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Jump Type Cahn-Hilliard Equations with Fractional Noises****

Lijun BO* Kehua SHI** Yongjin WANG***

Abstract The authors explore a class of jump type Cahn-Hilliard equations with fractional noises. The jump component is described by a (pure jump) Lévy space-time white noise. A fixed point scheme is used to investigate the existence of a unique local mild solution under some appropriate assumptions on coefficients.

Keywords Cahn-Hilliard equations, Fractional noises, Lévy space-time white noise, Local mild solution
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1 Introduction

It is well-known that a classical model for the process of the spinodal decomposition can be described by a Cahn-Hilliard equation on the domain $[0,T]\times[0,\pi]^d$ (see [7, 16] and their references therein) as follows:

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0, \tag{1.1}$$

which describes the complicated phase separation and coarsening phenomena in a melted alloy. Here the mapping f is the derivative of the homogeneous free energy $\widetilde{F}$, which contains a logarithmic term. In some cases, $\widetilde{F}$ can be approximated by an even-degree polynomial with positive dominant coefficient. A standard choice for f is a cubic polynomial such as $f(u)=u-u^3$.

This paper deals with the following jump type Cahn-Hilliard equations with fractional noise potentials:

$$\begin{cases}\Box u(t,x)=\Delta b(u(t,x))+\dot{B}^H(x,t)+a(u(t,x))\dot{F}(x,t), & \text{in } [0,T]\times D,\\ u(0)=\psi, & \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial\Delta u}{\partial n}=0, & \text{on } [0,T]\times\partial D,\end{cases} \tag{1.2}$$

where the operator $\Box:=\frac{\partial}{\partial t}+\Delta^2$ with the Laplace operator Δ, and the domain $D=[0,\pi]^d$. In addition, $\dot{B}^H$ denotes a fractional noise on $D\times[0,\infty)$ with Hurst parameter $H>\frac12$, and $\dot{F}$ is

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*Department of Mathematics, Xidian University, Xi'an 710071, China. E-mail: bolijunnk@yahoo.com.cn
**Corresponding author. School of Mathematical Sciences, Nankai University, Tianjin 300071, China. E-mail: kehuashink@gmail.com
***School of Mathematical Sciences, Nankai University, Tianjin 300071, China. E-mail: yjwang@nankai.edu.cn
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a (pure jump) Lévy space-time white noise on $D \times [0, \infty)$. The nonlinear drift $b : \mathbf{R} \to \mathbf{R}$ is a polynomial of degree 3 with positive dominant coefficients and $a : \mathbf{R} \to \mathbf{R}$ is a measurable map satisfying additional assumptions (see Section 4 below).

Second order heat equations with fractional noises have been investigated in the literature (see [9, 11, 14, 17, 21]). Among them, Duncan et al. [9] and Tindel et al. [21] investigated a class of parabolic equations with linear fractional noise terms, where the Hurst parameter H in [9] was restricted to $H > \frac{1}{2}$, and the later treats both cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$. The heat equations with a multiplicative fractional noise of Hurst parameter $H = (h_0, \cdots, h_d)$ on $[0, \infty) \times \mathbf{R}^d$ were proposed by Hu [11] and the author established the existence and uniqueness of mild solutions to the equation under some assumptions on H, through chaos expansion. For a nonlinear evolution equation in some Hilbert space, Maslowski and Nualart [14] proved the existence and uniqueness of mild solutions for the equation with a cylindrical fractional Brownian motion (FBM) under $H > \frac{1}{2}$. This leads one to define stochastic integrals with respect to FBM in a pathwise way (see also [18]). In [17], Nualart and Ouknine discussed a quasi-linear parabolic equation driven by an additive fractional noise on $[0, \infty) \times [0, 1]$.

Recently, a class of stochastic Cahn-Hilliard equations with Gaussian noise perturbations were introduced in [8, 5, 6], respectively. Furthermore, Bo and Wang [4] established a unique local solution to a stochastic Cahn-Hilliard equation driven by a Lévy space-time white noise, in which a new version of Burkholder-Davis-Gundy inequality (B-D-G inequality) played a key role (see also Proposition 3.2 below). In [2], Bo et al formulated a fourth-order Anderson model with double-parameter fractional noises on one-dimensional space by employing the Skorohod integral. In the present paper, we are going to develop several different versions of B-D-G inequalities for treating the jump component of (1.2). For the fractional noise term, we will limit our consideration on the linear additive fractional noises with Hurst parameter $H > \frac{1}{2}$ as proposed by Nualart and Ouknine [17]. Our aim is to establish the existence of a unique local mild solution to (1.2).

The outline of this paper is as follows. In the coming section, we will give the definitions of the fractional B^H and the pure jump Lévy noise F, respectively. In Section 3, several different B-D-G inequalities are presented. The statement of main result and its proof will be given in Section 4.

2 Fractional and Lévy Noises

In this section, we will present the definitions of the fractional noises, Lévy space-time white noises and stochastic integrals with respect to them in the respective subsections.

2.1 Fractional noises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a complete probability space with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, on which $(B^H(A \times [0, t]))_{(t,A) \in [0,T] \times \mathcal{B}(D)}$ is a centered Gaussian family of random variables with the covariance, for $H \in (0, 1)$,

$$\mathbf{E}[B^H(A \times [0, t])B^H(B \times [0, s])] = |A \cap B| R_H(t, s), \quad s, t \in [0, T], \ A, B \in \mathcal{B}(D),$$

with the covariance kernel

$$R_H(t,s) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}].$$

Here $|A|$ denotes Lebesgue measure of the set $A \in \mathcal{B}(D)$.

