Chin. Ann. of Math.
2 (1) 1981

A NOTE ON SOME METRICAL THEOREMS IN DIOPHANTINE APPROXIMATION

WANG YUAN YU KUNRUI
(*Institute of Mathematics, Academia Sinica*)

§ 1. Introduction. Suppose that A is a set in n-dimensional Euclidean space R_n. If $(x_1, \cdots, x_n) \in A$ implies that $(x'_1, \cdots, x'_n) \in A$ for any $0 \leqslant x'_i \leqslant x_i (1 \leqslant i \leqslant n)$, then A is said to have property $\mathscr{P}$ which was introduced by Gallagher[1]. Denote by $|A|$ the measure of a measurable set A and by G_n the n-dimensional unit cube, i. e., the set of all points $(x_1, \cdots, x_n)$ with $0 \leqslant x_i < 1 (1 \leqslant i \leqslant n)$. We also use ε to denote any pre-assigned positive number.

Theorem 1. *Let* $A_q (q=1, 2, \cdots)$ *be a sequence of measurable subsets of* G_n. *Suppose that each* A_q *has property* $\mathscr{P}$ *and* $\psi(q) = |A_q|$ *is a decreasing function of* q. *Let* $N(h, \theta_1, \cdots, \theta_n)$ *be the number of integers satisfying* $1 \leqslant q \leqslant h$ *and*

$$(\{q\theta_1\}, \cdots, \{q\theta_n\}) \in A_q. \tag{1}$$

Further let

$$\Psi(h) = \sum_{q=1}^{h} \psi(q) \quad \text{and} \quad \Omega(h) = \sum_{q=1}^{h} \psi(q) q^{-1}.$$

Then for almost all $(\theta_1, \cdots, \theta_n) \in R_n$, *we have*

$$N(h, \theta_1, \cdots, \theta_n) = \Psi(h) + O(\Psi(h)^{\frac{1}{2}} \Omega(h)^{\frac{1}{2}} (\log \Psi(h))^{2+\varepsilon}). \tag{2}$$

Let $\boldsymbol{q} = (q_1, \cdots, q_m)$ and $\boldsymbol{r} = (r_1, \cdots, r_m)$ denote the lattice points in R_m, where q_i's and r_i's are positive integers, $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_m)$ a point in R_m, $\boldsymbol{q\theta} = q_1\theta_1 + \cdots + q_m\theta_m$ the scalar product of $\boldsymbol{q}$ and $\boldsymbol{\theta}$ and $d(\boldsymbol{q}) = \sum_{\substack{d \mid q_i \\ 1 \leqslant i \leqslant m}} 1$. We also use $\boldsymbol{q} \leqslant h$ to denote $q = \max(q_1, \cdots, q_m) \leqslant h$. Similarly, we may define $\boldsymbol{q} < h$, $\boldsymbol{q} \geqslant h$ and $\boldsymbol{q} > h$.

Theorem 2. *Let* $\{A_{\boldsymbol{q}}\}$ *be a sequence of measurable subsets of* G_n *and each* $A_{\boldsymbol{q}}$ *have property* $\mathscr{P}$. *Put* $\psi(\boldsymbol{q}) = |A_{\boldsymbol{q}}|$ *and denote by* $N(h, \boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n)$ *the number of lattice points* $\boldsymbol{q}$ *satisfying* $\boldsymbol{q} \leqslant h$ *and*

$$(\{\boldsymbol{q\theta}_1\}, \cdots, \{\boldsymbol{q\theta}_n\}) \in A_{\boldsymbol{q}} \tag{3}$$

where $\boldsymbol{\theta}_i = (\theta_{i1}, \cdots, \theta_{im}) (1 \leqslant i \leqslant n)$. *Set*

$$\Psi(h) = \sum_{\boldsymbol{q} \leqslant h} \psi(\boldsymbol{q}) \quad \text{and} \quad \chi(h) = \sum_{\boldsymbol{q} \leqslant h} \psi(\boldsymbol{q}) \, d(\boldsymbol{q}).$$

Then

$$N(h, \boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n) = \Psi(h) + O(\chi(h)^{\frac{1}{2}} (\log \chi(h))^{\frac{3}{2}+\varepsilon}) \tag{4}$$

holds for almost all $(\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n) = (\theta_{11}, \cdots, \theta_{1m}, \cdots, \theta_{n1}, \cdots, \theta_{nm}) \in R_{nm}$.

Manuscript received May 28, 1980.

Theorem 1 gives a modification of a theorem of Gallagher[1]. Take A_q to be the set of points $(x_1, \cdots, x_n)$ satisfying $0 \leqslant x_i < \psi_i(q)$ $(1 \leqslant i \leqslant n)$ and suppose that $\psi(q) = \prod_{i=1}^{n} \psi_i(q)$ is a decreasing function of q in Theorem 1. Take A_q to be the set of points such that $0 \leqslant x_i < \psi_i(q)$ $(1 \leqslant i \leqslant n)$ in Theorem 2. Then we derive two theorems of Schmidt [2].

For any positive number τ, let $E(\tau)$ be the set of points $(x_1, \cdots, x_n)$ satisfying

$$0 \leqslant x_i < \frac{1}{2} \quad (1 \leqslant i \leqslant n) \quad \text{and} \quad \tau x_1 \cdots x_n < 1.$$

Then it is easily proved by mathematical induction that

$$|E(\tau)| = \begin{cases} 2^{-n}, \text{ for } \tau \leqslant 2^n \\ \tau^{-1} \sum_{s=0}^{n-1} \frac{1}{s!} \left(\log \frac{\tau}{2^n}\right)^s, \quad \text{otherwise.} \end{cases}$$

Take $A_q = E(q(\log q)^n)$. Then $q(\log q)^n > 2^n$ and

$$\psi(q) = |A_q| = (q(\log q)^n)^{-1} \sum_{s=0}^{n-1} \frac{1}{s!} \left(\log\left(\frac{q(\log q)^n}{2^n}\right)\right)^s$$

for $q \geqslant 8$. Hence

$$\Psi(h) = \sum_{s=0}^{n-1} \frac{1}{s!} \sum_{q=8}^{h} (q(\log q)^n)^{-1} \left(\log\left(\frac{q(\log q)^n}{2^n}\right)\right)^s + O(1)$$

$$= \frac{1}{(n-1)!} \sum_{q=2}^{h} \frac{1}{q \log q} + O(1) = \frac{1}{(n-1)!} \log\log h + O(1).$$

