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Holomorphic Lefschetz Fixed Point Formula for Non-compact Kähler Manifolds***

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Abstract The authors obtain a holomorphic Lefschetz fixed point formula for certain non-compact "hyperbolic" Kähler manifolds (e.g. Kähler hyperbolic manifolds, bounded domains of holomorphy) by using the Bergman kernel. This result generalizes the early work of Donnelly and Fefferman.

Keywords Lefschetz fixed point formula, Bergman kernel
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1 Introduction

In 1926, Lefschetz [8] published his celebrated fixed point formula, namely, if f is a holomorphic automorphism on a compact complex manifold of dimension n such that there are only a finite number of fixed points $p_1, \cdots, p_k$, then

$$L(f) = \sum_{f(p_j)=p_j} \frac{1}{\det(I - J_f(p_j))},$$

where $L(f)$ is the so-called Lefschetz number of f defined by

$$L(f) = \sum (-1)^q \text{Trace } f^*(H^{0,q}(M))$$

with $H^{p,q}(M)$ being Dolbeault cohomology groups.

In general, for non-compact complete Kähler manifolds, the Lefschetz number of L^2-Dolbeault cohomology groups with respect to some complete Kähler metric might depend on the choice of the Kähler metric. However, Donnelly and Fefferman discovered the following interesting fixed point formula:

Theorem 1.1 (cf. [2]) *Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ and let f be a holomorphic automorphism without fixed points on the boundary. Then*

$$(-1)^n \int_\Omega \overline{K_\Omega(z, f(z))} = \sum_{f(p_j)=p_j} \frac{1}{\det(I - J_f)(p_j)}, \tag{1.1}$$

where $K_\Omega(z, w)$ is the Bergman kernel form.

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Let us make a few remarks on Theorem 1.1. According to Fefferman [4], every holomorphic automorphism of Ω extends smoothly to $\partial\Omega$, which implies, if an automorphism f has not fixed points on $\partial\Omega$, then it has only finite isolated fixed points in Ω for otherwise the set of fixed points is an analytic subvariety with dimension at least one thus must intersects $\partial\Omega$. In fact, the above result is still a Lefschetz theorem since the left side of (1.1) equals to the Lefschetz number with respect to the Bergman metric. The argument used in [2] is the heat kernel approach, relying heavily on the bounded geometry of the Bergman metric, which seems not valid for general bounded domains. It seems worthwhile to generalize Theorem 1.1 through different methods.

Theorem 1.2 *Let $\Omega \subset\subset \mathbf{C}^n$ be a domain of holomorphy and let f be a holomorphic automorphism such that the closure of the graph Γ_f of f does not meet the diagonal at the boundary of $\Omega \times \Omega$. Then* (1.1) *holds.*

Similarly as the above remarks, we conclude that f has also only finite fixed points in Ω. The main ingredients are Hörmander's L^2 theory and Kerzman's representation of the Bergman kernel. This approach has the advantage to generalize to certain complete Kähler manifolds with slightly modifications.

Definition 1.1 *We call two subsets A, B of a metric space (M, d) do not meet at the ideal boundary of M if outside some compact subset K there is a positive constant C such that*

$$d_H(A \setminus K, B \setminus K) = \inf_{x\in A} \inf_{x\in B} d(x, y) > C.$$

Definition 1.2 (*cf.* [5]) *A complete Kähler manifold (M, ω) is said to be Kähler hyperbolic if ω is d-bounded, i.e., there is a 1-form θ such that $\omega = d\theta$ and $\sup_M |\theta|_\omega < \infty$.*

This is a large class of non-compact Kähler manifolds which includes all hyperconvex manifolds (i.e., there is a negative C^∞ strictly plurisubharmonic exhaustion function).

