argument in the proof of Lemma 3.4 in [14] shows that $\frac{\partial u^\lambda(r)}{\partial\lambda} > 0$ for all $r \in [0, 1)$ and $\lambda > \lambda_0$. Hence $\lambda \mapsto u^\lambda(r)$ is strictly increasing and $u^\lambda(r) > u^*(r)$.

If f satisfies (F1), the maximum principle tells us that $u^\lambda(r) \le c$. Hence there is a bounded function $\overline{u}(r) \le c$ such that $\lim\limits_{\lambda\to\infty} u^\lambda(r) = \overline{u}(r)$. Let ϕ be the unique solution of the equation

$$\begin{cases} \Delta\phi + 1 = 0, & \text{in } B^n, \\ \phi = 0, & \text{on } \partial B^n. \end{cases} \tag{2.16}$$

Then we have

$$\int_0^1 u^\lambda(r)\mathrm{d}r = \int_0^1 (-\Delta u^\lambda(r))\phi(r)\mathrm{d}r = \lambda\int_0^1 f(u^\lambda(r))\phi(r)\mathrm{d}r. \tag{2.17}$$

Let $\lambda \to \infty$. Then it follows from (2.17) that $f(\overline{u}(r)) \equiv 0$, $\forall\, r \in [0, 1)$. Hence $\overline{u}(r) \equiv c$, $\forall\, r \in [0, 1)$, and then $\zeta = c$.

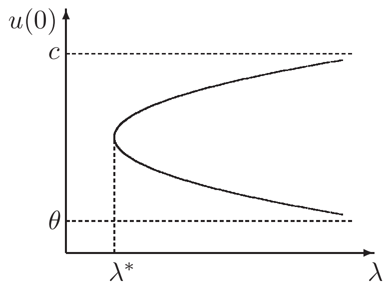
Now suppose that f satisfies (F2) with $f'(\infty) = 0$. If $\zeta < \infty$, by letting $\lim\limits_{\lambda\to\infty} u^\lambda(r) = \overline{u}(r)$, we get (2.17) as the prior paragraph, and then $f(\zeta) = 0$, a contradiction. Hence $\zeta = \infty$ for f satisfying (F2) with $f'(\infty) = 0$.

If f satisfies (F2) with $f'(\infty) > 0$, then by Step 4(ii), (II) can not occur, and hence (I) occurs. A similar argument as in the last part of Step 6 shows that $\lambda^\# = \lambda_\infty$.

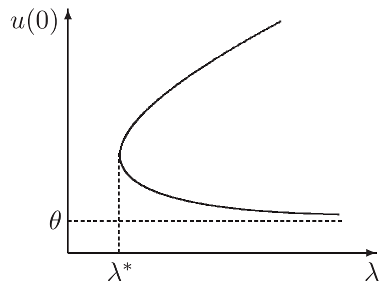
Step 9 Concluding of the proof.

We still need to show that all solutions are contained in the above solution curve. Suppose that there is a positive solution (λ^0, u^0) not lying on the above solution curve. Then by using a similar continuation argument as above, we obtain a second solution curve $(\lambda, \widetilde{u})$ containing (λ^0, u^0). The above argument shows that the curve $(\lambda, \widetilde{u})$ is "$\subset$"-shaped, and on its upper branch, $\widetilde{u}(0) \to \infty$ or c, as $\lambda \to \infty$ or λ_∞. This implies, however, that for any large $C > 0$ (or $C < c$ sufficiently close to c), there are at last two solutions u^λ and $\widetilde{u}$ with $u^\lambda(0) = \widetilde{u}(0) = C$, contradicting Lemma 2.5.

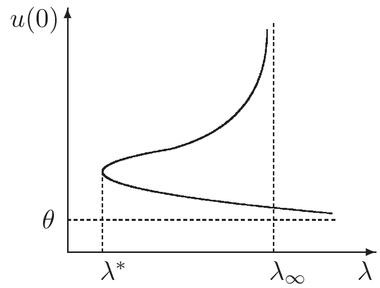
Theorem 2.2 *Suppose that $f'(0) = 0$ and f satisfies* (A1)–(A4) *and* (F1) *or* (F2). *Then there exist $\lambda^* > 0$ and $\lambda_\infty > 0$ such that* (1.1) *has no solution for $\lambda < \lambda^*$, has exactly one solution for $\lambda \ge \lambda_\infty$ (when λ_∞ is finite) or $\lambda = \lambda^*$, has exactly two solutions for $\lambda^* < \lambda < \lambda_\infty$. Furthermore, all positive solutions of* (1.1) *lie on a single smooth solution curve in the space $R^+ \times C^2(\overline{B^n})$, which for $\lambda > \lambda^*$ has two branches denoted by u^λ (the upper branch) and u_λ (the lower branch). u^λ continues to the right up to (∞, c) if f satisfies* (F1) *and to (λ_∞, ∞) if f satisfies* (F2)*; u_λ continues to the right down to (∞, θ), $\theta = 0$ if $n \le 2$ or $n \ge 3$ and $K_f(u) \le \frac{n+2}{n-2}$, $\theta > 0$ if $n \ge 3$ and $K_f(u) > \frac{n+2}{n-2}$ (see Figure* 3).