Obviously

$$\Omega(h) = O(1).$$

Let $N(h, \theta_1, \cdots, \theta_n)$ denote the number of integers satisfying $1 \leqslant q \leqslant h$ and the inequalities

$$0 \leqslant \{q\theta_i\} < \frac{1}{2} \ (1 \leqslant i \leqslant n) \quad \text{and} \quad \left(\prod_{i=1}^{n} \{q\theta_i\}\right) q(\log q)^n < 1.$$

Then it follows by Theorem 1 that

$$N(h, \theta_1, \cdots, \theta_n) = \frac{1}{(n-1)!} \log\log h + O((\log\log h)^{\frac{1}{2}} (\log\log\log h)^{2+\varepsilon}) \tag{5}$$

holds for almost all $(\theta_1, \cdots, \theta_n) \in R_n$. Let $\|x\|$ denote the distance from the real number x to the nearest integer. Then from (5) with some simple combinatorial considerations, we may derive

Theorem 3. *Let $N^*(h, \theta_1, \cdots, \theta_n)$ be the number of integers q satisfying*

$$\left(\prod_{i=1}^{n} \|q\theta_i\|\right) q(\log q)^n < 1, \quad 1 \leqslant q \leqslant h. \tag{6}$$

Then for almost all $(\theta_1, \cdots, \theta_n) \in R_n$, we have

$$N^*(h, \theta_1, \cdots, \theta_n) = \frac{2^n}{(n-1)!} \log\log h + O((\log\log h)^{\frac{1}{2}} (\log\log\log h)^{2+\varepsilon}). \tag{7}$$

Similarly, we may derive from Theorem 1 that for almost all $(\theta_1, \cdots, \theta_n) \in R_n$, the inequality

$$\left(\prod_{i=1}^{n} \|q\theta_i\|\right) q(\log q)^{n+\varepsilon} < 1 \tag{8}$$

has only finitely many positive integer solutions.

Let $f(1)=1$ and $f(k)=k(\log k)^n$ or $k(\log k)^{n+\varepsilon}(k>1)$. Put $\bar{x}=\max(1, |x|)$. Take $A_q=E(f(q_1)\cdots f(q_m))$ in Theorem 2. Then we may obtain for almost all $(\theta_{11}, \cdots, \theta_{nm})\in R_{nm}$ the asymptotic formula of the number of lattice points $\boldsymbol{q}$ satisfying $\max\limits_{1\leqslant i\leqslant m}|q_i|\leqslant h$ and

$$\prod_{i=1}^{n}\|\theta_{i1}q_1+\cdots+\theta_{im}q_m\|\prod_{j=1}^{m}f(\bar{q}_j)<1. \tag{9}$$

From this formula we may derive

Theorem 4. *The inequality*

$$\prod_{i=1}^{n}\|\theta_{i1}q_1+\cdots+\theta_{im}q_m\|\prod_{j=1}^{m}(\overline{q_j(\log \bar{q}_j)^{n+\varepsilon}})<1 \tag{10}$$

has only finitely many integral solutions for almost all $(\theta_{11}, \cdots, \theta_{nm})\in R_{nm}$, *but*

$$\prod_{i=1}^{n}\|\theta_{i1}q_1+\cdots+\theta_{im}q_m\|\prod_{j=1}^{m}(\overline{q_j(\log \bar{q}_j)^{n}})<1 \tag{11}$$

has infinitely many integral solutions for almost all $(\theta_{11}, \cdots, \theta_{nm})\in R$.

Especially, it follows from Theorem 4 that property A in Schmidt and Wang's[4] transference theorem holds for almost all $(\theta_{11}, \cdots, \theta_{nm})\in R_{nm}$.

The proofs of Theorems 1 and 2 are based on the method of Schmidt[1] and we may also treat the similar problems in non-linear diophantine approximation (Cf. Schmidt [3]).

Remark. We propose two conjectures.

1° Suppose that $\alpha_{11}, \cdots, \alpha_{nm}$ are nm real algebraic numbers such that $1, \alpha_{11}, \cdots, \alpha_{nm}$ are linearly independent over rational field Q. Then

$$\prod_{i=1}^{n}\|\alpha_{i1}q_1+\cdots+\alpha_{im}q_m\|\prod_{j=1}^{m}\bar{q}_j^{1+\varepsilon}\gg 1.$$

2° Suppose that $\beta_{ij}=e^{r_{ij}}$, where $r_{ij}(1\leqslant i\leqslant n, 1\leqslant j\leqslant m)$ are nm different rational numbers. Then

$$\prod_{i=1}^{n}\|\beta_{i1}q_1+\cdots+\beta_{im}q_{im}\|\prod_{j=1}^{m}q_j^{1+\varepsilon}\gg 1.$$

Put $n=1$ or $m=1$. Then 1° and 2° are Theorems of W. M. Schmidt[5] and A. Baker[6] respectively.

§ 2. Proof of Theorem 1. If $\sum\psi(q)<\infty$, the theorem can be proved by Borel-Cantelli's lemma (Cf. Gallagher [1]). Now we suppose that $\sum\psi(q)$ diverges. Evidently, we may confine ourselves to the case $(\theta_1, \cdots, \theta_n)\in G_n$. Let $\varphi(k, q)$ denote the number of integers l satisfying $0\leqslant l<q$ and $(l, q)\leqslant k$. Put

$$\beta(q, \theta_1, \cdots, \theta_n)=\begin{cases}1, & \text{if } (\theta_1, \cdots, \theta_n)\in A_q,\\ 0, & \text{otherwise},\end{cases}$$

$$\gamma(q, \theta_1, \cdots, \theta_n)=\sum_{p_i,\ 1\leqslant i\leqslant n}\beta(q, q\theta_1-p_1, \cdots, q\theta_n-p_n),$$

$$\gamma(k, q, \theta_1, \cdots, \theta_n)=\sum_{\substack{p_i,\ (p_i, q)\leqslant k\\ 1\leqslant i\leqslant n}}\beta(q, q\theta_1-p_1, \cdots, q\theta_n-p_n),$$