Theorem 1.3 *Let (M, ω) be an n-dimensional Kähler hyperbolic manifold and f is a holomorphic automorphism such that Γ_f does not meet the diagonal at the ideal boundary of $M \times M$. Then* (1.1) *also holds.*

2 Proof of Theorem 1.3

2.1 L^2-Hodge theory

Let (M, ω) be a complete Kähler manifold of dimension n and let $L_2^{p,q}(M)$ denote the space of L^2-forms of degree (p, q). The L^2-harmonic space is defined by

$$\mathcal{H}_2^{p,q}(M) = \{\psi \in L_2^{p,q}(M) : \overline{\partial}\psi = 0, \overline{\partial}^*\psi = 0\}.$$

The $\mathcal{H}_2^{n,0}(M)$ is just the space of square-integrable holomorphic n-forms. In the case when ω is d-bounded, it is known from [5] that there is a constant $\lambda_n > 0$ such that every $\psi \in L_2^{p,q}(M)$ with $p + q \neq n$ satisfies the inequality

$$(\psi, \Delta\psi) \geq \lambda_n(\psi, \psi),$$

where $\Delta = 2(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})$. Therefore, there exists a unique operator, the Green operator,

$$G : L_2^{p,q}(M) \to (\mathcal{H}_2^{p,q}(M))^\perp$$

such that $\overline{\partial}G = G\overline{\partial}$, $\overline{\partial}^* G = G\overline{\partial}^*$ and the following decomposition holds:

$$I = P + \Delta G,$$

where I is the identity endomorphism and P is the orthogonal projection from $L_2^{p,q}(M)$ to $\mathcal{H}_2^{p,q}(M)$. In particular, for any $g \in L_2^{n,0}(M)$ we have

$$P_g = g - \overline{\partial}^* \overline{\partial} G g = g - \overline{\partial}^* G \overline{\partial} g.$$

2.2 Bergman kernel form

Let (M, ω) be a complete Kähler manifold and let $\{\psi_i\}$ be a complete orthonormal system of $\mathcal{H}_2^{n,0}(M)$. The Bergman kernel form is given by

$$K_M(z, w) = \sum_i \psi_i(z) \wedge \overline{\psi_i(w)}.$$

K_M does not depend on the choice of basis and it enjoys the following reproducing property

$$\psi(w) = \int_M \psi(z) \wedge \overline{K_M(z, w)}, \quad w \in M, \, \psi \in \mathcal{H}_2^{n,0}(M).$$

By [1], $K_M(z, w)$ is not identically equal to zero if ω is d-bounded.

2.3 Lefschetz number

Let (M, ω) be a complete Kähler manifold and let $f : M \to M$ be a holomorphic map. The Lefschetz number is defined by

$$L_\omega(f) = \sum (-1)^q \text{Trace } f^*(\mathcal{H}_2^{0,q}(M)),$$

where the trace is given by

$$\sum_j \psi_j^{0,q} \wedge \overline{f^* \psi_j^{0,q}}$$

for a complete orthonormal system $\{\psi_j^{0,q}\}$ of $\mathcal{H}_2^{0,q}(M)$. Note that the space $\mathcal{H}_2^{0,n}(M)$ is conjugate to $\mathcal{H}_2^{n,0}(M)$. Thus by Subsection 2.1, if ω is d-bounded, we have

$$L_\omega(f) = (-1)^n \int_M \overline{K_M(z, f(z))}.$$

2.4 Bochner-Martinelli kernel

The Bochner-Martinelli kernel on $\mathbf{C}^n \times \mathbf{C}^n$ is given by

$$k(z, w) = C_n \frac{\sum_j \overline{\Phi_j(z - w)} \wedge \Phi(w)}{|z - w|^{2n}},$$

where

$$\Phi_j(\zeta) = (-1)^{j-1} \zeta_j \mathrm{d}\zeta_1 \wedge \cdots \wedge \widehat{\mathrm{d}\zeta_j} \wedge \cdots \wedge \mathrm{d}\zeta_n,$$
$$\Phi(\zeta) = \mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$$

and C_n is a constant depending only on n such that for any $\psi \in C_0^{n,0}(\mathbf{C}^n)$,

$$\psi(w) = \int \psi(z) \wedge \overline{\partial} k(z, w) \tag{2.1}$$

holds, i.e., the distributional derivative $\overline{\partial} k$ is supported on the diagonal.