(a) f satisfyinf (F1)



(b) f satisfying (F2) with $\lambda_\infty = \infty$



(c) f satisfying (F2) with $\lambda_\infty = \infty$

Figure 3 Bifurcation Diagram of (1.1) with $f'(0) = 0$

Proof The proof is just a modification of that of last theorem. Now since $f'(0) = 0$, $\lambda_0 = \infty$, there is no any bifurcation curve start off from the trivial solution, that is to say, Step 2 in the proof of Theorem 2.1 does not hold now. But it is easy to see that all the conclusions in the other steps are still true. By Steps 3–5, there exists a positive solution $(\overline{\lambda}, \overline{u})$. If $(\overline{\lambda}, \overline{u})$ is a non-degenerate solution, then the Implicit Function Theorem implies that there is a solution curve $(\lambda, u(\lambda))$ near $(\overline{\lambda}, \overline{u})$. We continue the solution curve in the direction of decreasing λ, still by using the Implicit Function Theorem. Step 1 tells us that the process of continuation towards smaller values of λ for the positive solution curve must stop at some $0 < \lambda^* < \lambda_0$, and the same argument as in the Step 6 tells us that $(\lambda^*, u(\lambda^*))$ is a degenerate solution. As in the argument in Step 7, the solution curve turns left at $(\lambda^*, u(\lambda^*))$. Denote the lower branch by u_λ and upper branch by u^λ respectively for $\lambda > \lambda^*$. As long as (λ, u_λ) or (λ, u^λ) is non-degenerate, we continue the solution curve in the direction of increasing λ, and the extended branches are still denoted by u_λ and u^λ, respectively. The upper branch's behavior is just the same as in Theorem 2.1 (see Step 8). Now we give a analysis for the lower branch (λ, u_λ).

As we extend the lower branch (λ, u_λ) towards larger value of λ, we will not meet any other degenerate solution. This is because at any degenerate solution (λ, u_λ), the solution curve turns right. That is impossible. By Lemma 2.5, $u_\lambda(0)$ is strictly decreasing for $\lambda > \lambda^*$. So the lower branch may be stopped at some finite $\overline{\lambda}$ only when $u_{\lambda_n}(0) \to 0$ for some $\lambda_n \to \overline{\lambda} - 0$. This cannot occur, as otherwise, denoting $u_n = u_{\lambda_n}$, we have

$$0 = \lambda_1\Big(- \Delta - \frac{\lambda_n f(u_n)}{u_n}\Big) \to \lambda_1(-\Delta - \lambda^* f'(0)) = \lambda_1(-\Delta) > 0,$$

a contradiction. Therefore the lower branch of solutions can extend to $\lambda = \infty$. Let

$$\lim_{\lambda\to\infty} u_\lambda(0) = \theta \in [0, u_0(0)).$$

The conclusion that $\theta = 0$ if $n \le 2$, or $n \ge 3$ and $K_f(u) \le \frac{n+2}{n-2}$, and $\theta > 0$ if $n \ge 3$ and $K_f(u) > \frac{n+2}{n-2}$ follows from the Proposition 6.6 in [19].

The same argument in Step 9 of the proof of last theroem shows that all solutions are contained in the solution curve $\{(\lambda, u^\lambda) \cup (\lambda, u_\lambda) \cup (\lambda^*, u(\lambda^*)) : \lambda > \lambda^*\}$.

3 Proof of Theorem 1.1

Let us first observe the following simple relationship between (1.1) and (1.2).

If (λ, u) is a positive solution of (1.1), and $u(0) > \epsilon > 0$, then we can find a unique $a \in (0, 1)$ such that $u(a) = \epsilon$. Define

$$\nu = \nu(\eta, r) = u(\lambda, ar) - \epsilon. \tag{3.1}$$

Clearly

$$\begin{cases} \nu'' + \frac{n-1}{r}\nu' + \eta f(\nu + \epsilon) = 0, & r \in (0, 1), \\ \nu'(0) = \nu(1) = 0, \end{cases}$$

where $\eta = a^2\lambda$. That is, (η, ν) is a positive solution of (1.2).

This relationship between (1.1) and (1.2) will be frequently used in this section. The following result will play a central role in this section.

Lemma 3.1 *If ν coming from* (3.1) *is a degenerate positive solution of* (1.2) *and w is a nontrivial solution to*

$$\begin{cases} \Delta w + \lambda f'(u + \epsilon)w = 0, & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{3.2}$$

where $\epsilon > 0$, then w does not change sign in B^n.

The proof of Lemma 3.1 is very long and also needs some other knowledge. Hence we put it in Section 5.

Using Lemma 3.1, we obtain a variant of Lemma 2.6, whose obvious proof we omit.

Lemma 3.2 *Suppose that u_0 is a degenerate solution of* (1.2) *with $\lambda = \lambda^*$. Then all positive solutions (λ, u) of* (1.2) *that are near (λ^*, u_0) in $R^+ \times C(\overline{B^n})$ lie on a smooth curve represented by*

$$(\lambda, u) = (\lambda^* + \tau(s), u_0 + sw + z(s)) \quad \textit{with } s \textit{ small},$$

where $z(0) = z'(0) = 0$, $\tau(0) = \tau'(0) = 0$, and w is the positive eigenfunction given in Lemma 3.1*. Moreover,*

$$\tau''(0) = -\lambda^* \frac{\int_{B^n} f''(u_0 + \epsilon)w^3 \mathrm{d}x}{\int_{B^n} f(u_0 + \epsilon)w \mathrm{d}x}. \tag{3.3}$$

Next we will give the proof of Theorem 1.1.