We denote by ϵ the set of step functions on $D \times [0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of ϵ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]\times A}, \mathbf{1}_{[0,s]\times B}\rangle_{\mathcal{H}} = |A \cap B| R_H(t,s).$$

Thus the mapping $\mathbf{1}_{[0,t]\times A} \to B^H(A \times [0,t])$ is an isometry between $\mathcal{E}$ and the linear space span$\{B^H(A\times[0,t]), A \in \mathcal{B}(D), t \in [0,T]\}$, a subspace of $L^2(\Omega)$. Moreover, the mapping can be extended to an isometry from $\mathcal{H}$ to Gaussian space associated with B^H. This isometry will be denoted by $\varphi \to B^H(\varphi)$ for $\varphi \in \mathcal{H}$. Therefore, we can regard $B^H(\varphi)$ as the stochastic integral with respect to B^H. In general, we use the notation

$$\int_{[0,T]\times D} \varphi(y,s) B^H(\mathrm{d}y, \mathrm{d}s)$$

to represent $B^H(\varphi)$. On the other hand, it is known that the covariance kernel $R_H(t,s)$ satisfies

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\mathrm{d}u,$$

where the kernel

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\Big(\frac{1}{2} - H\Big)\int_s^t (u-s)^{H-\frac{3}{2}}\Big(1 - \Big(\frac{s}{u}\Big)^{\frac{1}{2}-H}\Big)\mathrm{d}u$$

for some constant c_H. In particular, if $H > \frac{1}{2}$, then

$$R_H(t,s) = H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}\mathrm{d}u\mathrm{d}v. \tag{2.1}$$

Define a linear operator $K_H^* : \mathcal{E} \to L^2([0,T])$ by

$$(K_H^*\psi)(s,x) = K_H(T,s)\psi(s,x) + \int_s^T (\psi(u,x) - \psi(s,x))\frac{\partial K_H}{\partial u}(u,s)\mathrm{d}u. \tag{2.2}$$

Then the operator K_H^* gives an isometry from $\mathcal{H}$ to $L^2([0,T]\times D)$ (see [17, 21]). Consequently,

$$W(t,A) := B^H((K_H^*)^{-1}(\mathbf{1}_{[0,t]\times A})), \quad (t,A) \in [0,T]\times\mathcal{B}(D)$$

defines a space-time white noise. Moreover we can regard B^H as

$$B^H(A\times[0,t]) = \int_0^t\int_D K_H(t,s)W(\mathrm{d}y,\mathrm{d}s).$$

2.2 Lévy space-time white noises

Let $(E_i,\mathcal{E}_i,\mu_i)$, $i=1,2$ be two σ-finite measurable spaces. We call $N:(E_1,\mathcal{E}_1,\mu_1)\times(E_2,\mathcal{E}_2,\mu_2)\times(\Omega,\mathcal{F},\mathbf{P})\to\mathbf{N}\cup\{0\}\cup\{\infty\}$ a Poisson noise on $(E_1,\mathcal{E}_1,\mu_1)$, if for all $A\in\mathcal{E}_1$, $B\in\mathcal{E}_2$ and $n\in\mathbf{N}\cup\{0\}\cup\{\infty\}$,

$$\mathbf{P}(\omega\in\Omega:N(A,B,\omega)=n)=\frac{\mathrm{e}^{-\mu_1(A)\mu_2(B)}[\mu_1(A)\mu_2(B)]^n}{n!}. \tag{2.3}$$

In particular, when $(E_1,\mathcal{E}_1,\mu_1)=([0,\infty)\times D,\mathcal{B}([0,\infty)\times D),\mathrm{d}t\times\mathrm{d}x)$, we can define the compensated random martingale measure

$$M(B,A,t)=N([0,t]\times A,B)-\mu_1([0,t]\times A)\mu_2(B) \tag{2.4}$$

by assuming that $\mu_1([0,t]\times A)\mu_2(B)<\infty$ for all $(t,A,B)\in[0,\infty)\times\mathcal{B}(D)\times\mathcal{E}_2$. Moreover, let $f:E_1\times E_2\times\Omega\to\mathbf{R}$ be an $(\mathcal{F}_t)_{t\geq0}$-predictable random process satisfying

$$\mathbf{E}\Big[\int_0^t\int_A\int_B|f(s,x,y)|^2\mu_2(\mathrm{d}y)\mathrm{d}x\mathrm{d}s\Big]<\infty \tag{2.5}$$

for all $t>0$ and $(A,B)\in\mathcal{E}_1\times\mathcal{E}_2$. Then the stochastic integral process

$$\Big(R_t:=\int_0^{t+}\int_A\int_B f(s,x,y)M(\mathrm{d}y,\mathrm{d}x,\mathrm{d}s)\Big)_{t\geq0} \tag{2.6}$$

is a square integrable $(\mathcal{F}_t)_{t\geq0}$-martingale. It is well-known that a (pure jump) Lévy space-time white noise admits the following structure:

$$\dot F(x,t)=\int_{U_0}h_1(t,x,y)\dot M(\mathrm{d}y,x,t)+\int_{E_2\setminus U_0}h_2(t,x,y)\dot N(\mathrm{d}y,x,t) \tag{2.7}$$

for some $U_0\in\mathcal{E}_2$ such that $\mu_2(E_2\setminus U_0)<\infty$. Here $h_1,h_2:[0,\infty)\times D\times E_2\to\mathbf{R}$ are measurable maps; $\dot M$ and $\dot N$ denote the Radon-Nikodym derivatives

$$\dot M(\mathrm{d}y,x,t):=\frac{M(\mathrm{d}y,\mathrm{d}x,\mathrm{d}t)}{\mathrm{d}t\times\mathrm{d}x},\quad \dot N(\mathrm{d}y,x,t):=\frac{N(\mathrm{d}t\times\mathrm{d}x,\mathrm{d}y)}{\mathrm{d}t\times\mathrm{d}x} \tag{2.8}$$

with $(t,x,y)\in[0,\infty)\times D\times E_2$ (see [22]).

3 Burkholder-Davis-Gundy Inequalities

In order to estimate the higher order moments of mild solutions to (1.2), we need several different versions of Burkholder-Davis-Gundy inequalities. Those are quoted from [13, Theorem 4.1] and [10, Corollary 3.1], respectively. Let us first recall the usual Burkholder's inequality (see [19]).

Proposition 3.1 *Let $f:[0,\infty)\times D\times E_2\times\Omega\to\mathbf{R}$ be an $(\mathcal{F}_t)_{t\geq0}$-predictable process satisfying* (2.5). *Denote by X the integral process*

$$\Big(X_t:=\int_0^{t+}\int_D\int_{E_2}f(s,y,z)M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big)_{t\geq0}.$$

Then for $T > 0$ and $q \geq 1$, there exists a constant $C_1(q) > 0$ such that

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |X_t|^q\Big] \leq C_1(q)\mathbf{E}[X, X]_T^{\frac{q}{2}}, \tag{3.1}$$

where $[X, X]_t = \int_0^t\int_D\int_{E_2} |f(s, y, z)|^2 N(\mathrm{d}s \times \mathrm{d}y, \mathrm{d}z)$ is the quadratic variation process of X.