$$I(q)=\int_0^1\cdots\int_0^1\gamma(q, \theta_1, \cdots, \theta_n)d\theta_1\cdots d\theta_n,$$

$$I(k, q)=\int_0^1\cdots\int_0^1\gamma(k, q, \theta_1, \cdots, \theta_k)d\theta_1\cdots d\theta_n,$$

$$I(k,\ q,\ r)=\int_0^1\cdots\int_0^1\gamma(k,\ q,\ \theta_1,\ \cdots,\ \theta_n)\gamma(k,\ r,\ \theta_1,\ \cdots,\ \theta_n)d\theta_1\cdots d\theta_n,$$

$$\Psi(u,\ v)=\sum_{q=u+1}^{v}\psi(q),$$

$$N(k,\ u,\ v,\ \theta_1,\ \cdots,\ \theta_n)=\sum_{q=u+1}^{v}\gamma(k,\ q,\ \theta_1,\ \cdots,\ \theta_n)$$

and notice
$$N(v,\ \theta_1,\ \cdots,\ \theta_n)=\sum_{q=1}^{v}\gamma(q,\ \theta_1,\ \cdots,\ \theta_n).$$

Lemma 1.

$$I(q)=\psi(q),\quad I(k,\ q)=\psi(q)\varphi(k,\ q)^n q^{-n}, \tag{12}$$

$$I(k,\ q,\ r)\leqslant\psi(q)\psi(r)+(2^n-1)\psi(q)A(k,\ q,\ r)q^{-1}, \tag{13}$$

where $A(k,\ q,\ r)$ *is the number of integer pairs* $(p,\ s)$ *satisfying*

$$qs-rp=0,\quad 0\leqslant p<q,\quad (p,\ q)\leqslant k,\quad (s,\ r)\leqslant k.$$

Proof

$$\begin{aligned}I(q)&=\int_0^1\cdots\int_0^1\gamma(q,\ \theta_1,\ \cdots,\ \theta_n)d\theta_1\cdots d\theta_n\\&=\int_0^q\cdots\int_0^q q^{-n}\gamma(q,\ q^{-1}\theta_1,\ \cdots,\ q^{-1}\theta_n)d\theta_1\cdots d\theta_n\\&=q^{-n}\sum_{p_i,\ 1\leqslant i\leqslant n}\int_0^q\cdots\int_0^q\beta(q,\ \theta_1-p_1,\ \cdots,\ \theta_n-p_n)d\theta_1\cdots d\theta_n\\&=q^{-n}q^n\int_0^1\cdots\int_0^1\beta(q,\ \theta_1,\ \cdots,\ \theta_n)d\theta_1\cdots d\theta_n=\psi(q).\end{aligned}$$

Similarly

$$\begin{aligned}I(k,\ q)&=q^{-n}\sum_{\substack{p_i,\ (p_i,\ q)\leqslant k\\1\leqslant i\leqslant n}}\int_0^q\cdots\int_0^q\beta(q,\ \theta_1-p_1,\ \cdots,\ \theta_n-p_n)d\theta_1\cdots d\theta_n\\&=\psi(q)\varphi(k,\ q)^n q^{-n}.\end{aligned}$$

Now we proceed to prove (13). Divide the sum

$$\begin{aligned}I(k,\ q,\ r)=&\sum_{\substack{p_i,\ (p_i,\ q)\leqslant k\\s_i,\ (s_i,\ r)\leqslant k\\1\leqslant i\leqslant n}}\int_0^1\cdots\int_0^1\beta(q,\ q\theta_1-p_1,\ \cdots,\ q\theta_n-p_n)\\&\times\beta(r,\ r\theta_1-s_1,\ \cdots,\ r\theta_n-s_n)d\theta_1\cdots d\theta_n\end{aligned}$$

into $n+1$ parts

$$I(k,\ q,\ r)=I_0+\cdots+I_n, \tag{14}$$

where I_j is the sum of all the terms with exactly j indices $i_1,\ \cdots,\ i_j$ satisfying

$$qp_i-rs_i=0.$$

We first estimate I_0.

$$\begin{aligned}I_0\leqslant&\sum_{\substack{p_i,\ s_i\\qs_i-rp_i\neq 0\\1\leqslant i\leqslant n}}\int_0^1\cdots\int_0^1\beta(q,\ q\theta_1-p_1,\ \cdots,\ q\theta_n-p_n)\beta(r,\ r\theta_1-s_1,\ \cdots,\ r\theta_n-s_n)d\theta_1\cdots d\theta_n\\=&\sum_{\substack{p_i,\ s_i\\qs_i-rp_i\neq 0\\1\leqslant i\leqslant n}}\int_{-\frac{p_1}{q}}^{1-\frac{p_1}{q}}\cdots\int_{-\frac{p_n}{q}}^{1-\frac{p_n}{q}}\beta(q,\ q\theta_1',\ \cdots,\ q\theta_n')\\&\times\beta\left(r,\ r\theta_1'-\frac{qs_1-rp_1}{q},\ \cdots,\ r\theta_n'-\frac{qs_n-rp_n}{q}\right)d\theta_1'\cdots d\theta_n^1.\end{aligned}$$

Write $(q,\ r)=d,\ q=dq'$ and $r=dr'$. Then we have $qs_i-rp_i=h_id$. For given $h_i\neq 0,\ p_i$

is determined uniquely modulo q', so we have

$$I_0\leqslant d^n\sum_{\substack{h_i\neq 0\\ 1\leqslant i\leqslant n}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)\,\beta\left(r,\ r\theta_1-\frac{h_1d}{q},\ \cdots,\ r\theta_n-\frac{h_nd}{q}\right)d\theta_1\cdots d\theta_n. \tag{15}$$

Put

$$J(\lambda_1,\ \cdots,\ \lambda_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)\,\beta\left(r,\ r\theta_1-\frac{\lambda_1d}{q},\ \cdots,\ r\theta_n-\frac{\lambda_nd}{q}\right)\times d\theta_1\cdots d\theta_n.$$