Proof of Theorem 1.1 By Theorem 2.1 and Theorem 2.2, the solution curve of (1.1) is " $\subset$ "-shaped with exactly one turning point at (λ^*, u_0), where $u_0 = u_{\lambda^*} = u^{\lambda^*}$. Denote $\xi_0 = u_0(0)$ and denote the solution curve of (1.2) by $\Gamma(\epsilon)$.

Step 1 About $\Gamma(\epsilon)$.

Since $\epsilon > 0$ and sufficiently small, we have $\epsilon \in (0, \xi_0)$. Then for any $\lambda \geq \lambda^*$, we can find $a^\lambda \in (0,1)$ such that $u^\lambda(a^\lambda) = \epsilon$. Moreover, $\lambda \mapsto a^\lambda$ is strictly increasing and

$$\lim_{\lambda\to\infty} a^\lambda = 1.$$

As before, define

$$\eta^\lambda = (a^\lambda)^2\lambda$$

and

$$\nu^\lambda = \nu^\lambda(\eta, r) = u^\lambda(a^\lambda r) - \epsilon, \quad r \in (0,1).$$

Then

$$\Gamma^\epsilon = \{(\eta^\lambda, \nu^\lambda) : \lambda^* \leq \lambda < p\},$$

where $\lambda^* < p \leq \infty$ (for convenience, we give a constant p specified later since f satisfies different conditions), gives a smooth solution curve of (1.2). Since a^λ is increasing with λ, it follows that η^λ is strictly increasing with λ. Therefore, Γ^ϵ is a monotone curve connecting $(\eta^{\lambda^*}, \nu^{\lambda^*})$ and infinity; this will be explained in detail in the next step. Furthermore, since $\epsilon < \xi_0$, we can find a unique $\lambda_\epsilon > \lambda^*$ such that

$$u_{\lambda_\epsilon}(0) = \epsilon.$$

By Theorems 2.1 and 2.2, we see that λ_ϵ increases as ϵ decreases and $\lambda_\epsilon \to \infty$ as $\epsilon \to 0$.

For any $\lambda \in [\lambda^*, \lambda_\epsilon)$, we can find a unique $a_\lambda \in (0,1)$ such that

$$u_\lambda(a_\lambda) = \epsilon.$$

Clearly, for any fixed $\lambda \geq \lambda^*$,

$$\lim_{\epsilon\to 0} a_\lambda(\epsilon) = 1.$$

Now we define

$$\eta_\lambda = (a_\lambda)^2\lambda$$

and

$$\nu_\lambda = \nu_\lambda(\eta, r) = u_\lambda(a_\lambda r) - \epsilon, \quad r \in (0,1),$$

and find that

$$\Gamma_\epsilon = \{(\eta_\lambda, \nu_\lambda) : \lambda^* \le \lambda < \lambda_\epsilon\}$$

gives another piece of smooth solution curve to (1.2). Moreover, Γ_ϵ connects the end point $(\eta^{\lambda^*}, \nu^{\lambda^*})$ of Γ^ϵ (when $\lambda = \lambda^*$) and $(0, 0)$ (when $\lambda \to \lambda_\epsilon - 0$). Thus

$$\Gamma(\epsilon) = \Gamma^\epsilon \cup \Gamma_\epsilon$$

gives a smooth curve for (1.2) connecting $(0, 0)$ and infinity. By Lemma 2.5, we know that it contains all the positive solutions of (1.2).

Step 2 About Γ^ϵ.

If $f(u)$ satisfies (F1), then

$$\Gamma^\epsilon = \{(\eta^\lambda, \nu^\lambda) : \lambda^* \le \lambda < \infty\}$$

gives a smooth solution curve of (1.2) and connects $(\eta^{\lambda^*}, \nu^{\lambda^*})$ (when $\lambda = \lambda^*$) and $(\infty, c - \epsilon)$ (when $\lambda \to \infty$).

If $f(u)$ satisfies (F2), then

$$\Gamma^\epsilon = \{(\eta^\lambda, \nu^\lambda) : \lambda^* \le \lambda < \lambda_\infty\}$$

gives a smooth solution curve of (1.2) and connects $(\eta^{\lambda^*}, \nu^{\lambda^*})$ (when $\lambda = \lambda^*$) and (λ_∞, ∞) (when $\lambda \to \lambda_\infty$).

Step 3 About Γ_ϵ.

By (A2), we have known that $f''(u) > 0$ for $u \in (0, \alpha)$. We fix some $\xi_1 \in (0, \alpha)$ and suppose

$$\epsilon < \epsilon_1 \equiv \alpha - \xi_1.$$

Then clearly $f''(u + \epsilon) > 0$ for $u \in (0, \xi_1)$.