Remark 3.1 Note that, in Proposition 3.1,

$$\mathbf{E}[X, X]_t = \int_0^t\int_D\int_{E_2} |f(s, y, z)|^2\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s.$$

Then Jensen's inequality yields, for $q \in [1, 2]$,

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |X_t|^q\Big] \leq C_1(q)\Big[\int_0^T\int_D\int_{E_2} \mathbf{E}|f(s, y, z)|^2\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big]^{\frac{q}{2}}. \tag{3.2}$$

Proposition 3.2 *Let $(X_t)_{t\geq 0}$ be defined as in Proposition 3.1. Then for $T > 0$ and $q \geq 2$, there exists a constant $C_2(q) > 0$ such that*

$$\sup_{t\in[0,T]} \mathbf{E}[|X_t|^q] \leq C_2(q)\Big[\int_0^T\int_D\int_{E_2} (\mathbf{E}|f(s, y, z)|^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big]^{\frac{q}{2}}. \tag{3.3}$$

On the other hand, let $\mathcal{L}^{\mathrm{sym}}(E_2, \mathcal{E}_2)$ denote the total of all symmetric Lévy measure on $(E_2, \mathcal{E}_2)$ (see [10, Definition 2.2]). If the measure $\mu_2 \in \mathcal{L}^{\mathrm{sym}}(E_2, \mathcal{E}_2)$ for the separable Banach space E_2 and if $q \in [2, 4]$, then there exists $C_3(q) > 0$ such that

$$\sup_{t\in[0,T]} \mathbf{E}[|X_t|^q] \leq C_3(q)\int_0^T\int_D\int_{E_2} \mathbf{E}|f(s, y, z)|^q\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s. \tag{3.4}$$

In particular, if $q = p^n$ for some $n \in \mathbf{N}$ and $1 \leq p \leq 2$, then there exists $C_4(q) > 0$ such that

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |X_t|^q\Big] \leq C_4(q)\sum_{k=1}^{n}\Big[\int_0^T\int_D\int_{E_2} \mathbf{E}|f(s, y, z)|^{p^k}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big]^{p^{n-k}}. \tag{3.5}$$

In what follows, we turn to the definition of the solutions to (1.2). An $(\mathcal{F}_t)_{t\geq 0}$-adapted random field $u = (u(t, x))_{(t,x)\in[0,T]\times D}$ is called a weak solution of (1.2), if for all $\varphi \in C_0^\infty([0, T]\times \mathbf{R}^d)$ with $\frac{\partial\varphi}{\partial n}|_{[0,T]\times\partial D} = \frac{\partial\Delta\varphi}{\partial n}|_{[0,T]\times\partial D} = 0$, it holds that

$$\begin{aligned}\langle u(t), \varphi(t)\rangle = \langle\psi, \varphi(0)\rangle &+ \int_0^t \Big\langle\Big(\frac{\partial}{\partial t} - \Delta^2\Big)\varphi(s), u(s)\Big\rangle\mathrm{d}s\\ &+ \int_0^t \langle\Delta\varphi(s), b(u)(s)\rangle\mathrm{d}s + \int_0^t\int_D \varphi(s, x)B^H(\mathrm{d}x, \mathrm{d}s)\\ &+ \int_0^t\int_D \varphi(s, x)a(u(s, x))F(\mathrm{d}x, \mathrm{d}s),\end{aligned} \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product of $L^2(D)$.

Let $G(t, x, y) : [0, t] \times D^2 \to \mathbf{R}$ be the Green kernel of the operator $\frac{\partial}{\partial t} + \Delta^2$ with the homogeneous Neumann boundary condition as in (1.2) (see Appendix). Then by virtue of the proof of [20, Theorem 2.1] (or see [5, (1.9)–(1.10)]), (3.6) is equivalent to the following mild form in the sense of Walsh [23]. For $(t,x)\in[0,T]\times D$,

$$\begin{aligned}u(t,x)&=\int_D G(t,x,y)\psi(y)\mathrm{d}y+\int_0^t\int_D b(u(s,y))\Delta_y G(t-s,x,y)\mathrm{d}y\mathrm{d}s\\&\quad+\int_0^t\int_D G(t-s,x,y)a(u(s,y))g(s,y)\mathrm{d}y\mathrm{d}s+\int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s)\\&\quad+\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)a(u(s-,y))h(s,y,z)M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s),\end{aligned}\tag{3.7}$$

where the maps g,h are given respectively by

$$\begin{aligned}g(t,y)&=\int_{E_2\setminus U_0} h_2(t,y,z)\mu_2(\mathrm{d}z),\\h(t,y,z)&=h_1(t,y,z)\mathbf{1}_{U_0}(z)+h_2(t,y,z)\mathbf{1}_{E_2\setminus U_0}(z),\end{aligned}$$

with indicator $\mathbf{1}_A(\,\cdot\,)$ of the set $A\in\mathcal{E}_2$. On the other hand, as in Section 2, the fractional integral term in (3.7) can be represented as

$$\int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s)=\int_0^t\int_D[K_H^*G(t-\,\cdot\,,x,\,\cdot\,)](s,y)W(\mathrm{d}y,\mathrm{d}s),\tag{3.8}$$

with the space-time white noise $(W(t,x))_{(t,x)\in[0,T]\times D}$ mentioned in Section 2. We mainly study the existence of a local mild solution of (3.7). To achieve it, let $\|\cdot\|_q$ denote the usual norm of $L^q(D)$ with $q\in[1,\infty)$. Given $n\in\mathbf{N}$, define a C^1-function $\Psi_n:[0,\infty)\to[0,\infty)$ by

$$\Psi_n(x)=\begin{cases}1, & \text{if } x<n,\\ 0, & \text{if } x\geq n+1,\end{cases}\tag{3.9}$$

and $\|\Psi_n'\|_\infty:=\sup_{x\geq 0}|\Psi_n'(x)|\leq 2$. Let the random field $(u_n(t,x))_{(t,x)\in[0,T]\times D}$ be a unique solution of the following:

$$\begin{aligned}u_n(t,x)&=\int_D G(t,x,y)\psi(y)\mathrm{d}y+\int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s)\\&\quad+\int_0^t\int_D b(u_n(s,y))\Delta_y G(t-s,x,y)\Psi_n(\|u_n(s,\,\cdot\,)\|_q)\mathrm{d}y\mathrm{d}s\\&\quad+\int_0^t\int_D G(t-s,x,y)a(u_n(s,y))g(s,y)\Psi_n(\|u_n(s,\,\cdot\,)\|_q)\mathrm{d}y\mathrm{d}s\\&\quad+\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)a(u_n(s-,y))h(s,y,z)M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s).\end{aligned}\tag{3.10}$$

Define $\tau_n=\inf\{t>0;\|u_n(t,\,\cdot\,)\|_q\geq n\}$ with $n\in\mathbf{N}$. Then, on the event $\{t<\tau_n\}$, $u_1(t,x)=u_2(t,x)=\cdots=u_n(t,x)$ is a solution of (3.10). Let $\tau=\lim_{n\to\infty}\tau_n$, and define $u(t,\,\cdot\,)=u_n(t,\,\cdot\,)$, on the event $\{t<\tau_n<\tau\}$. Therefore $u(t,\,\cdot\,)$ is a solution of (3.10) on $\{t<\tau\}$. We call it a local mild solution of (1.2). In the following section, we will prove that such a local solution as in (3.10) exists and it is unique.

