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THE UNIFORM INTEGRABILITY OF A CLASS OF EXPONENTIAL MARTINGALES

WANG JIAGANG
(*Fudan University*)

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $F=(\mathscr{F}_t)_{t\in[0,b]}$ $(b\leqslant\infty)$ be a family of increasing sub-σ-fields of $\mathscr{F}$, satisfying the usual conditions. Suppose that $W=\{W_t, 0\leqslant t\leqslant b\}$ is the Wiener process with respect to F, $X=\{X_t, 0\leqslant t\leqslant b\}$ is progressively measurable and

$$P\left\{\int_0^b X_t^2\,dt<\infty\right\}=1,$$

then we can define the Ito-integral

$$(X\cdot W)_t=\int_0^t X_s\,dW_s.$$

If we denote

$$Z_t=\exp\left\{\int_0^t X_s\,dW_s-\frac{1}{2}\int_0^t X_s^2\,ds\right\}, \tag{1}$$

then it is an exponential (local) martingale. In order to apply Girsanov theorem to the problems of equivalence and transformation of measures[1,3,6], it is important to examine the condition for $E[Z_b]=1$, i. e., the uniform integrability of exponential martingale Z. In [5], Kazamaki has proved that, if $X\cdot W$ is a BMO martingale, then $E[Z_b]=1$. In [9], Yen extended the Novikov's results and proved that if

$$\lim_{a\uparrow}\left\{E\left[\exp\left(\frac{a}{2}\int_0^b X_t^2\,dt\right)\right]\right\}^{1-a}=1, \tag{2}$$

then $E[Z_b]=1$. While we discuss the equivalence of Gaussian measures, $X\cdot W$ may not be a BMO martingale even if X is Gaussian. In this case it is also difficult to verify condition (2). The purpose of this paper is to prove that if X is a progressively measurable Gaussian process and the integral of X with respect to W exists, then the Z given by (1) is always uniformly integrable.

In this paper, the definitions and notations accord with those in [8].

In the discussion of equivalence of Gaussian measures, the stochastic integrals with respect to infinite Gaussian processes with independent increments are necessary[3]. So we shall discuss the uniform integrability of exponential martingale, which is more general than (1).

Lemma 1. *Let $X=\{X_t, 0\leqslant t<b\}$ be a real measurable Gaussian process and m_t be*

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an increasing right-continuous function. If

$$\int_0^b \|X_t\| dm_t < \infty,$$

where $\|X_t\| = (E|X_t|^2)^{1/2}$, *then almost every sample function of* X *is integrable with respect to* m_t *and the integral of sample function*

$$\tilde{I} = \int_0^b X_t dm_t$$

is a Gaussian random variable.

Proof If we denote the integral in quadratic mean with respect to m by I (cf. [4] § 4), then I is a Gaussian random variable and for every random variable Y, for which EY^2 exists, we have

$$E\{IY\} = \int_0^b E(X_t Y) dm_t,$$

and in particular, for every bounded random variable Y this relation is true

$$E\{IY\} = \int_0^b E(X_t Y) dm_t = E\left\{\int_0^b X_t Y dm_t\right\} = E\{\tilde{I}Y\},$$

so $\tilde{I} = I$ a.s. and $\tilde{I}$ is a Gaussian random variable.

Lemma 2. *Let* $X = \{X_t, 0 \leqslant t < b\}$ *be a measurable Gaussian process, of which almost every sample function is integrable with respect to increasing right-continuous function* m, *then the integral in trajectory*

$$\tilde{I} = \int_0^b X_t dm_t$$

is a Gaussian random variable.

Proof First, we may assume that b is finite. Set $\|X\| = (E|X_t|^2)^{1/2}$ and define

$$Y_t^\varepsilon = \frac{X_t}{1 + \varepsilon\|X_t\|},$$

then Y_t^ε is also a measurable Gaussian process and $\|Y_t^\varepsilon\| < \frac{1}{\varepsilon}$, therefore

$$\int_0^b \|Y_t^\varepsilon\| dm_t < \infty.$$

By lemma 1, the integral of sample function of Y_t^ε with respect to m $\int_0^b Y_t^\varepsilon dm_t$ is a Gaussian random variable. Suppose $\varepsilon \downarrow 0$. Since

$$|Y_t^\varepsilon| = \frac{|X_t|}{1 + \varepsilon\|X_t\|} < |X_t|,$$

$$\lim_{\varepsilon \downarrow 0} Y_t^\varepsilon = X_t, \quad \text{a. s.}$$

by Lebesgue's convergence theorem, we have

$$\lim_{\varepsilon \downarrow 0} \int_0^b Y_t^\varepsilon dm_t = \int_0^b X_t dm_t, \quad \text{a. s.}$$

hence from lemma 16.10 [7], $\int_0^b X_t dm_t$ is a Gaussian random variable.