Now we choose $\lambda_{\xi_1} > \lambda^*$ such that

$$u_\lambda(0) < \xi_1, \quad \text{when } \lambda \ge \lambda_{\xi_1}.$$

By shrinking ϵ_1 we may assume that $\lambda_{\xi_1} < \lambda_\epsilon$ for any $\epsilon \in (0, \epsilon_1)$. We can now divide Γ_ϵ into two parts:

$$\Gamma_\epsilon^1 = \{(\eta_\lambda, \nu_\lambda) : \lambda_{\xi_1} \le \lambda < \lambda_\epsilon\}, \quad \Gamma_\epsilon^2 = \{(\eta_\lambda, \nu_\lambda) : \lambda^* \le \lambda \le \lambda_{\xi_1}\}.$$

We first analyze the shape of Γ_ϵ^1. Define

$$\Lambda_\epsilon^* = \sup_{\lambda \in [\lambda_{\xi_1}, \lambda_\epsilon)} \eta_\lambda.$$

One easily shows that there exists $\epsilon_2 \in (0, \epsilon_1]$ such that when $\epsilon \in (0, \epsilon_2)$,

$$\Lambda_\epsilon^* \text{ is achieved at some } \lambda_* \in (\lambda_{\xi_1}, \lambda_\epsilon) \text{ and } \lim_{\epsilon \to \infty} \Lambda_\epsilon^* = \infty.$$

By the Implicit Function Theorem, $(\eta_{\lambda_*}, \nu_{\lambda_*})$ must be a degenerate solution of (1.2). Then by Lemma 3.2, (3.3), and our choice of ξ_1, the solution of (1.2) near $(\eta_{\lambda_*}, \nu_{\lambda_*})$ turns to the left.

Therefore, we have an upper branch and a lower branch of positive solutions starting from this point, and both branches can be continued towards smaller values of λ. The lower branch can be continued to reach $(0,0)$, because (1) we cannot meet a degenerate solution in the way of continuation due to Lemma 3.2 and $u(0) < \xi_1$ on Γ_ϵ^1, and (2) the branch goes along Γ_ϵ^1. For the same reason, the upper branch can be continued until it reaches $(\eta_{\lambda_{\xi_1}}, \nu_{\lambda_{\xi_1}})$. This implies that Γ_ϵ^1 is exactly "$\supset$"-shaped.

Next we analyze the shape of Γ_ϵ^2. It is more convenient for our discussion if we consider a bigger piece of solution curve

$$\Gamma_\epsilon^3 = \Gamma_\epsilon^2 \cup \{(\eta^\lambda, \nu^\lambda) : \lambda^* \le \lambda \le \lambda_{\xi_1}\},$$

which contains part of Γ^ϵ. We observe that any $(\lambda, u) \in \Gamma_\epsilon^3$ satisfies

$$0 < \lambda_\epsilon^* \le \lambda \le \lambda_{\xi_1}, \quad u_{\lambda_{\xi_1}}(0) - \epsilon \le \|u\|_\infty = u(0) \le u^{\lambda_{\xi_1}}(0) - \epsilon, \tag{3.4}$$

where

$$\lambda_\epsilon^* = \inf\{\lambda : (\lambda, u) \in \Gamma_\epsilon^3\}.$$

It is easy to find that λ_ϵ^* is achieved at some $\eta_{\lambda'}$, $\lambda' \in [\lambda^*, \lambda_{\xi_1})$. Therefore $(\lambda_\epsilon^*, v_{\lambda'})$ must be a degenerate solution of (1.2). Clearly

$$\lambda_\epsilon^* \le \eta_{\lambda^*} = (a_{\lambda^*}(\epsilon))^2 \lambda^* < \lambda^*.$$

On the other hand, it is easy to see that $a_\lambda(\epsilon) \to 1$ as $\epsilon \to 0$ uniformly for $\lambda \in [\lambda^*, \lambda_{\xi_1}]$. Hence

$$\lim_{\epsilon \to 0} \lambda_\epsilon^* = \lim_{\epsilon \to 0} \min\{(a_\lambda(\epsilon))^2 \lambda : \lambda^* \le \lambda_{\xi_1}\} = \lambda^*.$$

We know from the discussion above that Γ_ϵ^3 contains at least one degenerate solution $(\lambda^*, \nu_{\lambda'})$. If we can show that there exists $\epsilon_3 \in (0, \epsilon_2)$ such that whenever $\epsilon \in (0, \epsilon_3)$, any degenerate solution on Γ_ϵ^3 must make $\tau''(0) > 0$ in (3.3) of Lemma 3.2, then a continuation argument as before shows that Γ_ϵ^3 contains exactly one degenerate solution at $\lambda = \lambda_\epsilon^*$ and the curve makes a turn to right at this point. Hence Γ_ϵ^3 must be "$\subset$"-shaped. This tells us that the entire solution curve $\Gamma(\epsilon)$ is exactly S-shaped with two turning points at $\lambda = \lambda_\epsilon^*$ and $\lambda = \Lambda_\epsilon^*$, respectively. Clearly, this would finish the proof of Theorem 1.1.

It remains to show that there exists $\epsilon_3 \in (0, \epsilon_2)$ such that any degenerate solution on Γ_ϵ^3 must make $\tau''(0) > 0$ in (3.3) of Lemma 3.2 as long as $\epsilon \in (0, \epsilon_3)$. We argue indirectly. Suppose that for some $\epsilon_k \to 0$, we can find a degenerate solution $(\lambda^k, \lambda^k) \in \Gamma_{\epsilon_k}^3$ such that

$$\tau_k''(0) = -\lambda^k \frac{\int_{B^n} f''(u^k + \epsilon_k) w_k^3 \mathrm{d}x}{\int_{B^n} f(u^k + \epsilon_k) w_k \mathrm{d}x} \le 0,$$

where w_k is the positive eigenfunction given in Lemma 3.1 when $(\lambda, u) = (\lambda^k, u^k)$. We may assume that $\|w_k\|_\infty = 1$.