4 Main Results and Proofs

At the beginning, we state the main result of the paper as follows. Let the interval $(\alpha, \beta] = (\alpha, \beta]$ if $\alpha < \beta$, and $\emptyset$ otherwise.

Theorem 4.1 *Let $H \in (\frac{1}{2}, 1)$ and $d < \frac{4H}{2-H}$ with $d \in \mathbf{N}$. Suppose that the following conditions are satisfied:*

(i) *b is a polynomial of degree 3 with positive dominant coefficients.*

(ii) *a is Lipschitzian and has linear growth on $\mathbf{R}$, i.e. there exists a constant $C > 0$ such that $|a(x)| \leq C(1 + |x|)$, for all $x \in \mathbf{R}$.*

(iii) *For g, h and μ_2,*

$$\sup_{t\in[0,T]} \|g(t, \cdot)\|_q < \infty \quad \text{with } q > d + 2. \tag{4.1}$$

(1) *For $q \in (d + 2, 4]$,*

(a) $V_q := \sup_{(s,y)\in[0,T]\times D} \|h(s, y, \cdot)s^{-\frac{d}{4}}\|^q_{L^q(E_2,\mu_2)} < \infty$;

(b) *$\mu_2 \in \mathcal{L}^{\rm sym}(E_2, \mathcal{E}_2)$ with separable Banach space E_2.*

(2) *For $q > (d + 2) \vee 4$,*

(a′) *$G(t - s, x, y)h(s, y, z)$ is $L^q([0, t] \times D^2 \times E_2, \mathrm{d}s \times \mathrm{d}x \times \mathrm{d}y \times \mu_2(\mathrm{d}z))$ integrable, for $0 \leq t \leq T$;*

(b′) $\mu_2(E_2) < \infty$.

Then for every $\mathcal{F}_0$-adapted initial process $\psi : D \times \Omega \to \mathbf{R}$ satisfying $\mathbf{E}\|\psi(\cdot)\|^q_q < \infty$, there exists a unique local solution $(u(t, x))_{(t,x)\in[0,T]\times D}$ for (3.7) and there exists a stopping time τ such that

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t \wedge \tau, \cdot)\|^q_q < \infty \quad \text{for all } q > d + 2.$$

Let Λ_q be the space of all $L^q(D)$-valued $\mathcal{F}_t$-adapted RCLL processes $u(t, \cdot)$. For fixed $\lambda > 0$ and $q \in [2, \infty)$, define a norm $\| \cdot \|_{\Lambda_q}$ (depending on (λ, q)) on Λ_q by

$$\|u\|_{\Lambda_q} = \Big[\sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\mathbf{E}\|u(t, \cdot)\|^q_q\Big]^{\frac{1}{q}} < \infty, \tag{4.2}$$

with $\| \cdot \|_q$ the usual norm of $L^q(D)$. Then $(\Lambda_q, \| \cdot \|_{\Lambda_q})$ forms a Banach space. Let $\psi \in \Lambda_q$. Recall (1.2) or (3.10). For $(t, x) \in [0, T] \times D$,

$$\begin{aligned} u_n(t, x) &= \int_D G(t, x, y)\psi(y)\mathrm{d}y + \int_0^t\int_D G(t - s, x, y)B^H(\mathrm{d}y, \mathrm{d}s) \\ &\quad + \int_0^t\int_D b(u_n(s, y))\Delta_y G(t - s, x, y)\Psi_n(\|u_n(s, \cdot)\|_q)\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t\int_D G(t - s, x, y)a(u_n(s, y))g(s, y)\Psi_n(\|u_n(s, \cdot)\|_q)\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^{t+}\int_D\int_{E_2} G(t - s, x, y)a(u_n(s-, y))h(s, y, z)M(\mathrm{d}z, \mathrm{d}y, \mathrm{d}s) \\ &:= \mathcal{A}_0(\phi)(t, x) + \sum_{i=1}^4 \mathcal{A}_i(u_n)(t, x). \end{aligned} \tag{4.3}$$

According to (4.3), we have

Proposition 4.1 *Under the assumptions of Theroem* 4.1*, for each* $q > d+2$ *and* $u \in \Lambda_q$*, it holds that* $\mathcal{A}_i(u) \in \Lambda_q$*,* $i = 0, \cdots, 4$.

Proof From (A.2), Minkovski's inequality and Young's inequality for $\frac{1}{q} = 1 + \frac{1}{q} - 1$, it follows that

$$\begin{aligned}\|\mathcal{A}_0(\phi)(t,\cdot)\|_q &\le Kt^{-\frac{d}{4}}\Big\|\int_D \exp\Big(-C\frac{|\cdot-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big)\psi(y)\mathrm{d}y\Big\|_q \\ &\le Kt^{-\frac{d}{4}}\Big\|\Big(\exp\Big(-C\frac{|\cdot|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big)*\psi(\cdot)\Big)(\cdot)\Big\|_q \\ &\le Kt^{-\frac{d}{4}}\Big\|\exp\Big(-C\frac{|\cdot|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big)\Big\|_1\|\psi(\cdot)\|_q \\ &= C\|\psi(\cdot)\|_q. \end{aligned} \tag{4.4}$$

Therefore $\mathcal{A}_0(\phi) \in \Lambda_q$ if $\mathbf{E}\|\psi(\cdot)\|_q^q < \infty$. Next we turn to $\mathcal{A}_1(u)$. Let $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0,1]$. Applying (A.6), we conclude that

$$\|\mathcal{A}_1(u)(t,\cdot)\|_q \le C\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r_1}}\|b(u(s,\cdot))\Psi_n(\|u(s,\cdot)\|_q)\|_\rho \mathrm{d}s. \tag{4.5}$$