We proceed to prove that J decreases as a function of λ_i when $\lambda_i\geqslant 0$ and increases when $\lambda_i\leqslant 0$ for $i=1,\ \cdots,\ n$. Without loss of generality, we may suppose $i=1$. In fact, for fixed $\theta_2,\ \cdots,\ \theta_n$ and $\lambda_2,\ \cdots,\ \lambda_n$, $\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)$ and $\beta\left(r,\ r\theta_1-\frac{\lambda_1d}{q},\ \cdots,\ r\theta_n-\frac{\lambda_nd}{q}\right)$ are the characteristic functions of two intervals on θ_1-axis which have fixed lengths and start from 0 and $\frac{\lambda_1d}{qr}$ respectively. The measure of the overlap of the two intervals is

$$\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)\,\beta\left(r,\ r\theta_1-\frac{\lambda_1d}{q},\ \cdots,\ r\theta_n-\frac{\lambda_nd}{q}\right)d\theta_1.$$

It is a decreasing function of λ_1 for $\lambda_1\geqslant 0$ and an increasing function for $\lambda_1\leqslant 0$. Hence J decreases for $\lambda_1\geqslant 0$ and increases for $\lambda_1\leqslant 0$. From this fact and (15) we have

$$I_0\leqslant d^n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)\,\beta\left(r,\ r\theta_1-\frac{\lambda_1d}{q},\ \cdots,\ r\theta_n-\frac{\lambda_nd}{q}\right)\times d\theta_1\cdots d\theta_n d\lambda_1\cdots d\lambda_n=\psi(q)\psi(r). \tag{16}$$

As for $I_j\,(j\geqslant 1)$, since

$$\sum_{\substack{(p_i,q)\leqslant k\\ (s_i,r)\leqslant k\\ 1\leqslant i\leqslant n}}\ \sum_{\substack{qs_i-rp_i\neq 0\\ 1\leqslant i\leqslant n-j}}\ \sum_{\substack{qs_i-rp_i=0\\ n-j<i\leqslant n}}\int_0^1\cdots\int_0^1\beta(q,\ q\theta_1-p_1,\ \cdots,\ q\theta_n-p_n)$$
$$\times\beta(r,\ r\theta_1-s_1,\ \cdots,\ r\theta_n-s_n)\,d\theta_1\cdots d\theta_n$$
$$=\sum_{\substack{(p_i,q)\leqslant k\\ (s_i,r)\leqslant k\\ 1\leqslant i\leqslant n}}\ \sum_{\substack{qs_i-rp_i\neq 0\\ 1\leqslant i\leqslant n-j}}\ \sum_{\substack{qs_i-rp_i=0\\ n-j<i\leqslant n}}\int_{-\frac{p_1}{q}}^{1-\frac{p_1}{q}}\cdots\int_{-\frac{p_n}{q}}^{1-\frac{p_n}{q}}\beta(q,\ q\theta_1',\ \cdots,\ q\theta_n')$$
$$\times\beta\left(r,\ r\theta_1'-\frac{qs_1-rp_1}{q},\ \cdots,\ r\theta_{n-j}'-\frac{qs_{n-j}-rp_{n-j}}{q},\ r\theta_{n-j+1}',\ \cdots,\ r\theta_n'\right)d\theta_1'\cdots d\theta_n'$$
$$\leqslant A(k,\ q,\ r)^j\sum_{\substack{h_i\neq 0\\ 1\leqslant i\leqslant n-j}}d^{n-j}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)$$
$$\times\beta\left(r,\ r\theta_1-\frac{dh_1}{q},\ \cdots,\ r\theta_{n-j}-\frac{dh_{n-j}}{q},\ r\theta_{n-j+1},\ \cdots,\ r\theta_n\right)d\theta_1\cdots d\theta_n$$
$$\leqslant A(k,\ q,\ r)^j d^{n-j}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)$$
$$\times\beta\left(r,\ r\theta_1-\frac{d\lambda_1}{q},\ \cdots,\ r\theta_{n-j}-\frac{d\lambda_{n-j}}{q},\ r\theta_{n-j+1},\ \cdots,\ r\theta_n\right)d\theta_1\cdots d\theta_n d\lambda_1\cdots d\lambda_{n-j}$$
$$\leqslant A(k,\ q,\ r)^j d^{n-j}\left(\frac{q}{d}\right)^{n-j}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\beta(q,\ q\theta_1,\ \cdots,\ q\theta_n)\,d\theta_1\cdots d\theta_n$$
$$=\psi(q)A(k,\ q,\ r)^j q^{-j},$$

hence

$$I_j \leqslant \binom{n}{j} \psi(q) A(k, q, r) q^{-1}. \tag{17}$$

(13) follows by combining (14), (16) and (17). The lemma is proved.

Lemma 2.

$$\int_0^1 \cdots \int_0^1 N(v, \theta_1, \cdots, \theta_n) d\theta_1 \cdots d\theta_n = \Psi(v), \tag{18}$$

$$\int_0^1 \cdots \int_0^1 N(k, u, v, \theta_1, \cdots, \theta_n) d\theta_1 \cdots d\theta_n = \sum_{q=u+1}^{v} \psi(q) \varphi(k, q)^n q^{-n}, \tag{19}$$

$$\int_0^1 \cdots \int_0^1 N(k, u, v, \theta_1, \cdots, \theta_n)^2 d\theta_1 \cdots d\theta_n \leqslant \Psi(u, v)^2 + 2(2^n - 1) \sum_{q=u+1}^{v} \psi(q) d_k(q), \tag{20}$$

where $d_k(q) = \sum_{d|q, d\leqslant k} 1$.

Proof (18) and (19) follow from (12) immedeately. (20) follows from (13).

The rest part of the proof of Theorem 1 is similar to the proof of Theorem 1 in Schmidt [2], and we omit it here.

§ 3. Proof of Theorem 2. If $\sum \psi(q) < \infty$, the theorem can be proved by Borel-Cantelli's lemma (cf. Gallagher [1]). Now we suppose that $\sum \psi(q)$ diverges. We may also confine ourselves to the case $(\theta_1, \cdots, \theta_n) \in G_{nm}$. Let $\omega(0) = 0$ and $\omega(h)$, $h \geqslant 1$, be an increasing integral-valued function which tends to infinity. Set $S' = \{0\} \cup \{h > 0 \mid \omega(h-1) < \omega(h)\}$, $S'' = \{h \geqslant 0 \mid \omega(h) < \omega(h+1)\}$ and $S = \{\omega(h) \mid h \geqslant 0\}$. For integer $t > 0$, we define intervals of order t to be

$$(u2^t + v_1, \ (u+1)2^t + v_2],$$

where u, v_1, v_2 are non-negative integers such that $v_1 < 2^t$ and v_1, v_2 are the smallest non-negative integers satisfying $u2^t + v_1 \in S$, $(u+1)2^t + v_2 \in S$.