By (3.4), we may assume that $\lambda^k \to \lambda^0 \in [\lambda^*, \lambda_{\xi_1}]$. The second part of (3.4) implies that $\|f(u^k + \epsilon_k)\|_\infty$ is uniformly bounded. Therefore, by the equation for u^k and a standard regularity and compactness argument, u^k has a convergent subsequence in C^1. We may assume $u^k \to u^0$ in C^1. Moreover, from

$$\begin{cases} \Delta w_k + \lambda^k f'(u^k + \epsilon_k) w_k = 0, & \text{in } B^n, \\ w_k = 0, & \text{on } \partial B^n, \end{cases}$$

we can use a similar regularity and compactness argument to obtain a C^1 convergent subsequence of w_k. We may assume $w_k \to w^0$. Then we easily deduce

$$\begin{cases} \Delta u^0 + \lambda^0 f(u^0) = 0, & \text{in } B^n, \\ u^0 \geq 0,\ u^0 \neq 0, & \text{in } B^n, \\ u^0 = 0, & \text{on } \partial B^n, \end{cases}$$

and

$$\begin{cases} \Delta w^0 + \lambda^0 f'(u^0) w^0 = 0, & \text{in } B^n, \\ w^0 \geq 0,\ \|w^0\|_\infty = 1, & \text{in } B^n, \\ w^0 = 0, & \text{on } \partial B^n. \end{cases}$$

This implies that (λ^0, u^0) is a degenerate positive solution of (1.1) and w^0 is the corresponding positive eigenfunction. By Theorem 2.1 and Theorem 2.2, (1.1) has a unique degenerate positive solution which is (λ^*, u_0), and by Lemma 2.6 and (2.7),

$$\tau''(0) = -\lambda^* \frac{\int_{B^n} f''(u_0) w^3 \mathrm{d}x}{\int_{B^n} f(u_0) w \mathrm{d}x} > 0.$$

Therefore, we must have $\lambda^k \to \lambda^*$, $u^k \to u_0$ and $w^0 = w$ (note that positive eigenfunction is unique if it is normalized). Then we deduce, however,

$$0 \geq \tau_k''(0) = -\lambda^k \frac{\int_{B^n} f''(u^k + \epsilon_k) w_k^3 \mathrm{d}x}{\int_{B^n} f(u^k + \epsilon_k) w_k \mathrm{d}x} \to \tau''(0) = -\lambda^* \frac{\int_{B^n} f''(u_0) w^3 \mathrm{d}x}{\int_{B^n} f(u_0) w \mathrm{d}x} > 0.$$

This contradiction finishes our proof.

4 Proof of Lemma 2.7

Suppose that u is a degenerate positive solution of (1.1) and w is a nontrivial solution to the linearized equation (2.3). By Lemma 2.4, u and w are radially symmetric on Ω and satisfy (2.2) and (2.4). We rewrite (2.2) and (2.4) in the form

$$\begin{cases} (r^{n-1} u')' + \lambda r^{n-1} f(u) = 0, & r \in (0,1), \\ u'(0) = u(1) = 0; \end{cases} \tag{4.1}$$

$$\begin{cases} (r^{n-1} w')' + \lambda r^{n-1} f'(u) w = 0, & r \in (0,1), \\ w'(0) = w(1) = 0. \end{cases} \tag{4.2}$$

By the Harnack inequality (or by the well-known uniqueness result for the second order differential equation), $w(0) \neq 0$.

Proof of Lemma 2.7 Without loss of generality, we assume that $w(0) > 0$. If $u(0) > \beta$, since $u(r)$ is strictly decreasing for r in $[0,1]$, there exists uniquely $0 < r_1 < r_2 < 1$ such that

$u(r_1) = \beta$, $u(r_2) = \rho$ (remember that $0 < \rho < \alpha < \beta$, see Remark 1.1). If $u(0) \le \beta$, then we let $r_1 = 0$, and if $u(0) \le \rho$, then we let $r_1 = r_2 = 0$. For clearness, we divide the proof into several steps in the following.

Step 1 Firstly, we show that w has no zeros on $[0, r_2]$.

Let $\phi(r) = u(r) - \rho$. Then

$$(r^{n-1}\phi')' + \lambda r^{n-1} f'(u)\phi = \lambda r^{n-1}[f'(u(r))(u(r) - \rho) - f(u(r))]. \tag{4.3}$$

Let $p(u) = (u - \rho)f'(u) - f(u)$. Then we have

$$p'(u) = (u - \rho)f''(u),$$

which is negative on $(0, \rho)$, positive on $[\rho, \alpha]$, and negative on (α, ∞). By the definition of ρ, $p(\alpha) = f'(\alpha)(\alpha - \rho) - f(\alpha) = 0$, and $p(0) = -\rho f'(0) \le 0$, so $p(u) \le 0$ for all $u \ge 0$.