In particular, let $\rho = \frac{q}{3}$. Then by the assumption (i) in Theorem 4.1,

$$\|b(u(s,\cdot))\|_\rho \le C[\|u(s,\cdot)\|_q + \|u(s,\cdot)\|_q^2 + \|u(s,\cdot)\|_q^3].$$

Consequently,

$$\|\mathcal{A}_1(u)(t,\cdot)\|_q \le C_n\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r_1}}\mathrm{d}s, \tag{4.6}$$

which is finite if $\frac{d}{4r_1} - \frac{d+2}{4} > -1$. Since $\frac{1}{r_1} = \frac{q-2}{q}$, we have $\mathcal{A}_1(u) \in \Lambda_q$ for $q > d$. As for $\mathcal{A}_2(u)$, by virtue of (A.5), we have for $\frac{1}{r_2} = \frac{1}{q} - \frac{2}{q} + 1 = -\frac{1}{q} + 1 \in [0,1]$,

$$\begin{aligned}\|\mathcal{A}_2(u)(t,\cdot)\|_q &\le C\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}\|g(s,\cdot)a(u(s,\cdot))\Psi_n(\|u(s,\cdot)\|_q)\|_{\frac{q}{2}}\mathrm{d}s \\ &\le C\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}\|g(s,\cdot)(1+|u(s,\cdot)|)\Psi_n(\|u(s,\cdot)\|_q)\|_{\frac{q}{2}}\mathrm{d}s \\ &\le C\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}[\|g(s,\cdot)\|_{\frac{q}{2}} + \|g(s,\cdot)u(s,\cdot)\|_{\frac{q}{2}}]\Psi_n(\|u(s,\cdot)\|_q)\mathrm{d}s \\ &\le C_q\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}[\|g(s,\cdot)\|_q + \|g(s,\cdot)\|_q\|u(s,\cdot)\|_q]\Psi_n(\|u(s,\cdot)\|_q)\mathrm{d}s \\ &\le C_q\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}[\|g(s,\cdot)\|_q + (n+1)\|g(s,\cdot)\|_q]\mathrm{d}s \\ &\le C_{n,q}\sup_{t\in[0,T]}\|g(t,\cdot)\|_q\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}\mathrm{d}s \\ &< \infty, \end{aligned} \tag{4.7}$$

provided $q > \frac{d}{4}$. Therefore $\mathcal{A}_2(u) \in \Lambda_q$ as $q > \frac{d}{4}$. In what follows, let us consider $\mathcal{A}_3(u)$. Applying the B-D-G inequality (3.1), we conclude that,

$$\begin{aligned}
\mathbf{E}\|\mathcal{A}_3(u)(t,\cdot)\|_q^q &= \int_D \mathbf{E}\Big|\int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s)\Big|^q \mathrm{d}x \\
&= \int_D \mathbf{E}\Big|\int_0^t\int_D (K_H^* G(t-\cdot,x,\cdot))(s,y)W(\mathrm{d}y,\mathrm{d}s)\Big|^q \mathrm{d}x \\
&\le C_q \int_D \mathbf{E}\Big(\int_0^t\int_D (K_H^* G(t-\cdot,x,\cdot))^2(s,y)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}} \mathrm{d}x \\
&= C_q \int_D \langle (K_H^* G(t-\cdot,x,\cdot))(\cdot,\cdot), (K_H^* G(t-\cdot,x,\cdot))(\cdot,\cdot)\rangle_{L^2([0,T]\times D)}^{\frac{q}{2}} \mathrm{d}x \\
&= C_q \int_D \langle G(t-\cdot,x,\cdot), G(t-\cdot,x,\cdot)\rangle_{\mathcal{H}}^{\frac{q}{2}} \mathrm{d}x \\
&\le C_q \int_D \|G(t-\cdot,x,\cdot)\|^q_{L^{\frac{2}{H}}([0,T]\times D)} \mathrm{d}x,
\end{aligned} \tag{4.8}$$

where we have used the fact that $L^{\frac{2}{H}}([0,T]\times D) \subset \mathcal{H}$ when $H > \frac{1}{2}$ (see [15, Theorem 1], but we need to modify their proof which is given in Appendix below). Note that

$$\begin{aligned}
\|G(t-\cdot,x,\cdot)\|^q_{L^{\frac{2}{H}}([0,T]\times D)} &= \Big[\int_0^T\int_D |G(t-s,x,y)|^{\frac{2}{H}}\mathrm{d}y\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&= \Big[\int_0^t\int_D |G(t-s,x,y)|^{\frac{2}{H}}\mathrm{d}y\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&\le \Big[\int_0^t (t-s)^{-\frac{d}{2H}} \int_D \exp\Big(-C_H\frac{|x-y|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}}\Big)\mathrm{d}y\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&\le C_H\Big[\int_0^t (t-s)^{\frac{d}{4}-\frac{d}{2H}}\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&\le C_H T^{\frac{qH}{2}(1+\frac{d}{4}-\frac{d}{2H})} \\
&< \infty,
\end{aligned} \tag{4.9}$$

under the assumption $d < \frac{4H}{2-H}$ of Theorem 4.1. So we have $A_3(u) \in \Lambda_q$ for $q \ge 2$. Now we estimate $\mathcal{A}_4(u)$. In the case of $q \in (d+2, 4]$, the inequality (3.4) of Proposition 3.2 yields

$$\begin{aligned}
\|\mathcal{A}_4(u)\|^q_{\Lambda_q} &= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)\|_q^q \\
&= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \mathbf{E}\Big(\Big|\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)h(s,y,z) \\
&\quad \times a(u(s-,y))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\Big)\mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \Big(\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \\
&\quad \times \mathbf{E}[|a(u(s,y))|^q]\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)\mathrm{d}x \\
&= C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_0^t\int_D \Big(\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q\mu_2(\mathrm{d}z)\mathrm{d}x\Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \mathbf{E}[|a(u(s,y))|^q]\mathrm{d}y\mathrm{d}s \\
&\le C_q V_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_0^t |t-s|^{\frac{d}{4}}\mathbf{E}\Big[\int_D |a(u(s,y))|^q\mathrm{d}y\Big]\mathrm{d}s \\
&\le C_q V_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_0^t |t-s|^{\frac{d}{4}}\mathbf{E}[\|a(u(s,\cdot))\|_q^q]\mathrm{d}s \\
&\le C_q V_q \sup_{t\in[0,T]} \int_0^t |t-s|^{\frac{d}{4}}\mathrm{e}^{-\lambda(t-s)}(1+\mathrm{e}^{-\lambda s}\mathbf{E}[\|u(s,\cdot)\|_q^q])\mathrm{d}s \\
&\le C_q V_q (1+\|u\|_{\Lambda_q}^q)\int_0^T s^{\frac{d}{4}}\mathrm{e}^{-\lambda s}\mathrm{d}s \\
&\le C_q V_q (1+\|u\|_{\Lambda_q}^q)\frac{\Gamma(\frac{d}{4}+1)}{\lambda^{\frac{d}{4}+1}} \\
&< \infty, \qquad (4.10)
\end{aligned}
$$