2.5 Representation of $K_M(z,w)$

Let (M,ω) be a complete Kähler manifold. Let $w\in M$ be fixed and take a coordinate ball $B_{2r}=\{|\zeta|<2r\}$ around w. Let $\rho_w\in C_0^\infty(B_r)$, $\varrho_w\in C_0^\infty(B_{2r})$ such that $\rho_w|_{B_{\frac{r}{2}}}=1$, $\varrho_w|_{B_{\frac{r}{3}}}=0$ and $\varrho_w=1$ on $B_r-B_{\frac{r}{2}}$ (if w is changed, r might be changed). Applying Stock's theorem, we obtain the following formula for any $\psi\in\mathcal{H}_2^{n,0}(M)$:

$$\begin{aligned}\psi(w)=\rho_w(w)\psi(w)&=\int\rho_w(\zeta)\psi(\zeta)\wedge\overline{\partial}k(\zeta,w)\\&=(-1)^{n+1}\int\overline{\partial}(\rho_w(\zeta)\psi(\zeta))\wedge k(\zeta,w)\\&=(-1)^{n+1}\int\varrho_w(\zeta)\overline{\partial}(\rho_w(\zeta)\psi(\zeta))\wedge k(\zeta,w)\\&=\int\psi(\zeta)\wedge\rho_w(\zeta)\overline{\partial}(\varrho_w(\zeta)k(\zeta,w)).\end{aligned}$$

It follows that

$$\psi(w)=(\psi,P(\overline{\rho_w\overline{\partial}(\varrho_w k(\,\cdot\,,w))})),\quad\forall\psi\in\mathcal{H}_2^{n,0}(M),$$

where P is the Bergman projection. The uniqueness of reproducing kernel guarantees

$$K_M(z,w)=P(\overline{\rho_w(z)\overline{\partial}(\varrho_w(z)k(\,\cdot\,,w))})$$

(compare [7]).

2.6 Proof of Theorem 1.3

By the hypothesis of the theorem, we may choose the r in Subsection 2.5 sufficiently small so that the support of $\varrho_w|_{\Gamma_f}$ is contained in $\bigcup_j B_\epsilon(p_j,p_j)$ for some $\epsilon>0$, where p_j are fixed points and $B_\epsilon(p_j,p_j)$ are geodesic balls around (p_j,p_j) in $M\times M$. Set $\eta_w=\rho_w\overline{\partial}(\varrho_w k(\,\cdot\,,w))$. Note that for each fixed w, $\overline{\eta}_w(z)$ is a form of type $(n,0)$ w.r.t. z, it follows from Subsection 2.1 that

$$(-1)^n\int_M\overline{K_M(z,f(z))}=(-1)^n\int_M\eta_{f(z)}(z)-2(-1)^n\int_M\overline{\overline{\partial}^*G\overline{\partial}\overline{\eta}_{f(z)}(z)}.$$

Now if we set $w_j=z_j-f(z_j)$ in some local coordinate z_j around p_j, then

$$\Phi(w_j)=\det(I-J_f)\Phi(z_j).$$

If ϵ is sufficiently small, then the support of $\eta_{f(z)}(z)$ is a finite number of small balls centered at p_j. Therefore, for $r\ll\epsilon$ we have

$$\begin{aligned}(-1)^n\int_M\eta_{f(z)}(z)&=(-1)^n\int_{\{|z-f(z)|<\frac{r}{2}\}}\overline{\partial}(\varrho_\zeta k(\,\cdot\,,\zeta))|_{\zeta=f(z)}\\&=(-1)^nC_n\sum_j\int_{\{|w_j|=\frac{r}{2}\}}\frac{\sum_i\overline{\Phi_i(w_j)}\wedge\Phi(w_j+f(z_j))}{|w_j|^{2n}}\quad\text{(by Stokes theorem)}\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n C_n \sum_j \int_{\{|w_j|=\frac{r}{2}\}} \frac{\sum_i \overline{\Phi_i(w_j)} \wedge \Phi(z_j)}{|w|^{2n}} \\
&= (-1)^n C_n \sum_j \int_{\{|w_j|=\frac{r}{2}\}} \frac{\sum_i \overline{\Phi_i(w_j)} \wedge \Phi(w_j)}{|w_j|^{2n} \det(I - J_f)} \\
&\to \sum_j \frac{1}{\det(I - J_f)(p_j)}, \quad r \to 0,
\end{aligned}
$$