If $b=+\infty$, we have $\lim_{c\uparrow+\infty}\int_0^c X_t dm_t=\int_0^\infty X_t dm_t$,

so $\int_0^{+\infty} X_t dm_t$ is also a Gaussian random variable.

Remark. In the same way, we can also prove that if $X=(X_t^{(1)}, X_t^{(2)}, \cdots)$ is a Gaussian process with infinite components, $\{\varphi_j(t), j\geqslant 1\}$ is a system of measurable functions and the integrals in trajectory

$$\xi_{ij}=\int_0^b X_t^{(i)}\varphi_j(t)\,dm_t$$

exist, then $\{\xi_{ij}, i\geqslant 1, j\geqslant 1\}$ is a system of Gaussian random variables.

Lemma 3. *If $\{\xi_n, n\geqslant 1\}$ is a sequence of Gaussian random variables, then*

$$\sum_{i=1}^{\infty} E\xi_i^2\leqslant\left[E\exp\left(-\frac{1}{2}\sum_{i=1}^{\infty}\xi_i^2\right)\right]^{-2}. \tag{3}$$

Furthermore, if $\sum_{i=1}^{\infty}E\xi_i^2<1$, then

$$E\left[\exp\left(\frac{1}{2}\sum_{i=1}^{\infty}\xi_i^2\right)\right]\leqslant\left(1-\sum_{i=1}^{\infty}E\xi_i^2\right)^{-\frac{1}{2}}. \tag{4}$$

Proof First, we suppose that $\xi=(\xi_1, \cdots, \xi_n)^\tau$ is an n-dimensional Gaussian random vector, where $(\ )^\tau$ denotes the transition of vector or matrix. In this case, there exists an n-dimensional random vector η with Gaussian distribution $N(m, I)$ and an $n\times n$ matrix C such that $\xi=C\eta$, so

$$\sum_{i=1}^{n}\xi_i^2=\xi^\tau\xi=\eta^\tau C^\tau C\eta.$$

If necessary, we can transform η by an orthogonal transformation, so we may assume that $C^\tau C=D$ is a diagonal matrix, let $D=\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, then

$$\sum_{i=1}^{n}\xi_i^2=\eta^\tau D\eta=\sum_{j=1}^{n}\lambda_j\eta_j^2,$$

so $\eta_1, \cdots, \eta_n$ are mutually independent and every η_j is a Gaussian random variable with distribution $N(m_j, 1)$. Therefore

$$\begin{aligned}\left\{E\left[\exp\left(-\frac{1}{2}\sum_{j=1}^{n}\xi_j^2\right)\right]\right\}^{-2}&=\left[\prod_{j=1}^{n}E\exp\left(-\frac{1}{2}\eta_j^2\right)\right]^{-2}\\&=\prod_{j=1}^{n}(1+\lambda_j)\exp\left(\frac{\lambda_j m_j^2}{1+\lambda_j}\right)\geqslant\prod_{j=1}^{n}(1+\lambda_j)\left(1+\frac{\lambda_j m_j^2}{1+\lambda_j}\right)\\&=\prod_{j=1}^{n}(1+\lambda_j+\lambda_j m_j^2)\geqslant 1+\sum_{j=1}^{n}\lambda_j(1+m_j^2)\\&=1+E\left(\sum_{j=1}^{n}\lambda_j\eta_j^2\right)=1+\sum_{j=1}^{n}E\xi_j^2.\end{aligned} \tag{5}$$