where V_q is defined by Theorem 4.1(iii)(1)(a) and $\Gamma(\,\cdot\,)$ denotes the Gamma function.

As for $q > (d+2)\vee 4$, by the hypotheses Theorem 4.1(ii), (iii) and the inequality (3.3), it follows that

$$
\begin{aligned}
\|\mathcal{A}_4(u)\|_{\Lambda_q}^q &= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)\|_q^q \\
&= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \mathbf{E}\Big(\Big|\int_0^{t+}\!\!\int_D\int_{E_2} G(t-s,x,y)h(s,y,z) \\
&\quad \times a(u(s-,y))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\Big)\mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \Big(\int_0^t\!\!\int_D\int_{E_2} (\mathbf{E}|G(t-s,x,y)h(s,y,z)a(u(s,y))|^q)^{\frac{2}{q}} \\
&\quad \times \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \Big(\int_0^t\!\!\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^2\mathrm{e}^{\frac{2\lambda s}{q}} \\
&\quad \times (\mathrm{e}^{-\lambda s}\mathbf{E}(1+|u(s,y)|)^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \int_D \Big(\int_0^t\!\!\int_D\int_{E_2} \mathrm{e}^{-\lambda s}\mathbf{E}(1+|u(s,y)|)^q\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big) \\
&\quad \times \Big(\int_0^t\!\!\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}}\mathrm{e}^{-\frac{2\lambda(t-s)}{q-2}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q-2}{2}}\mathrm{d}x \\
&\le C_q\mu_2(E_2)\Big(T|D| + \int_0^T \mathrm{e}^{-\lambda s}\mathbf{E}\|u(s,\cdot)\|_q^q\mathrm{d}s\Big) \\
&\quad \times \sup_{t\in[0,T]}\int_D \Big(\int_0^t\!\!\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}}\mathrm{e}^{-\frac{2\lambda(t-s)}{q-2}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q-2}{2}}\mathrm{d}x \\
&\le C_q\mu_2(E_2)(T|D| + T\cdot\|u\|_{\Lambda_q}^q) \\
&\quad \times \int_D\int_0^T\!\!\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q\mathrm{e}^{-\lambda(t-s)}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\mathrm{d}x \\
&< \infty. \qquad (4.11)
\end{aligned}
$$

This shows that $\mathcal{A}_4(u)\in\Lambda_q$. Thus we complete the proof of the proposition.

We can now define an operator $\mathcal{K}$ on Λ_q by

$$\mathcal{K}(u)(t,x)=\mathcal{A}_0(\phi)(t,x)+\sum_{i=1}^4\mathcal{A}_i(u)(t,x),\quad (t,x)\in[0,T]\times D. \tag{4.12}$$

In what follows, we will prove that the operator $\mathcal{K}:\Lambda_q\to\Lambda_q$ is a contract mapping.

Theorem 4.2 *For $q>d+2$, the operator $\mathcal{K}$ defined by* (4.12) *is a contraction on Λ_q under the conditions of Theorem* 4.1. *In other words, there exists a constant $\varrho\in(0,1)$ such that $\|\mathcal{K}(u)-\mathcal{K}(v)\|_{\Lambda_q}\le\varrho\|u-v\|_{\Lambda_q}$ for $u,v\in\Lambda_q$.*

Proof Suppose that ψ_1,ψ_2 are initials of $(\mathcal{F}_t)_{t\ge0}$-adapted random fields $u,v\in\Lambda_q$ such that $\psi_1=\psi_2$. Let us begin by considering $\mathcal{A}_1$. Note that for $\rho=\frac{q}{3}$,

$$\|\Psi_n(\|u(s,\cdot)\|_q)b(u(s,\cdot))-\Psi_n(\|v(s,\cdot)\|_q)b(v(s,\cdot))\|_\rho\le C_n\|u(s,\cdot)-v(s,\cdot)\|_\rho. \tag{4.13}$$

By virtue of (A.6), we have for $\frac{1}{r}=\frac{1}{q}-\frac{1}{\rho}+1=-\frac{2}{q}+1\in[0,1]$,

$$\begin{aligned}
\|\mathcal{A}_1(u)-\mathcal{A}_1(v)\|_{\Lambda_q}^q&=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}(\|\mathcal{A}_1(u)(t,\cdot)-\mathcal{A}_1(v)(t,\cdot)\|_q^q)\\
&\le\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\Big(\int_0^t(t-s)^{\frac{d}{4r}-\frac{d+2}{4}}\\
&\quad\times\|\Psi_n(\|u(s,\cdot)\|_q)b(u(s,\cdot))-\Psi_n(\|v(s,\cdot)\|_q)b(v(s,\cdot))\|_\rho\mathrm{d}s\Big)^q\\
&\le C_n\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\Big(\int_0^t(t-s)^{\frac{d}{4r}-\frac{d+2}{4}}\|u(s,\cdot)-v(s,\cdot)\|_q\mathrm{d}s\Big)^q\\
&\le C_{n,q}\sup_{t\in[0,T]}\mathbf{E}\Big(\int_0^t\mathrm{e}^{\frac{-\lambda(t-s)}{q}}(t-s)^{\frac{d}{4r}-\frac{d+2}{4}}\mathrm{e}^{\frac{-\lambda s}{q}}\|u(s,\cdot)-v(s,\cdot)\|_q\mathrm{d}s\Big)^q\\
&\le C_{n,q}\sup_{t\in[0,T]}\mathbf{E}\Big(\int_0^t\mathrm{e}^{-\lambda s}\|u(s,\cdot)-v(s,\cdot)\|_q^q\mathrm{d}s\\
&\quad\times\Big[\int_0^t(\mathrm{e}^{\frac{-\lambda(t-s)}{q}}(t-s)^{\frac{d}{4r}-\frac{d+2}{4}})^{\frac{q}{q-1}}\mathrm{d}s\Big]^{q-1}\Big)\\
&\le C_{n,q}\sup_{t\in[0,T]}\Big(\int_0^t\mathrm{e}^{-\lambda s}\mathbf{E}\|u(s,\cdot)-v(s,\cdot)\|_q^q\mathrm{d}s\\
&\quad\times\Big[\int_0^t(\mathrm{e}^{\frac{-\lambda(t-s)}{q-1}}(t-s)^{\frac{q}{q-1}(\frac{d}{4r}-\frac{d+2}{4})})\mathrm{d}s\Big]^{q-1}\Big)\\
&\le C_{n,q}T\|u-v\|_{\Lambda_q}^q\Psi(d,q,T),
\end{aligned} \tag{4.14}$$