It follows from (4.2) and (4.3) that

$$[r^{n-1}(\phi' w - \phi w')]' = \lambda r^{n-1}[f'(u(r))(u(r) - \rho) - f(u(r))]w. \tag{4.4}$$

If $w(r)$ has a zero in $[0, r_2]$, then we can find $r_0 \in (0, r_2]$, such that $w(r) > 0$ on $[0, r_2)$ and $w(r_0) = 0$. Integrating (4.4) over the interval $[0, r_0]$, we get

$$0 < -r_0^{n-1}(u(r_0) - \rho)w'(r_0) = \lambda \int_0^{r_0} r^{n-1} p(u(r)) w(r) \mathrm{d}\, r < 0. \tag{4.5}$$

This contradiction shows that w has no zero in $[0, r_2]$.

In the following we show that w has no zeros in $[r_2, 1)$. To do this, we use the test function

$$v(r) = r u_r(r) + \mu u(r), \tag{4.6}$$

where $\mu > 0$ is a constant to be specified later. It is easy to verify that

$$(r^{n-1}v')' + \lambda r^{n-1} f'(u)v = \lambda r^{n-1}[\mu(f'(u)u - f(u)) - 2f(u)] = \lambda r^{n-1} g(u), \tag{4.7}$$

where $g(u) = \mu(f'(u)u - f(u)) - 2f(u)$.

Define

$$h(r) = -\frac{r u'(r)}{u(r)}. \tag{4.8}$$

Then

$$\begin{aligned} h'(r) &= \frac{(n-2)u u_r + r u_r^2 + \lambda f r u}{u^2} \\ &= \frac{2H(r) - 2\lambda r F + \lambda f r u}{u^2} = \frac{2H(r)}{u^2} + \lambda r \frac{f u - 2F}{u^2}, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} H(r) &= \frac{1}{2}[r u_r^2(r) + (n-2) u_r(r) u(r)] + \lambda F(u(r)), \\ F(u) &= \int_0^u f(s) \mathrm{d}\, s. \end{aligned} \tag{4.10}$$

Step 2 $K_f(u)$ is strictly decreasing in (α,β).

In fact, by a simple computation, we obtain

$$K_f'(u)=\frac{f''(u)f(u)u-f'(u)[f'(u)u-f(u)]}{f^2(u)}<0.$$

By (A2), $f''(u)\le 0$ in (α,β), by (A3) $f'(\beta)=\frac{f(\beta)}{\beta}>0$, and then $f'(u)>0$ in (α,β). By (A3) again, $f'(u)u-f(u)>0$ in (α,β). Hence $K_f'(u)<0$ in (α,β).

Step 3 $H(r)>0$ for all $r\in(r_2,1]$.

We will show that $J(r)=r^{n-1}H(r)>0$ in $(r_2,1]$. When $n=1$ or 2, it is true from (4.10), so we may assume that $n\ge 3$. Note that $J(0)=0$, $J(1)=H(1)=\frac{1}{2}[u'(1)]^2>0$. It is easy to verify that

$$J'(r)=\lambda r^{n-1}G(u(r)),\tag{4.11}$$

where $G(u)=nF(u)-\frac{n-2}{2}\cdot uf(u)$. We study the property of $G(u)$. The function $G(u)$ satisfies

$$G'(u)=\frac{n+2}{2}f-\frac{n-2}{2}f'u,\tag{4.12}$$

$$G''(u)=2f'-\frac{n-2}{2}f''u.\tag{4.13}$$

To prove $J(r)\ge 0$ on $[r_2,1]$, we consider three cases.

Case 1 $K_f(\rho)\ge\frac{n+2}{n-2}$.

By (A4), $K_f(u)\ge K_f(\rho)\ge\frac{n+2}{n-2}$ for all $u\in(0,\rho)$. Thus

$$G'(u)=\frac{n+2}{2}f-\frac{n-2}{2}f'u=\frac{n-2}{2}f\Big[\frac{n+2}{n-2}-K_f(u)\Big]\le 0$$

for all $u\in(0,\rho)$. Since $G(0)=0$, we have $G(u)\le 0$ in $(0,\rho)$, which implies that $J'(r)<0$ on $[r_2,1]$, and since $J(1)>0$, we have $J(r)>0$ on $[r_2,1]$.

Case 2 $K_f(\rho)<\frac{n+2}{n-2}$ and $\lim\limits_{u\to+0}K_f(u)\le\frac{n+2}{n-2}$.

By (A4) and Step 2, $K_f(u)\le\frac{n+2}{n-2}$ in $(0,\beta]$, which implies that $G'(u)>0$ in $(0,\beta]$. We claim that $G'(u)>0$ for $u>\beta$. In fact, if $f(u)$ satisfies (F1), then there exists $\eta>\beta$ such that $f'(\eta)=0$. Then for $u\in(\beta,\eta)$, $f'>0$ and $f''<0$ by (A2), so we have

$$G''(u)=2f'-\frac{n-2}{2}f''u>0,$$

and by (A3)

$$\begin{aligned}G'(\beta)&=\frac{n+2}{2}f(\beta)-\frac{n-2}{2}f'(\beta)\beta\\&=\frac{n-2}{2}[f(\beta)-f'(\beta)\beta]+2f(\beta)\\&=2f(\beta)>0,\end{aligned}$$

so $G'(u)>0$. And for $u\in(\eta,c)$, $f'(u)<0$, $f(u)>0$, so

$$G'(u)=\frac{n+2}{2}f-\frac{n-2}{2}f'u>0.$$

On the other hand, if $f(u)$ satisfies (F2), then $f'(u) > 0$ for $u > \beta$, then $G''(u) > 0$ for $u > \beta$. Since $G'(\beta) > 0$, we have $G'(u) > 0$ for any $u > \beta$. Then it follows from (4.11) that $J'(r) > 0$ for $r \in (0,1)$. Since $J(0) = 0$, we have $J(r) > 0$ in $(0,1]$ and hence $H(r) > 0$ in $(0,1]$. In particular, $H(r) \geq 0$ in $[r_2, 1]$.