Lemma 3. *Every interval* $(0, x]$ *with* $x \in S$ *can be expressed as union of intervals* $\bigcup I_i$ *of the type described above, where no two of intervals* I_i *are of the same order.*

Proof (cf. [2]).

Put

$$\beta(q, \alpha_1, \cdots, \alpha_n) = \begin{cases} 1, & \text{if } (\alpha_1, \cdots, \alpha_n) \in A_q, \\ 0, & \text{otherwise.} \end{cases}$$

$$\gamma(q, \theta_1, \cdots, \theta_n) = \sum_{p_i, 1 \leqslant i \leqslant n} \beta(q, q\theta_1 - p_1, \cdots, q\theta_n - p_n),$$

$$I(q) = \int_{G_m} \cdots \int_{G_m} \gamma(q, \theta_1, \cdots, \theta_n) d\theta_1 \cdots d\theta_n,$$

$$I(q, r) = \int_{G_m} \cdots \int_{G_m} \gamma(q, \theta_1, \cdots, \theta_n) \gamma(r, \theta_1, \cdots, \theta_n) d\theta_1 \cdots d\theta_n,$$

$$\Psi(u, v) = \sum_{u < q \leqslant v} \psi(q),$$

$$N(u, v, \theta_1, \cdots, \theta_n) = \sum_{u < q \leqslant v} \gamma(q, \theta_1, \cdots, \theta_n),$$

and notice $$N(v, \boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n) = \sum_{q \le v} \gamma(\boldsymbol{q}, \boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n).$$

Lemma 4.
$$I(\boldsymbol{q}) = \psi(\boldsymbol{q}). \tag{21}$$
If $\boldsymbol{q}$ and $\boldsymbol{r}$ are linearly independent (this fact is abbreviated to $\boldsymbol{q}, \boldsymbol{r}, l. i.$), then
$$I(\boldsymbol{q}, \boldsymbol{r}) = \psi(\boldsymbol{q})\psi(\boldsymbol{r}). \tag{22}$$
If $\boldsymbol{q}$ and $\boldsymbol{r}$ are linearly dependent (this fact is abbreviated to $\boldsymbol{q}, \boldsymbol{r}, l. d.$), then
$$I(\boldsymbol{q}, \boldsymbol{r}) \le \psi(\boldsymbol{q})\psi(\boldsymbol{r}) + (2^n - 1) A(q_1, r_1)\psi(\boldsymbol{q}) q_1^{-1}, \tag{23}$$
where $A(q_1, r_1)$ is the number of the integral solutions (p, s) of the equation
$$q_1 s - r_1 p = 0, \quad 0 \le p < q_1.$$

Proof 1) Suppose that $\boldsymbol{q}, \boldsymbol{r}, l. i.$ Without loss of generality we may assume that $\begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix} \ne 0$. Let
$$T = \begin{pmatrix} q_1 & q_2 \cdots q_m \\ r_1 & r_2 \cdots r_m \\ 0 & I^{(m-2)} \end{pmatrix},$$
where $I^{(l)}$ is the $l \times l$ identity matrix. Obviously, $\det T = q_1 r_2 - q_2 r_1$. Write $T\boldsymbol{\theta}_i = \boldsymbol{\xi}_i = (\xi_{i1}, \cdots, \xi_{im})$ $(1 \le i \le n)$ and $MG_m = \{(x_1, \cdots, x_m) \mid 0 \le x_i < M, 1 \le i \le m\}$. Then
$$\begin{aligned} I(\boldsymbol{q}, \boldsymbol{r}) &= M^{-nm} \int_{MG_m} \cdots \int_{MG_m} \gamma(\boldsymbol{q}, \boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n) \gamma(\boldsymbol{r}, \boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n) d\boldsymbol{\theta}_1 \cdots d\boldsymbol{\theta}_n \\ &= M^{-nm} |q_1 r_2 - q_2 r_1|^{-n} \sum_{\substack{p_i, s_i \\ 1 \le i \le n}} \int_{T(MG_m)} \cdots \int_{T(MG_m)} \beta(\boldsymbol{q}, \xi_{11} - p_1, \cdots, \xi_{n1} - p_n) \\ &\quad \times \beta(\boldsymbol{r}, \xi_{12} - s_1, \cdots, \xi_{n2} - s_n) d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_n. \end{aligned} \tag{24}$$
Put $T(MG_m) \times \cdots \times T(MG_m) = D(M)$. Let P_1 and P_2 $(P_1 \subset D(M) \subset P_2)$ be two nm-dimensional parallelepipeds whose surfaces are parallel to the corresponding surfaces of $D(M)$ with distance $\sqrt{nm}$. Since
$$\int_{G_m} \cdots \int_{G_m} \beta(\boldsymbol{q}, \xi_{11}, \cdots, \xi_{n1}) \beta(\boldsymbol{r}, \xi_{12}, \cdots, \xi_{n2}) d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_n = \psi(\boldsymbol{q})\psi(\boldsymbol{r}),$$
$$\begin{aligned} N(P_1)\psi(\boldsymbol{q})\psi(\boldsymbol{r}) &\le \sum_{\substack{p_i, s_i \\ 1 \le i \le n}} \int_{T(MG_m)} \cdots \int_{T(MG_m)} \beta(\boldsymbol{q}, \xi_{11} - p_1, \cdots, \xi_{n1} - p_n) \\ &\quad \times \beta(\boldsymbol{r}, \xi_{12} - s_1, \cdots, \xi_{n2} - s_n) d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_n \\ &\le N(P_2)\psi(\boldsymbol{q})\psi(\boldsymbol{r}), \end{aligned} \tag{25}$$
where $N(P_i)$ is the number of the lattice points in $P_i (i = 1, 2)$. Evidently, there exists a constant c which is independent of M such that
$$N(P_1) \ge |D(M - c)| = (M - c)^{nm} |q_1 r_2 - q_2 r_1|^n, \tag{26}$$
$$N(P_2) \le |D(M + c)| = (M + c)^{nm} |q_1 r_2 - q_2 r_1|^n. \tag{27}$$
It follows from (24)—(27) that
$$\left(\frac{M-c}{M}\right)^{nm} \psi(\boldsymbol{q})\psi(\boldsymbol{r}) \le I(\boldsymbol{q}, \boldsymbol{r}) \le \left(\frac{M+c}{M}\right)^{nm} \psi(\boldsymbol{q})\psi(\boldsymbol{r}).$$
Let $M \to \infty$, then we have (22).