Furthermore, suppose $\sum_{j=1}^{n}E\xi_j^2<1$, we have also that

$$\begin{aligned}E\left[\exp\left(\frac{1}{2}\sum_{j=1}^{n}\xi_j^2\right)\right]&=\prod_{j=1}^{n}E\left[\exp\left(\frac{1}{2}\lambda_j\eta_j^2\right)\right]=\prod_{j=1}^{n}\left[\frac{1}{1-\lambda_j}\exp\frac{\lambda_j m_j^2}{1-\lambda_j}\right]^{\frac{1}{2}}\\&\leqslant\prod_{j=1}^{n}\left[\frac{1}{1-\lambda_j}\left(1-\frac{\lambda_j m_j^2}{1-\lambda_j}\right)^{-1}\right]^{\frac{1}{2}}\\&\leqslant\left[1-\sum_{j=1}^{n}\lambda_j(1+m_j^2)\right]^{-\frac{1}{2}}=\left(1-\sum_{j=1}^{n}E\xi_j^2\right)^{-\frac{1}{2}}.\end{aligned} \tag{6}$$

By monotone convergence theorem let $N\to\infty$ in (5), (6), we can obtain (3), (4) respectively.

Lemma 4. *Suppose that m_s is a right-continuous increasing function on $[0, b)$ and $X=(X_t^{(i)},\ 0\leqslant t<b,\ 1\leqslant i<\infty)$ is a Gaussian process, if X satisfies*

$$P\Big[\sum_{j=1}^{\infty}\int_0^b (X_s^{(i)})\,dm_s<\infty\Big]=1, \tag{7}$$

then

$$\sum_{i=1}^{\infty}E\int_0^b (X_s^{(i)})^2\,dm_s<\infty. \tag{8}$$

Furthermore, if $\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t<1$, then

$$E\Big\{\exp\Big(\frac{1}{2}\sum_{i=1}^{\infty}\int_0^b (X_t^{(i)})^2\,dm_t\Big)\Big\}\leqslant\Big(1-\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t\Big)^{\frac{1}{2}}.$$

Proof Let $\{\varphi_j(t),\ j\geqslant 1\}$ be an arbitrary completely orthogonal system in $L^2([0, b), dm)$ and define

$$\xi_{ij}=\int_0^b X_t^{(i)}\varphi_j(t)\,dm_t,\quad i\geqslant 1,\ j\geqslant 1,$$

where integrals are taken in trajectory of X. According to the remark of lemma 2, $\{\xi_{ij},\ i\geqslant 1,\ j\geqslant 1\}$ is a system of Gaussian random variables and by Paserval's equality we have

$$\int_0^b (X_t^{(i)})^2\,dm_t=\sum_{j=1}^{\infty}\xi_{ij}^2,\qquad \sum_{i=1}^{\infty}\int_0^b (X_t^{(i)})^2\,dm_t=\sum_{i,j=1}^{\infty}\xi_{ij}^2.$$

Also, by using the monotone convergence theorem we have

$$E\int_0^b (X_t^{(i)})^2\,dm_t=\sum_{j=1}^{\infty}E\xi_{ij}^2,\qquad \sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t=\sum_{i,j=1}^{\infty}E\xi_{ij}^2.$$

So from (3) we hav

$$\begin{aligned}\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t&=\sum_{i,j=1}^{\infty}E\xi_{ij}^2\leqslant\Big[E\exp\Big(-\frac{1}{2}\sum_{i,j=1}^{\infty}\xi_{ij}^2\Big)\Big]^{-2}\\&=\Big[E\exp\Big(-\frac{1}{2}\sum_{i=1}^{\infty}\int_0^b (X_t^{(i)})^2\,dm_t\Big)\Big]^{-2}<\infty.\end{aligned}$$

Furthermore, if $\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t<1$, from (4) we can deduce that

$$\begin{aligned}E\exp\Big(\frac{1}{2}\sum_{i=1}^{\infty}\int_0^b (X_t^{(i)})^2\,dm_t\Big)&=E\exp\Big(\frac{1}{2}\sum_{i,j=1}^{\infty}\xi_{ij}^2\Big)\leqslant\Big(1-\sum_{i,j=1}^{\infty}E\xi_{ij}^2\Big)^{-\frac{1}{2}}\\&=\Big(1-\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t\Big)^{-\frac{1}{2}}.\end{aligned}$$

This completes the proof of the lemma.

Remarks. 1. These lemmas are the extensions and improvements of lemma 7.2 in [6].