because

$$
\begin{aligned}
(-1)^n C_n \int_{\{|w|=\frac{r}{2}\}} \frac{\sum \overline{\Phi_i(w)} \wedge \Phi(w)}{|w|^{2n}} &= (-1)^n C_n \int_{\{|w|=\frac{r}{2}\}} \frac{\Phi(w) \wedge \sum \overline{\Phi_i(w)}}{|w|^{2n}} \\
&= \int_{\{|w|=\frac{r}{2}\}} \Phi(w) \wedge \overline{\partial} \Big\{ C_n \frac{\sum \overline{\Phi_i(w)}}{|w|^{2n}} \Big\} \\
&= \int \chi_{\{|w|<\frac{r}{2}\}} \Phi(w) \wedge \overline{\partial} \Big\{ C_n \frac{\sum \overline{\Phi_i(w)}}{|w|^{2n}} \Big\} \\
&= 1,
\end{aligned}
$$

where $\chi_{\{\,\cdot\,\}}$ denotes the characteristic function and the last equality follows from (2.1) (passing to a C^∞ regularization if necessary). On the other hand,

$$
\begin{aligned}
\int_M \overline{\overline{\partial}^* G \overline{\partial} \overline{\eta}_{f(z)}(z)} &= \int_{M_z} \int_{M_w} \overline{\overline{\partial}^* G \overline{\partial} \overline{\eta}_w(z)} \wedge \overline{\partial} k(w, f(z)) \\
&= \int_{M_w} \int_{M_z} \overline{\overline{\partial}^* G \overline{\partial} \overline{\eta}_w(z)} \wedge \overline{\partial} k(w, f(z)) \quad \text{(by Fubini's theorem)} \\
&= \int_{M_w} \int_{M_z} \overline{G \overline{\partial} \overline{\eta}_w(z)} \wedge \overline{\partial}^2 k(w, f(z)) \\
&= 0.
\end{aligned}
$$

The proof is completed.

3 Proof of Theorem 1.2

Let $L^2_{p,q}(\Omega)$ denote the space of square-integrable (p,q)-forms with respect to the Lebesgue measure and let $\mathcal{H}^2_{p,q}(\Omega)$ be the corresponding L^2-harmonic spaces. Set $\mathcal{D}_{p,q}$ the space of compactly supported smooth forms. We always omit the lower subscript when $p = q = 0$. It follows from Hörmander [6] that there is a bounded operator $N : L^2_{p,q}(\Omega) \to L^2_{p,q}(\Omega)$, the Neumann operator, such that

(1) $N(\mathcal{H}^2_{p,q}(\Omega)) = 0$, $\overline{\partial} N = N \overline{\partial}$, $\overline{\partial}^* N = N \overline{\partial}^*$;

(2) $N(\mathcal{D}_{p,q}) \subset \mathcal{D}_{p,q}$;

(3) $Pg = g - \overline{\partial}^* N \overline{\partial} g$ holds for any $g \in L^2(\Omega)$ with $\overline{\partial} g \in L^2_{0,1}(\Omega)$.

Here $P : L^2(\Omega) \to \mathcal{H}^2(\Omega)$ denotes the Bergman projection and $\overline{\partial}^*$ denotes the adjoint of $\overline{\partial}$ with respect to the Lebesgue measure.

Set $\mathrm{d}V_z = \overline{\Phi(z)} \wedge \Phi(z)$ and denote by $K^*_\Omega(z,w)$ the Bergman kernel function of Ω. From the argument in Subsection 2.5, we see

$$
K^*_\Omega(z,w) = P(\overline{\lambda}_w)(z)
$$

with

$$\lambda_w(z)=\frac{\eta_w(z)}{\overline{\Phi(z)}\wedge\Phi(w)}.$$

Thus

$$\begin{aligned}&(-1)^n\int_\Omega\overline{K_\Omega(z,f(z))}\\&=(-1)^n\int_\Omega\overline{K^*_\Omega(z,f(z))}\det J_f(z)\mathrm{d}V_z\\&=(-1)^n\int_\Omega\overline{P(\overline{\lambda}_{f(z)})(z)}\det J_f(z)\mathrm{d}V_z\\&=(-1)^n\int_\Omega\lambda_{f(z)}(z)\det J_f(z)\mathrm{d}V_z-(-1)^n\int_\Omega\overline{\overline{\partial}^*N\overline{\partial}\,\overline{\lambda}_{f(z)}(z)}\det J_f(z)\mathrm{d}V_z.\end{aligned}$$