Case 3 $K_f(\rho) < \frac{n+2}{n-2}$ and $\lim\limits_{u\to+0} K_f(u) > \frac{n+2}{n-2}$.

By (A4), $K_f(u)$ is non-increasing in $(0,\rho)$, so there exists $d \in (0,\rho)$ such that $K_f(d) = \frac{n+2}{n-2}$, $K_f(u) > \frac{n+2}{n-2}$ for $u \in (0,d)$ and $K_f(u) \leq \frac{n+2}{n-2}$ for $u \in (d,\rho)$, and then $G'(u) < 0$ in $(0,d]$, $G'(u) > 0$ in $(d,\rho]$. Since $G(0) = 0$, we have $G(u) < 0$ in $(0,d]$.

By (A4) again, $K_f(u) \leq K_f(\rho) < \frac{n+2}{n-2}$ for $u \in (\rho,\alpha)$. So

$$G'(u) = \frac{n+2}{2} f - \frac{n-2}{2} f' u = \frac{n-2}{2} f \Big[\frac{n+2}{n-2} - K_f(u)\Big] > 0$$

in (ρ,α). By Step 2, $K_f(u) < K_f(\alpha) < \frac{n+2}{n-2}$ for $u \in (\alpha,\beta)$. So $G'(u) > 0$ in $(\alpha,\beta]$. The same argument as in the proof of Case 2 ensures that $G'(u) > 0$ for $u > \beta$. Furthermore, by the definition of G, $G(c) > 0$ if f satisfies (F1), and by the proof in Case 2, $G(u) > 0$ for sufficiently large u.

Then there exists a point $p > d$ such that $G(p) = 0$. We claim that $u(0) > p$ for any solution of (2.2). If not, then for $r \in [0,1]$, $0 \leq u(r) \leq p$, so $J'(r) = \lambda r^{n-1} G(u(r)) < 0$ for $r \in (0,1)$. We have known that $J(1) = H(1) = (\frac{1}{2}) u_r^2(1) \geq 0$, but $J(1) = \int_0^1 J'(r)\mathrm{d}r = \lambda \int_0^1 r^{n-1} G(u(r))\mathrm{d}r < 0$, that is a contradiction, so $u(0) > p$.

Now since $u(0) > p$, $u(1) = 0$ and $u_r < 0$, there exists $r_0 \in (0,1)$ such that $u(r_0) = p$. For $r \in [0, r_0]$, $G(u(r)) \leq 0$, so $J'(r) > 0$ and $J(u) \geq 0$ since $J(0) = 0$. For $r \in [r_0, 1]$, $G(u(r)) \leq 0$, so $J'(u) \leq 0$; since $J(1) \geq 0$, we have $J(r) \geq 0$, and hence $H(r) > 0$ in $[0,1]$. In particular, $H(r) \geq 0$ in $[r_2, 1]$.

Step 4 $h'(r) > 0$ in $(r_2, 1)$ and $\lim\limits_{r\to1^-} h(r) = +\infty$.

Let $Q(u) = uf(u) - 2F(u)$. Then by (A3), $Q'(u) = uf'(u) - f(u) > 0$ in $(0,\beta)$. Since $Q(0) = 0$, we have $Q(u(r)) > 0$ in $(r_1, 1)$. Then by Step 2, we have $h'(r) > 0$ in $(r_2, 1)$. By the definition of $h(r)$, we have

$$\lim_{r\to1^-} h(r) = \lim_{r\to1^-} -\frac{ru'(r)}{u(r)} = +\infty.$$

Step 5 We conclude the proof by proving that w has no zeros in $[r_2, 1)$.

Define

$$\mu(r) = \frac{2f(u(r))}{f'(u(r))u(r) - f(u(r))} = \frac{2}{K_f(u(r)) - 1}. \tag{4.14}$$

Then $\mu(r)$ is decreasing in $(r_2, 1)$ by (A4). By Step 4, $h(r)$ is strictly increasing for r in $(r_2, 1)$ and $\lim\limits_{r\to1^-} h(r) = +\infty$. So there are only two possibilities.

(1) There exists a unique $r^* \in (r_2, 1)$ such that $h(r^*) = \mu(r^*) = \mu^*$.

In this case, we take the test function

$$v(r) = ru_r + \mu^* u = [\mu^* - h(r)]u(r).$$

It is easy to see that $v(r) > 0$ in (r_2, r^*) and $v(r) < 0$ in $(r^*, 1)$.