2. Lemma 4 means that (7) and (8) are equivalent mutually.

3. From lemma 3 it is easy to see that although $\{\xi_j, j\geqslant 1\}$ may not be mutually independent, the zero-one law is still valid for the determination of whether the series $\sum_{j=1}^{\infty}\xi_j^2$ is convergent or divergent, because if $\sum_{j=1}^{\infty}\xi_j^2$ is convergent with positive probability, then the right side of (3) is finite, hence $\sum_{j=1}^{\infty}E\xi_j^2<\infty$ and so $\sum_{j=1}^{\infty}\xi_j^2<\infty$ a.s. The $\sum_{i=1}^{\infty}\int_0^T(X_t^{(i)})^2dm_t$ is the same as the series $\sum_{j=1}^{\infty}\xi_j^2$.

According to [2], let $W_t=(W_t^{(1)}, \cdots, W_t^{(n)}, \cdots)$ be a process with infinite components, which are continuous Gaussian processes with independent increments. We also suppose that W_t satisfies the following conditions

$$E(W_t^{(i)}-W_s^{(i)})=0,$$
$$E(W_t^{(i)}-W_s^{(i)})(W_t^{(j)}-W_s^{(j)})=\delta_{ij}(m_t^{(i)}-m_s^{(i)}), \tag{9}$$

where every $m_t^{(i)}$ is an increasing continuous function (usually it is also assumed that $dm^{(1)}\gg dm^{(2)}\gg\cdots\gg dm^{(n)}\gg\cdots$, but here we do not make this assumption). Let $f_t=(f_t^{(1)}, \cdots, f_t^{(n)}, \cdots)$, where $f_t^{(i)}(i=1, 2, \cdots, n, \cdots)$ is progressively measurable with respect to $F=(\mathscr{F}_t)$, then under

$$P\left\{\sum_{i=1}^{\infty}\int_0^b(f_t^{(i)})^2dm_t^{(i)}<\infty\right\}=1,$$

we can define the following stochastic integral $f\cdot W$ as in [2].

$$(f\cdot W)_t=\sum_{i=1}^{\infty}\int_0^t f_s^{(i)}dW_s^{(i)},$$

which is a F-local square-integrable martingale and

$$\langle f\cdot W, f\cdot W\rangle_t=\sum_{i=1}^{\infty}\int_0^t(f_s^{(i)})^2dm_s.$$

Theorem. *Let $W=(W_t^{(1)}, \cdots, W_t^{(n)}, \cdots)$ be a process with infinite components which are continuous Gaussian processes with independent increments and satisfy* (9). *Also suppose that $\{f_t=(f_t^{(1)}, \cdots, f_t^{(n)}, \cdots)\}$ is a Gaussian process and for every i, $f_t^{(i)}$ is progressively measurable. If*

$$P\left[\sum_{i=1}^{\infty}\int_0^b(f_t^{(i)})^2dm_t^{(i)}<\infty\right\}=1, \tag{10}$$

then
$$Z_t=\exp\left\{\sum_{i=1}^{\infty}\int_0^t f_s^{(i)}dW_s^{(i)}-\frac{1}{2}\sum_{i=1}^{\infty}\int_0^t(f_s^{(i)})^2dm_s^{(i)}\right\}$$

is a uniformly integrable martingale, in particular, we have

$$E\{Z_b\}=1.$$

Proof On $[0, b)$ every $m^{(i)}$ corresponds to a σ-finite measure which is equivalent to a finite measure $n^{(i)}$. We set

$$m=\sum_{i=1}^{\infty}\frac{1}{n^{(i)}([0, b))2^i}n^{(i)}.$$

Therefore $m^{(i)}\ll m$ and we write $\rho^{(i)}=\frac{dm^{(i)}}{dm}$, which is a nonnegative function on $[0, b)$. Now we define

$$X_t^{(i)}=\sqrt{\rho_t^{(i)}}f_t^{(i)},$$

then we have

$$\sum_{i=1}^{\infty}\int_0^b (X_t^{(i)})^2 dm_t=\sum_{i=1}^{\infty}\int_0^b (f_t^{(i)})^2 dm_t^{(i)}<\infty. \quad \text{a. s.}$$