2) Take $\boldsymbol{r}=(1, 0, \cdots, 0)$ and $A_r=G_m$, then $\psi(\boldsymbol{r})=1$. Since all the components of $\boldsymbol{q}$ are positive integers, $\boldsymbol{q}$ and $\boldsymbol{r}$ are linearly independent. Hence (21) follows from (22). (Notice that the proof of (22) depends only on the linear independence of $\boldsymbol{q}$ and $\boldsymbol{r}$.)

3) Suppose $\boldsymbol{q}$, $\boldsymbol{r}$, *l.d.*. Put

$$W=\begin{pmatrix} 1 & \frac{q_2}{q_1} & \cdots & \frac{q_m}{q_1} \\ 0 & & I^{(m-1)} & \end{pmatrix}, \quad \boldsymbol{\xi}_i=W\boldsymbol{\theta}_i\ (1\leqslant i\leqslant n).$$

Evidently, $\det W=1$. We have

$$I(\boldsymbol{q}, \boldsymbol{r})=\sum_{\substack{p_i, s_i\\ 1\leqslant i\leqslant n}}\int_{W(G_m)}\cdots\int_{W(G_m)}\beta(\boldsymbol{q}, q_1\xi_{11}-p_1, \cdots, q_1\xi_{n1}-p_n)$$
$$\times\beta(\boldsymbol{r}, r_1\xi_{11}-s_1, \cdots, r_1\xi_{n1}-s_n)\,d\boldsymbol{\xi}_1\cdots d\boldsymbol{\xi}_n=I_0+\cdots+I_n, \tag{28}$$

where I_j is the sum of all the terms with exactly j indices $i_1, \cdots, i_j$ having $q_1s_i-r_1p_i=0$.

For real $\alpha_1, \cdots, \alpha_n$, by the method similar to the proof of (13), we have

$$\sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n}}\int_{\alpha_1}^{1+\alpha_1}\cdots\int_{\alpha_n}^{1+\alpha_n}\beta(\boldsymbol{q}, q_1\eta_1-p_1, \cdots, q_1\eta_n-p_n)\beta(\boldsymbol{r}, r_1\eta_1-s_1, \cdots, r_1\eta_n-s_n)\,d\eta_1\cdots d\eta_n$$
$$\leqslant\sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n}}\int_{\alpha_1-\frac{p_1}{q_1}}^{1+\alpha_1-\frac{p_1}{q_1}}\cdots\int_{\alpha_n-\frac{p_n}{q_1}}^{1+\alpha_n-\frac{p_n}{q_1}}\beta(\boldsymbol{q}, q_1\eta_1', \cdots, q_1\eta_n')$$
$$\times\beta\left(\boldsymbol{r}, r_1\eta_1'-\frac{q_1s_1-r_1p_1}{q_1}, \cdots, r_1\eta_n'-\frac{q_1s_n-r_1p_n}{q_1}\right)d\eta_1'\cdots d\eta_n'\leqslant\psi(\boldsymbol{q})\psi(\boldsymbol{r}).$$

Taking $\alpha_i=q_1^{-1}\sum_{j=2}^{m}q_j\xi_{ij}\,(1\leqslant i\leqslant n)$, we have

$$I_0\leqslant\int_0^1\cdots\int_0^1 d\xi_{12}\cdots d\xi_{1m}\cdots d\xi_{n2}\cdots d\xi_{nm}\sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n}}\int_{\alpha_1}^{1+\alpha_1}\cdots$$
$$\int_{\alpha_n}^{1+\alpha_n}\beta(\boldsymbol{q}, q_1\xi_{11}-p_1, \cdots, q_1\xi_{n1}-p_n)\beta(\boldsymbol{r}, r_1\xi_{11}-s_1, \cdots, r_1\xi_{n1}-s_n)\,d\xi_{11}\cdots d\xi_{n1}$$
$$\leqslant\psi(\boldsymbol{q})\psi(\boldsymbol{r}). \tag{29}$$

As for $I_j\,(j\geqslant 1)$, we first have

$$\sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n-j}}\ \sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i=0.\\ n-j<i\leqslant n}}\int_{\alpha_1}^{1+\alpha_1}\cdots\int_{\alpha_n}^{1+\alpha_n}\beta(\boldsymbol{q}, q_1\eta_1-p_1, \cdots, q_1\eta_n-p_n)$$
$$\times\beta(\boldsymbol{r}, r_1\eta_1-s_1, \cdots, r_1\eta_n-s_n)\,d\eta_1\cdots d\eta_n$$
$$=\sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n-j}}\ \sum_{\substack{p_i, s_i\\ q_1s_i-r_1p_i=0\\ n-j<i\leqslant n}}\int_{\alpha_1-\frac{p_1}{q_1}}^{1+\alpha_1-\frac{p_1}{q_1}}\cdots\int_{\alpha_n-\frac{p_n}{q_1}}^{1+\alpha_n-\frac{p_n}{q_1}}\beta(\boldsymbol{q}, q_1\eta_1', \cdots, q_1\eta_n')$$
$$\times\beta\left(\boldsymbol{r}, r_1\eta_1'-\frac{q_1s_1-r_1p_1}{q_1}, \cdots, r_1\eta_{n-j}'-\frac{q_1s_{n-j}-r_1p_{n-j}}{q_1}, r_1\eta_{n-j+1}', r_1\eta_n'\right)d\eta_1'\cdots d\eta_n'$$
$$\leqslant\psi(\boldsymbol{q})A(q_1, r_1)^j q_1^{-j}\leqslant\psi(\boldsymbol{q})A(q_1, r_1)q_1^{-1}.$$