By the definition of g,

$$g(u(r)) = (\mu^* - \mu(r))(f'(u(r))u - f(u(r))),$$

it follows that $g(u(r)) < 0$ in $(0, r^*)$ and $g(u(r)) > 0$ in $(r^*, 1)$. From (4.2) and (4.7), we obtain

$$[r^{n-1}(v'w - vw')]' = \lambda r^{n-1} g(u) w. \tag{4.15}$$

By Step 1, w has no zeros in $[0, r_2]$. If w has a zero in $(r_2, r^*]$, then there exists $r_0 \in (r_2, r^*)$, such that $w(r_0) = 0$, and $w(r) > 0$ in $[0, r_0)$. Integrating (4.15) over the interval $[0, r_0]$ yields a contradiction

$$0 < -r_0^{n-1} w'(r_0) v(r_0) = \int_0^{r_0} \lambda r^{n-1} g w \mathrm{d} r < 0.$$

If w has zeros in $[r^*, 1)$, we may assume that r_0 is the biggest zero of $w(r)$ in $[r^*, 1)$ and $w > 0$ in $[r_0, 1)$. Integrating (4.15) over the interval $[r_0, 1]$ yields another contradiction

$$0 > -w'(1)v(1) + r_0^{n-1} w'(r_0) v(r_0) = \int_{r_0}^{1} \lambda r^{n-1} g w \mathrm{d} r > 0.$$

So w has no zero in $[r^*, 1)$.

(2) For all $r \in (r_2, 1)$, $\mu(r) < h(r)$.

In this case, we choose $\mu^* = h(r_2)$ and $v = r u_r + \mu^* u$. Then on $[r_2, 1]$, $g(u(r)) > 0$, and $v(r) < 0$. The same technique in prior (1) implies that w has no zero in $[r_2, 1)$.

5 Proof of Lemma 3.1

First, we should introduce the Morse index. We define the Morse index $M(u)$ of a solution (λ, u) to be the number of negative eigenvalues of the following eigenvalue problem

$$\begin{cases} (r^{n-1}\phi')' + \lambda r^{n-1} f'(u)\phi = -\mu\phi, \quad r \in (0,1), \\ \phi'(0) = \phi(1) = 0. \end{cases} \tag{5.1}$$

It is well-known that the eigenvalues $\mu_1, \mu_2, \cdots$ of (5.1) are all simple, and the eigenfunction ϕ_i corresponding to μ_i has exactly $i-1$ simple zeros in $(0,1)$ for $i \in N$.

To consider the Morse indices of the solution, we introduce an auxiliary equation

$$\begin{cases} (r^{n-1}w')' + \lambda r^{n-1} f'(u) w = 0, \quad r \in (0,1), \\ w'(0) = 0, \ w(0) = 1, \end{cases} \tag{5.2}$$

where u is a solution to (1.1). Let $w(\lambda, \cdot)$ be the solution of (5.2). Then $w(\lambda, \cdot)$ has the following relation with the Morse index of u (see Lemma 5.2 in [19]):

Lemma 5.1 *Suppose that u is a solution of* (1.1)*, and $w(\lambda, \cdot)$ is the solution of* (5.2)*. Then $M(u) = k$ if and only if $w(\lambda, \cdot)$ has exactly k zeros in* $(0,1)$.

Next we will give the proof of Lemma 3.1.

Proof of Lemma 3.1 From (3.1), we know that $\nu = \nu(\eta, r) = u(\lambda, ar) - \epsilon$ with corresponding $\eta = a^2\lambda$, where $u(\lambda, \cdot)$ is a degenerate solution of (1.1). First, we claim that $M(\nu(\eta, \cdot)) \le M(u(\lambda, \cdot))$. By Lemma 5.1, the Morse index of a radial solution u to (2.4) is the number of zeros of the solution to (5.2). Let $\phi(\lambda, r)$ be the solution of (5.2) associated with $u = u(\lambda, \cdot)$ and corresponding λ. Define

$$\psi(\eta, r) = \phi(\lambda, ar).$$

Then ψ is the solution of (5.2) associated with $u = u(\lambda, \cdot) + \epsilon$ and $\lambda = \eta$. In particular, the number of zeros of $\psi(\eta, \cdot)$ in $(0, 1)$ is the number of zeros of $\phi(\lambda, \cdot)$ in $(0, a) \subset (0, 1)$. Thus from Lemma 5.1, $M(\nu(\eta, \cdot)) \le M(u(\lambda, \cdot))$, so the claim is true.

From Lemma 2.7, the Morse index of $u(\lambda, \cdot)$ is either 0 or 1. Thus $M(\nu(\eta, \cdot)) \le M(u(\lambda, \cdot)) \le 1$. If $M(\nu(\eta, \cdot)) = 1$, then $\psi(\eta, \cdot)$ has at least one zero in $(0, 1)$ and another zero at $r = 1$, where $\psi(\lambda, \cdot)$ is the solution of (5.2) associated with $\nu(\eta, \cdot)$. But $a < 1$, so the solution $\phi(\lambda, \cdot)$ of (5.2) associated with $u(\lambda, \cdot)$ has at least two zeros in $(0, 1)$, which implies that $M(u(\lambda, \cdot)) \ge 2$, that is a contradiction. Hence $M(\nu(\eta, \cdot)) = 0$ and $w(r)$ does not change sign in $(0, 1)$, i.e., w does not change sign in B^n.

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