Lemma 4 implies that

$$\int_0^b \sum_{i=1}^{\infty} E(X_t^{(i)})^2 dm_t=\sum_{i=1}^{\infty} E\int_0^b (X_t^{(i)})^2 dm_t<\infty,$$

hence there exists a finite partition of $[0, b]: 0=t_0<t_1<\cdots<t_n=b$ such that, for every k

$$\int_{t_{k-1}}^{t_k} \sum_{i=1}^{\infty} E(X_t^{(i)})^2 dm_t<1,$$

also by lemma 4, we obtain

$$E\left\{\exp\left(\frac{1}{2}\sum_{i=1}^{\infty}\int_{t_{k-1}}^{t_k}(f_t^{(i)})^2 dm_t^{(i)}\right)\right\}=E\left\{\exp\left(\frac{1}{2}\sum_{i=1}^{\infty}\int_{t_{k-1}}^{t_k}(X_t^{(i)})^2 dm_t\right)\right\}$$
$$\leqslant\left(1-\sum_{i=1}^{\infty}E\int_{t_{k-1}}^{t_k}(X_t^{(i)})^2 dm_t\right)^{-2}<\infty.$$

Write

$$\tilde{Z}_u^v=\exp\left(\sum_{i=1}^{\infty}\int_u^v f_s^{(i)} dW_s^{(i)}-\frac{1}{2}\int_u^v (f_s^{(i)})^2 dm_s^{(i)}\right)$$
$$=\exp\left\{(f\cdot W)_v-(f\cdot W)_u-\frac{1}{2}(\langle f\cdot W, f\cdot W\rangle_v-\langle f\cdot W, f\cdot W\rangle_u)\right\}.$$

For fixed $u=t_{k-1}$, $\{\tilde{Z}_{t_{k-1}}^t, t_{k-1}\leqslant t\leqslant t_k\}$ is an exponential martingale, so by the results of [9], we conclude that $\{\tilde{Z}_{t_{k-1}}^t, t_{k-1}\leqslant t\leqslant t_k\}$ is a uniformly integrable martingale, in particular

$$E(\tilde{Z}_{t_{k-1}}^t)=1, \quad t_{k-1}\leqslant t\leqslant t_k,$$
$$E(\tilde{Z}_{t_{k-1}}^{t_k}|\mathscr{F}_{t_{k-1}})=1, \quad 1\leqslant k\leqslant n.$$

Thus $\quad E(Z_b)=E(\tilde{Z}_0^b)=E(E(\tilde{Z}_{t_{n-1}}^{t_n}|\mathscr{F}_{t_{n-1}})\tilde{Z}_0^{t_{n-1}})=E(\tilde{Z}_0^{t_{n-1}})=\cdots=E(\tilde{Z}_0^{t_1})=1.$

Therefore $\{Z_t, 0\leqslant t<b\}$ is a uniformly integrable martingale, which completes the proof of the theorem.

Corollary. *If $K^{(i)}(u, v)=(K_{ij}(u, v), j\geqslant 1)$, $i\geqslant 1$ are such that*

$$K_{ij}(u, v)=0, \quad u<v,$$
$$\sum_i\sum_j\int_0^b\int_0^u K_{ij}^2(u, v)\, dm_v^{(i)} dm_u^{(j)}<\infty.$$

Let
$$f_t^{(i)}=\sum_{j=1}^{\infty}\int_0^t K_{ij}(t, v)\, dW^{(j)}(v),$$

then
$$Z_t=\exp\left\{\sum_{i=1}^{\infty}\int_0^t f_s^{(i)} dW_s^{(i)}-\frac{1}{2}\sum_{i=1}^{\infty}\int_0^t (f_s^{(i)})^2 dm_s^{(i)}\right\}, \quad 0\leqslant t<b,$$

is a uniformly integrable martingale.

Proof Since

$$\sum_{i=1}^{\infty}E\int_0^b (f_t^{(i)})^2 dm_t^{(i)}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^b\int_0^b K_{ij}^2(u, v)\, dm_v^{(i)} dm_u^{(j)}<\infty,$$

so (10) is satisfied and from the theorem we deduce that $\{Z_t, 0\leqslant t<b\}$ is uniformly integrable.

By using former results and the Hida-Cramer decomposition of Gaussian process, as in [1] we may discuss the equivalence of Gaussian measures and obtain their Radon-Nikodym derivatives.