We may choose r sufficiently small so that the support of $\lambda_w|_{\Gamma_f}$ is contained in $\bigcup_j B_\epsilon(p_j,p_j)$ for some $\epsilon>0$, where $B_\epsilon(p_j,p_j)$ denotes the Euclidean ball centered at (p_j,p_j) with radius ϵ. Observe that

$$(-1)^n\int_\Omega\lambda_{f(z)}(z)J_f(z)\mathrm{d}V_z=(-1)^n\int_\Omega\eta_{f(z)}(z)\to\sum_{f(p_j)=p_j}\frac{1}{\det(I-J_f)(p_j)},\quad r\to0$$

by Subsection 2.6, while

$$\begin{aligned}\int_\Omega\overline{\overline{\partial}^*N\overline{\partial}\,\overline{\lambda}_{f(z)}(z)}\det J_f(z)\mathrm{d}V_z&=\int_\Omega\int_\Omega\overline{\overline{\partial}^*N\overline{\partial}\,\overline{\lambda}_w(z)}\overline{\partial}\Big(\frac{\det J_f(z)k(w,f(z))}{\mathrm{d}V_w}\Big)\mathrm{d}V_w\mathrm{d}V_z\\&=\int_\Omega\int_\Omega\overline{\overline{\partial}^*N\overline{\partial}\,\overline{\lambda}_w(z)}\overline{\partial}\Big(\frac{\det J_f(z)k(w,f(z))}{\mathrm{d}V_w}\Big)\mathrm{d}V_z\mathrm{d}V_w\\&=0\end{aligned}$$

for all sufficiently small r, since N maps $\mathcal{D}_{0,1}$ to $\mathcal{D}_{0,1}$. The proof is completed.

Remark 3.1 The reason why we call Theorem 1.2 a Lefschetz fixed point formula lies in the following: Fix a positive C^∞ strictly plurisubharmonic exhaustion function ρ on Ω and set $\omega=\partial\overline{\partial}\rho^2$. Then ω is a complete Kähler metric such that

$$|\partial\rho^2|^2_\omega\le2\rho^2$$

which is Kähler convex in the sense of McNeal [9], hence L^2-harmonic forms vanish outside the middle degree, and we still have

$$L_\omega(f)=(-1)^n\int_\Omega\overline{K_\Omega(z,f(z))}.$$

4 Applications and Remarks

4.1 Variations of Theorems 1.2 and 1.3

Corollary 4.1 *Let $\Omega\subset\subset\mathbf{C^n}$ be a domain of holomorphy and let f,g be holomorphic automorphisms such that the closure of the graph $\Gamma_{f\circ g^{-1}}$ of $f\circ g^{-1}$ does not meet the diagonal at the boundary of $\Omega\times\Omega$. Then*

$$(-1)^n\int_\Omega\overline{K_\Omega(g(z),f(z))}=\sum_{g^{-1}\circ f(p_j)=p_j}\frac{1}{\det(I-J_{g^{-1}\circ f})(p_j)}.\tag{4.1}$$

Proof In fact, from (1.1),

$$\begin{aligned}(-1)^n \int_\Omega \overline{K_\Omega(g(z), f(z))} &= (-1)^n \int_\Omega \overline{K^*_\Omega(g(z), f(z)) \det J_g(z)} \det J_f(z) \mathrm{d}V_z \\ &= (-1)^n \int_\Omega \overline{K^*_\Omega(z, g^{-1} \circ f(z))} \det J_{g^{-1}\circ f}(z) \mathrm{d}V_z \\ &= (-1)^n \int_\Omega \overline{K_\Omega(z, g^{-1} \circ f(z))} \\ &= \sum_{f(p_j)=g(p_j)} \frac{1}{\det(I - J_{g^{-1}\circ f})(p_j)},\end{aligned}$$

where the second equality follows from the well-known translation formula for the Bergman kernel function.

Similarly, we have

Corollary 4.2 *Let (M, ω) be an n-dimensional Kähler hyperbolic manifold and let f, g be holomorphic automorphisms such that the closure of the graph $\Gamma_{f\circ g^{-1}}$ of $f \circ g^{-1}$ does not meet the diagonal at the ideal boundary of $M \times M$. Then* (4.1) *holds.*

4.2 Examples

A basic difference between the fixed point formulas of the classical and ours is that in some cases, the left side of (1.1) is computable. This enables us calculate some seemingly complicated integrals.