where

$$\Psi(d,q,T)=\Big[\int_0^T\mathrm{e}^{\frac{-\lambda(t-s)}{q-1}}(t-s)^{\frac{q}{q-1}(\frac{d}{4r}-\frac{d+2}{4})}\mathrm{d}s\Big]^{q-1}.$$

Let

$$k=\frac{q}{q-1}\Big(\frac{d}{4r}-\frac{d+2}{4}\Big).$$

Then

$$\Psi(d,q,T)\leq\Big[\frac{(q-1)^{k+1}}{\lambda^{k+1}}\int_0^\infty \mathrm{e}^{-s}s^k\mathrm{d}s\Big]^{q-1}=\Big[\frac{(q-1)^{k+1}\Gamma(k+1)}{\lambda^{k+1}}\Big]^{q-1}<\infty, \tag{4.15}$$

when $q>d+2$. Therefore,

$$\begin{aligned}\|\mathcal{A}_1(u)-\mathcal{A}_1(v)\|_{\Lambda_q}&\leq C(n,q)T^{\frac{1}{q}}\Big[\frac{(q-1)^{\frac{k+1}{q}}\Gamma(k+1)^{\frac{1}{q}}}{\lambda^{\frac{k+1}{q}}}\Big]^{q-1}\|u-v\|_{\Lambda_q}\\&\leq \varrho_1\|u-v\|_{\Lambda_q},\end{aligned} \tag{4.16}$$

where $\varrho_1\in(0,1)$ by choosing λ large enough.

Next we consider $\mathcal{A}_4(u)$. In the case of $q\in(d+2,4]$, from a similar argument as in (4.10), it follows that

$$\begin{aligned}\|\mathcal{A}_4(u)-\mathcal{A}_4(v)\|^q_{\Lambda_q}&=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)-\mathcal{A}_4(v)(t,\cdot)\|^q_q\\&=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\mathbf{E}\Big[\Big|\int_0^{t+}\int_D\int_{E_2}G(t-s,x,y)h(s,y,z)\\&\quad\times a((u(s-,y))-a(v(s-,y)))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\Big]\mathrm{d}x\\&=C_q\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_0^t\int_D\Big(\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^q\mu_2(\mathrm{d}z)\mathrm{d}x\Big)\\&\quad\times\mathbf{E}|u(s,y)-v(s,y)|^q\mathrm{d}y\mathrm{d}s\\&\leq C_qV_q\|u-v\|^q_{\Lambda_q}\frac{\Gamma(\frac{d}{4}+1)}{\lambda^{\frac{d}{4}+1}}\\&<\infty.\end{aligned} \tag{4.17}$$

For $q>(d+2)\vee 4$, thanks to Proposition 3.1, we derive from the assumptions (ii) and (iii) of Theorem 4.1 that

$$\begin{aligned}&\|\mathcal{A}_4(u)-\mathcal{A}_4(v)\|^q_{\Lambda_q}\\&=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)-\mathcal{A}_4(v)(t,\cdot)\|^q_q\\&=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\mathbf{E}\Big|\int_0^{t+}\int_D\int_{E_2}G(t-s,x,y)h(s,y,z)\\&\quad\times(a(u(s-,y))-a(v(s-,y)))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\mathrm{d}x\\&\leq C_q\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\Big(\int_0^t\int_D\int_{E_2}(\mathbf{E}|G(t-s,x,y)h(s,y,z)\\&\quad\times(a(u(s,y))-a(v(s,y)))|^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x\\&\leq C_q\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\Big(\int_0^t\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^2\\&\quad\times(\mathbf{E}|u(s,y)-v(s,y)|^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \int_D \Big(\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^2 \mathrm{e}^{\frac{2\lambda s}{q}} \\
&\quad \times (\mathrm{e}^{-\lambda s}\mathbf{E}|u(s,y)-v(s,y)|^q)^{\frac{2}{q}} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}} \mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \int_D \Big(\int_0^t\int_D\int_{E_2} \mathrm{e}^{-\lambda s}\mathbf{E}|u(s,y)-v(s,y)|^q \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big) \\
&\quad \times \Big(\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}} \mathrm{e}^{-\frac{2\lambda(t-s)}{q-2}} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q-2}{2}} \mathrm{d}x \\
&\le C_q T\mu_2(E_2)\|u-v\|_{\Lambda_q}^q \\
&\quad \times \sup_{t\in[0,T]} \int_D\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mathrm{e}^{-\lambda(t-s)} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\mathrm{d}x. \qquad (4.18)
\end{aligned}
$$

Take into account

$$V_q \frac{\Gamma(\frac{d}{4}+1)}{\lambda^{\frac{d}{4}+1}} \to 0, \quad \text{as } \lambda\to\infty$$

and

$$\sup_{t\in[0,T]} \int_D\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mathrm{e}^{-\lambda(t-s)} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\mathrm{d}x \to 0, \quad \text{as } \lambda\to\infty.$$

Then by (4.17) and (4.18), together with the assumption (iii) of Theorem 4.1, $\mathcal{A}_4$ is a contraction on Λ_q, for $\lambda>0$ large enough.

For $q>d+2$, a similar procedure as (4.14)–(4.16) yields that $\mathcal{A}_2$ is a contraction on Λ_q, by letting $\lambda>0$ large enough. Therefore, it follows from (4.12) that $\mathcal{K}(\cdot)$ is a contraction on Λ_q if $\lambda>0$ large enough. Thus the proof of Theorem 4.2 is completed.