Taking $\alpha_i = q_1^{-1}\sum_{j=2}^{m} q_j\xi_{ij}\ (1\leqslant i\leqslant n)$, we have

$$\sum_{\substack{p_i,\, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n-j}} \sum_{\substack{p_i,\, s_i\\ q_1s_i-r_1p_i=0\\ n-j<i\leqslant n}} \int_{W(G_m)}\cdots\int_{W(G_m)} \beta(\boldsymbol{q},\ q_1\xi_{11}-p_1,\ \cdots,\ q_1\xi_{n1}-p_n)$$
$$\times\beta(\boldsymbol{r},\ r_1\xi_{11}-s_1,\ \cdots,\ r_1\xi_{n1}-s_n)\,d\boldsymbol{\xi}_1\cdots d\boldsymbol{\xi}_n$$
$$=\int_0^1\cdots\int_0^1 d\xi_{12}\cdots d\xi_{nm} \sum_{\substack{p_i,\, s_i\\ q_1s_i-r_1p_i\neq 0\\ 1\leqslant i\leqslant n-i}} \sum_{\substack{p_i,\, s_i\\ q_1s_i-r_1p_i=0\\ n-i<i\leqslant n}} \int_{\alpha_1}^{1+\alpha_1}\cdots$$
$$\int_{\alpha_n}^{1+\alpha_n}\beta(\boldsymbol{q},\ q_1\xi_{11}-p_1,\ \cdots,\ q_1\xi_{n1}-p_n)\,\beta(\boldsymbol{r},\ r_1\xi_{11}-s_1,\ \cdots,\ r_1\xi_{n1}-s_n)\,d\xi_{11}\cdots d\xi_{n1}$$
$$\leqslant\psi(\boldsymbol{q})A(q_1,\ r_1)q_1^{-1}.$$

Hence

$$I_j\leqslant\binom{n}{j}\psi(\boldsymbol{q})A(q_1,\ r_1)q_1^{-1}. \tag{30}$$

(23) follows from (28), (29) and (30). The lemma is proved.

Lemma 5.

$$\int_{G_m}\cdots\int_{G_m} N(u,\ v,\ \boldsymbol{\theta}_1,\ \cdots,\ \boldsymbol{\theta}_n)\,d\boldsymbol{\theta}_1\cdots d\boldsymbol{\theta}_n=\Psi(u,\ v), \tag{31}$$

$$\int_{G_m}\cdots\int_{G_m} N(u,\ v,\ \boldsymbol{\theta}_1,\ \cdots,\ \boldsymbol{\theta}_n)^2\,d\boldsymbol{\theta}_1\cdots d\boldsymbol{\theta}_n\leqslant\Psi(u,\ v)^2+2(2^n-1)\sum_{u<\boldsymbol{q}\leqslant v}\psi(\boldsymbol{q})d(\boldsymbol{q}). \tag{32}$$

Proof (31) follows from (21) immediately. Let $\boldsymbol{r}\leqslant\boldsymbol{q}$ denote $r\leqslant q$. Then the left-hand side of (32) is equal to

$$\sum_{u<\boldsymbol{q},\,\boldsymbol{r}\leqslant v}\int_{G_m}\cdots\int_{G_m}\gamma(\boldsymbol{q},\ \boldsymbol{\theta}_1,\ \cdots,\ \boldsymbol{\theta}_n)\,\gamma(\boldsymbol{r},\ \boldsymbol{\theta}_1,\ \cdots,\ \boldsymbol{\theta}_n)\,d\boldsymbol{\theta}_1\cdots d\boldsymbol{\theta}_n$$
$$=\sum_{\substack{u<\boldsymbol{q},\,\boldsymbol{r}\leqslant v\\ \boldsymbol{q},\,\boldsymbol{r},\ l.\,i.}} I(\boldsymbol{q},\ \boldsymbol{r})+\sum_{\substack{u<\boldsymbol{q},\,\boldsymbol{r}\leqslant v\\ \boldsymbol{q},\,\boldsymbol{r},\ l.\,d.}} I(\boldsymbol{q},\ \boldsymbol{r})$$
$$\leqslant\Psi(u,\ v)^2+2(2^n-1)\sum_{\substack{u<\boldsymbol{r}\leqslant\boldsymbol{q}\leqslant v\\ \boldsymbol{q},\,\boldsymbol{r},\ l.\,d.}}\psi(\boldsymbol{q})A(q_1,\ r_1)q_1^{-1} \tag{33}$$

by Lemma 4. If $\boldsymbol{r}$ is linearly dependent of $\boldsymbol{q}$ and satisfies $u<\boldsymbol{r}\leqslant\boldsymbol{q}$, we put

$$\frac{r_1}{q_1}=\cdots=\frac{r_m}{q_m}=\frac{a}{b},\quad (a,\ b)=1,$$

then $b\,|\,q_i(1\leqslant i\leqslant m)$, $\frac{u}{q}<\frac{a}{b}\leqslant 1$. Conversely, if $(a,\ b)=1$, $b\,|\,q_i(1\leqslant i\leqslant m)$, $\frac{u}{q}<\frac{a}{b}\leqslant 1$, then $\boldsymbol{r}=\left(\frac{a}{b}q_1,\ \cdots,\ \frac{a}{b}q_m\right)$ is a lattice point which is linearly dependent of $\boldsymbol{q}$ and satisfies $u<\boldsymbol{r}\leqslant\boldsymbol{q}$.

Hence

$$\sum_{\substack{u<\boldsymbol{r}\leqslant\boldsymbol{q}\leqslant v\\ \boldsymbol{q},\,\boldsymbol{r},\ l.\,d.}}\psi(\boldsymbol{q})\,A(q_1,\ r_1)q_1^{-1}=\sum_{u<\boldsymbol{q}\leqslant v}\psi(\boldsymbol{q})\sum_{\substack{b|q_i\\ 1\leqslant i\leqslant m}}\sum_{\substack{a\\ (a,\,b)=1\\ \frac{u}{q}<\frac{a}{b}\leqslant 1}} A\left(q_1,\ \frac{a}{b}q_1\right)q_1^{-1}$$
$$=\sum_{u<\boldsymbol{q}\leqslant v}\psi(\boldsymbol{q})\sum_{\substack{b|q_i\\ 1\leqslant i\leqslant m}}\sum_{\substack{a\\ (a,\,b)=1\\ \frac{u}{q}<\frac{a}{b}\leqslant 1}}\frac{q_1}{b}q_1^{-1}\leqslant\sum_{u<\boldsymbol{q}\leqslant v}\psi(\boldsymbol{q})\sum_{\substack{b|q_i\\ 1\leqslant i\leqslant m}}\frac{1}{b}b=\sum_{u<\boldsymbol{q}\leqslant v}\psi(\boldsymbol{q})d(\boldsymbol{q}).$$

Combining (33) and the above inequality, we obtain the lemma.