Example 4.1 Let D denote the unit disk in $\mathbf{C}$. We claim

$$\int_D \frac{1}{(1 - \frac{3}{5}z - \frac{3}{5}\overline{z} + |z|^2)^2} \mathrm{d}V_z = \frac{25\pi}{32}.$$

Indeed, we can take $f(z) = \frac{\frac{3}{5} - z}{1 - \frac{3}{5}z} \in \mathrm{Aut}(D)$, $\mathrm{Aut}(D)$ being the automorphism group of D. It is easy to see that $\frac{1}{3}$ is the only fixed point of f in D and f has not fixed points on the boundary of D. Thus by (1.1), we have

$$\begin{aligned}-\int_\Omega \overline{K_\Omega(z, f(z))} &= -\frac{1}{\pi} \int_D \frac{1}{(1 - \overline{z} f(z))^2} J_f(z) \mathrm{d}V_z \\ &= \frac{16}{25\pi} \int_D \frac{1}{(1 - \frac{3}{5}z - \frac{3}{5}\overline{z} + |z|^2)^2} \mathrm{d}V_z \\ &= \frac{1}{1 + \frac{16}{25(1 - \frac{3}{5}z)^2|_{z=\frac{1}{3}}}} = \frac{1}{2}.\end{aligned}$$

Example 4.2 Let $B_2^* := \{z = (z_1, z_2) \in \mathbf{C}^2, |z|^2 + |z \cdot z| < 1\}$ be the minimal ball in $\mathbf{C}^2$, where $z \cdot z = z_1^2 + z_2^2$. We shall show

$$\int_{B_2^*} \frac{3(1 - \phi(z))^2(1 + \phi(z)) + (z_1^2 + z_2^2)(\overline{w_1^2 + w_2^2})(5 - 3\phi(z))}{((1 - \phi(z))^2 - (z_1^2 + z_2^2)(\overline{w_1^2 + w_2^2}))^3} \mathrm{d}V_z = \frac{\pi^2}{4 - 2\sqrt{2}},$$

where $\phi(z) = \frac{\sqrt{2}}{2}(|z_1|^2 - \overline{z}_1 z_2 \mathrm{i} - \overline{z}_2 z_1 \mathrm{i} + |z_2|^2)$, $w_1 = \frac{\sqrt{2}}{2}(z_1 + z_2\mathrm{i})$, $w_2 = \frac{\sqrt{2}}{2}(z_2 + z_1\mathrm{i})$. It is verified as follows: clearly, the automorphism

$$z = (z_1, z_2), \quad f(z) = (z_1, z_2)\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$

satisfies the hypothesis of Theorem 1.3. On the other hand, it is known from [10] that the Bergman kernel function has the following explicit expression:

$$K_{B_2^*}(z,w)=\frac{2}{\pi^2}\frac{3(1-\langle z,w\rangle)^2(1+\langle z,w\rangle)+(z\cdot z)\overline{w\cdot w}(5-3\langle z,w\rangle)}{((1-\langle z,w\rangle)^2-(z\cdot z)\overline{w\cdot w})^3}.$$

Theorem 1.3 implies

$$\begin{aligned}\int_{B_2^*}\overline{K_\Omega(z,f(z))}&=\int_{B_2^*}K_{B_2^*}(z,f(z))J_f(z)\mathrm{d}V_z\\&=\frac{2}{\pi^2}\int_{B_2^*}\frac{3(1-\phi(z))^2(1+\phi(z))+(z_1^2+z_2^2)(\overline{w_1^2+w_2^2})(5-3\phi(z))}{((1-\phi(z))^2-(z_1^2+z_2^2)(\overline{w_1^2+w_2^2}))^3}\mathrm{d}V_z\\&=\frac{1}{2-\sqrt{2}},\end{aligned}$$

where $\phi(z)=\frac{\sqrt{2}}{2}(|z_1|^2-\overline{z}_1z_2\mathrm{i}-\overline{z}_2z_1\mathrm{i}+|z_2|^2)$, $w_1=\frac{\sqrt{2}}{2}(z_1+z_2\mathrm{i})$, $w_2=\frac{\sqrt{2}}{2}(z_2+z_1\mathrm{i})$.

Remark 4.1 Our theorems exclude the case when the automorphism has fixed points on the boundary. It is probable that the latter might be settled by using Kohn's theory for $\overline{\partial}$-Neumann problem, at least for the special case of the strongly pseudoconvex domains.

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