We note that $h\to\frac{4h}{2-h}$ is an increasing function with the range $(\frac{4}{3},4)$ on $h\in(\frac12,1)$. Hence for $q>d+2$, applying the fixed point principal on the set $\{u\in\Lambda_q : u(0)=\psi\}$, we conclude that (3.10) admits a unique solution $u\in\Lambda_q$. Thus the conclusion of Theorem 4.1 follows.

Appendix

Firstly, we will give a short proof for the assertion that $L^{\frac{2}{H}}([0,T]\times D)\subset\mathcal{H}$ if $H>\frac12$.

Let

$$f\in L^{\frac{2}{H}}([0,T]\times D) \quad \text{and} \quad d_H = H(2H-1).$$

Then Theorem 1 in [15] implies that there exists a positive constant $C(H,T,d)$ depending on H,T,d such that

$$
\begin{aligned}
\|f\|_{\mathcal{H}}^2 &:= d_H \int_0^T\int_0^T\int_D f(u,x)f(v,x)|u-v|^{2H-2}\mathrm{d}x\mathrm{d}u\mathrm{d}v \\
&= d_H \int_D \Big[\int_0^T\int_0^T f(u,x)f(v,x)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\Big]\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le C_H \int_D \Big[\int_0^T |f(u,x)|^{\frac{1}{H}}\mathrm{d}u\Big]^{2H}\mathrm{d}x\\
&\le C_{H,T} \int_D \Big[\int_0^T |f(u,x)|^{\frac{2}{H}}\mathrm{d}u\Big]^{H}\mathrm{d}x\\
&\le C_{H,T,d} \Big[\int_D\int_0^T |f(u,x)|^{\frac{2}{H}}\mathrm{d}u\mathrm{d}x\Big]^{H}\\
&= C_{H,T,d} \Big[\int_D\int_0^T |f(u,x)|^{\frac{2}{H}}\mathrm{d}u\mathrm{d}x\Big]^{\frac{H}{2}\times 2}\\
&= C_{H,T,d}\|f\|^2_{L^{\frac{2}{H}}([0,T]\times D)},
\end{aligned} \tag{A.1}
$$

where we have used Hölder inequality twice. This shows the continuity of the embedding.

In the following, we will give some estimates on the Green kernel $G(t,x,y)$ corresponding to the operator $\frac{\partial}{\partial t}+\Delta^2$ on the domain $[0,\infty)\times D$. As in [8], the Green function $G(t,x,y)$ admits the following expansion. Let $A=-\Delta$ be defined on $D(A)=\{u\in H^2(D):\frac{\partial u}{\partial n}|_{\partial D}=0\}$ and let $(\Theta_k)_{k\in\mathbf{N}^d}$ be the basis of eigenfunctions of A in $L^2(D)$, which can be written as

$$\Theta_k(x)=\prod_{i=1}^d \theta_{k_i}(x_i),$$

where $k=(k_1,\cdots,k_d)\in\mathbf{N}^d$, $x=(x_1,\cdots,x_d)\in D$. Moreover,

$$
\begin{cases}
\theta_{k_i}(x_i)=\sqrt{\dfrac{2}{\pi}}\cos(k_ix_i), & k_i\neq 0,\\
\theta_0(x_i)=\dfrac{1}{\sqrt{\pi}}, & k_i=0
\end{cases}
$$

with $i=1,\cdots,d$, and $\Big(\lambda_k=\sum_{i=1}^d k_i^2\Big)_{k\in\mathbf{N}^d}$ are the eigenvalues corresponding to the eigenfunctions. Therefore the Green function $G(t,x,y)$ on $[0,\infty)\times D^2$ can be expressed as

$$G(t,x,y)=\sum_{k\in\mathbf{N}^d}\mathrm{e}^{-\lambda_k^2 t}\Theta_k(x)\Theta_k(y)$$

with $(t,x,y)\in[0,\infty)\times D^2$.

Lemma A.1 *There exist $K>0$ and $C>0$ such that for all $t\in(0,T]$, $x,y\in D$,*

$$|G(t,x,y)|\le \frac{K}{t^{\frac{d}{4}}}\exp\Big(-C\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big), \tag{A.2}$$

$$|\Delta_y G(t,x,y)|\le \frac{K}{t^{\frac{d+2}{4}}}\exp\Big(-C\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big), \tag{A.3}$$

$$\Big|\frac{\partial G(t,x,y)}{\partial t}\Big|\le \frac{K}{t^{\frac{d+4}{4}}}\exp\Big(-C\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big). \tag{A.4}$$

Lemma A.2 *For $v\in L^1([0,T],L^\rho(D))$, $0\le t_0\le t\le T$ and $x\in D$, define*

$$J(v)(t_0,t,x)=\int_{t_0}^t\int_D H(t-s,x,y)v(s,y)\mathrm{d}y\mathrm{d}s.$$

Then for any $\rho \in [1,\infty)$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0,1]$. J *is a bounded operator from* $L^1([0,T], L^\rho(D))$ *to* $L^\infty([0,T], L^q(D))$. *Furthermore,*

(1) *If* $H(t-s,x,y) = G(t-s,x,y)$, *there exists a constant* $C > 0$ *such that*

$$\|J(v)(t_0,t,\cdot)\|_q \le C \int_{t_0}^{t} (t-s)^{\frac{d}{4r}-\frac{d}{4}} \|v(s,\cdot)\|_\rho \mathrm{d}s. \tag{A.5}$$

(2) *If* $H(t-s,x,y) = \Delta_y G(t-s,x,y)$, *there exists a constant* $C > 0$ *such that*

$$\|J(v)(t_0,t,\cdot)\|_q \le C \int_{t_0}^{t} (t-s)^{\frac{d}{4r}-\frac{d+2}{4}} \|v(s,\cdot)\|_\rho \mathrm{d}s, \tag{A.6}$$

where we assume that $r \neq \infty$ *if* $d = 2$, *and* $r < 3$, *if* $d = 3$.

(3) *If* $H(t-s,x,y) = G^2(t-s,x,y)$, *there exists a constant* $C > 0$ *such that*

$$\|J(v)(t_0,t,\cdot)\|_q \le C \int_{t_0}^{t} (t-s)^{\frac{d}{4r}-\frac{d}{2}} \|v(s,\cdot)\|_\rho \mathrm{d}s, \tag{A.7}$$

where we assume that $r \neq \infty$ *if* $d = 2$, *and* $r < \frac{3}{2}$, *if* $d = 3$.

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