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一类指数鞅的一致可积性

汪 嘉 冈

(复旦大学)

摘 要

若 $(\Omega, \mathscr{F}, P)$ 为完备概率空间，$F=(\mathscr{F}_t)_{t\in[0,b]}$ 为 $\mathscr{F}$ 的递增子 σ 域族，且满足通常条件，$b\leqslant\infty$. 又 $W=\{W_t,\ 0\leqslant t<b\}$ 为关于 F 的 Wiener 过程，$X=\{X_t,\ 0\leqslant t<b\}$ 为循序可测过程，且

$$P\left\{\int_0^b X_t^2 dt<\infty\right\}=1,$$

则可定义 X 关于 W 的 Ito 随机积分

$$(X\cdot W)_t=\int_0^t X_s dW_s,\quad 0\leqslant t\leqslant b.$$

这时若记

$$Z_t=\exp\left\{\int_0^t X_s dW_s-\frac{1}{2}\int_0^t X_s^2 ds\right\},$$

它便是一个指数(局部)鞅. 本文的目的在于证明当 X 为循序可测正态过程时，只要 X 关于 W 的积分存在，$\{Z_t,\ 0\leqslant t<b\}$ 总是一致可积的.

引理1 若 $X=\{X_t,\ 0\leqslant t<b\}$ 为实可测正态过程，且

$$\int_0^b \|X_t\| dm_t<\infty,$$

其中 $\|X_t\|=(E|X_t|^2)^{1/2}$, m_t 为 $[0, b)$ 上右连续递增函数，则 X 的几乎所有样本函数关于 m_t 可积，且其轨道积分

$$\tilde{I}=\int_0^b X_t\,dm_t$$

为正态分布随机变量.

引理 2 若 $X=\{X_t,\ 0\leqslant t<b\}$ 为可测正态过程，其几乎所有样本函数关于右连续增函数 m_t 可积，即

$$P\left(\int_0^b |X_t|\,dm_t<\infty\right)=1,$$

则按轨道积分

$$\tilde{I}=\int_0^b X_t\,dm_t$$

是正态分布随机变量.

引理 3 若 $\{\xi_n,\ n\geqslant 1\}$ 为正态分布随机变量序列，则

$$\sum_{j=1}^{\infty} E\xi_j^2\leqslant\left[E\exp\left(-\frac{1}{2}\sum_{j=1}^{\infty}\xi_j^2\right)\right]^{-2},$$

进而若 $\sum_{j=1}^{\infty} E\xi_j^2<1$, 则

$$E\left[\exp\left(\frac{1}{2}\sum_{j=1}^{\infty}\xi_j^2\right)\right]\leqslant\left(1-\sum_{j=1}^{\infty}E\xi_j^2\right)^{-\frac{1}{2}}.$$

引理 4 若 m_s 为 $[0, b)$ 上右连续增函数，又 $X=\{X_t^{(i)},\ 0\leqslant t<b,\ 1\leqslant i<\infty\}$ 为正态过程，则当 $P\left\{\sum_{i=1}^{\infty}\int_0^b (X_t^{(i)})^2\,dm_t<\infty\right\}=1$ 时必有

$$\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t<\infty,$$

进而若 $\sum_{i=1}^{\infty}E\int_0^b (X_t^{(i)})^2\,dm_t<1$, 必有

$$E\exp\left(\frac{1}{2}\sum_{i=1}^{\infty}\int_0^b (X_s^{(i)})^2\,dm_s\right)\leqslant\left(1-\sum_{i=1}^{\infty}E\int_0^b (X_s^{(i)})^2\,dm_s\right)^{-\frac{1}{2}}.$$

定理 若 $W=(W_t^{(1)}, \cdots, W_t^{(n)}, \cdots)$ 为一个具有无限个分量的过程，其分量都是连续正态独立增量过程且满足

$$E\{W_t^{(i)}-W_s^{(i)}\}=0,$$
$$E\{(W_t^{(i)}-W_s^{(i)})(W_t^{(j)}-W_s^{(j)})\}=\delta_{ij}(m_t^{(i)}-m_s^{(i)}).$$

又 $\{f_t=(f_t^{(1)}, \cdots, f_t^{(n)}, \cdots)\}$ 为循序可测正态过程，若

$$P\left\{\sum_{i=1}^{\infty}\int_0^b (f_t^{(i)})^2\,dm_t^{(i)}<\infty\right\}=1,$$

则

$$Z_t=\exp\left\{\sum_{i=1}^{\infty}\int_0^t f_s^{(i)}\,dW_s^{(i)}-\frac{1}{2}\int_0^t (f_s^{(i)})^2\,dm_s^{(i)}\right\},\quad 0\leqslant t<b,$$

是一致可积鞅，特别有 $EZ_b=1$.

利用上述结果及正态过程的 Hida-Cramer 分解，可以象 [1] 一样方便地讨论正态测度的等价性问题并求出其 Radon-Nikodym 导数.