Take $\omega(h)=[\chi(h)]$. Let

$L_s=\{(u, v)\,|\,u\in S', v\in S', (\omega(u), \omega(v)]$ is an interval of any order t, $\omega(v)\leqslant 2^s\}$ and let $h^*=h^*(s)$ be the greatest integer satisfying $\omega(h)\leqslant 2^s$.

Lemma 6.

$$\sum_{(u,v)\in L_s}\int_{G_m}\cdots\int_{G_m}(N(u, v, \theta_1, \cdots, \theta_n)-\Psi(u, v))^2 d\theta_1\cdots d\theta_n=O(s2^s).$$

Proof By Lemma 5, every term in the above sum has upper bound

$$2(2^n-1)\sum_{u<q\leqslant v}\psi(q)d(q).$$

We first sum over all the $(u, v)\in L_s$ for which $(\omega(u), \omega(v)]$ is an interval of fixed order t. Since intervals of order t cover the positive axis exactly once, we have the upper bound

$$2(2^n-1)\chi(h)=O(2^s).$$

Summing over t, the lemma follows.

Lemma 7. *There exists a sequence of subsets $\sigma_1, \sigma_2\cdots$ in G_{nm} with measures*

$$\mu_s=\int_{\sigma_s}d\theta_1\cdots d\theta_n=O(s^{-1-\varepsilon})$$

such that

$$N(h, \theta_1, \cdots, \theta_n)=\Psi(h)+O(2^{\frac{s}{2}}s^{\frac{3}{2}+\varepsilon})$$

holds for any $h\in S'$, $\omega(h)\leqslant 2^s$ and $(\theta_1, \cdots, \theta_n)\in G_{nm}\backslash\sigma_s$.

Proof We define σ_s to be the set of $(\theta_1, \cdots, \theta_n)\in G_{nm}$ for which

$$\sum_{(u,v)\in L_s}(N(u, v, \theta_1, \cdots, \theta_n)-\Psi(u, v))^2\leqslant s^{2+\varepsilon}2^s \tag{34}$$

does not hold. By Lemma 6, we have

$$\mu_s=O(s^{-1-\varepsilon}).$$

If $h\in S'$, $\omega(h)\leqslant 2^s$, then by Lemma 3, $(0, \omega(h)]$ can be expressed as union of at most s intervals $(\omega(u), \omega(v)]$, where $(u, v)\in L_s$. For $(\theta_1, \cdots, \theta_n)\in G_{nm}\backslash\sigma_s$, summing over these (u, v), we obtain

$$(N(h, \theta_1, \cdots, \theta_n)-\Psi(h))^2=(\sum_{(u,v)}(N(u, v, \theta_1, \cdots, \theta_n)-\Psi(u, v)))^2$$
$$\leqslant s\sum_{(u,v)\in L_s}(N(u, v, \theta_1, \cdots, \theta_n)-\Psi(u, v))^2\leqslant s^{3+\varepsilon}2^s$$

by (34) and Cauchy's inequality. The lemma follows.

Proof of Theorem 2. Since $\sum s^{-1-\varepsilon}<\infty$, there exists for almost all $(\theta_1, \cdots, \theta_n)\in G_{nm}$ a $s_0=s_0(\theta_1, \cdots, \theta_n)$ such that $(\theta_1, \cdots, \theta_n)\overline{\in}\sigma_s$ for $s\geqslant s_0$. Suppose that $(\theta_1, \cdots, \theta_n)$ has such a s_0 and h is so large that $\omega(h)\geqslant 2^{s_0}$. Pick s satisfying $2^{s-1}\leqslant\omega(h)<2^s$. Then $s>s_0$.

Suppose that $h\in S'$. Since $(\theta_1, \cdots, \theta_n)\overline{\in}\sigma_s$, Lemma 7 yields

$$N(h, \theta_1, \cdots, \theta_n)=\Psi(h)+O(2^{\frac{1}{2}s}s^{\frac{3}{2}+\varepsilon})=\Psi(h)+O(\chi(h)^{\frac{1}{2}}(\log\chi(h))^{\frac{3}{2}+\varepsilon}),$$

that is, the theorem holds for $h\in S'$. We can prove similarly the theorem for $h\in S''$. For every h there exist $h'\in S'$ and $h''\in S''$ auch that

$$h'\leqslant h\leqslant h'', \quad \omega(h')=\omega(h)=\omega(h'').$$

Then we have

$$|\Psi(h)-\Psi(h')|\leqslant|\chi(h)-\chi(h')|\leqslant 1$$

Similarly

$$|\Psi(h)-\Psi(h'')|\leqslant 1$$

Since $$N(h', \theta_1, \cdots, \theta_n)\leqslant N(h, \theta_1, \cdots, \theta_n)\leqslant N(h'', \theta_1, \cdots, \theta_n),$$

the theorem follows.

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关于丢番图逼近论中的若干测度定理

王　元　于坤瑞
(中国科学院数学研究所)

摘　要

本文证明了两条关于丢番图逼近论中的测度定理. (详细叙述见本文§1.) 这些定理是 P. X. Gallagher 定理的改进,并包有 W. M. Schmidt 的测度定理. 还可以导出,例如:

1° 对于几乎所有的 $(\theta_1, \cdots, \theta_n)\in R_n$, 适合于

$$\prod_{i=1}^{n}\|q\theta_i\| q(\log q)^n<1,\quad 1\leqslant q\leqslant h$$

的整数 q 的个数为

$$\frac{2^n}{(n-1)!}\log\log h+O((\log\log h)^{1/2+\varepsilon}),$$

此处 $\|X\|$ 表示实数 X 至最近整数的距离, ε 为任意正常数, 而与"O"有关的常数依赖于 ε 与诸 θ_i.

2° W. M. Schmidt 与王元的转换定理中的性质 A 对于几乎所有的 $(\theta_{11}, \cdots, \theta_{nm})\in R_{nm}$ 都成立.