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拟线性双曲抛物耦合方程组的第二边值问题

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§1. 引 言

在[1]中，我们讨论了拟线性双曲抛物耦合方程组的柯西问题．在本文中，进一步讨论它的第二边值问题．我们仍用[1]中的记号，凡类似的推导不再赘述．

在区域

$$R(\delta)=\{0\leqslant t\leqslant\delta,\ 0\leqslant x\leqslant 1\} \tag{1.1}$$

上考察拟线性双曲抛物耦合方程组

$$\begin{cases}\displaystyle\sum_{j=1}^{n}\zeta_{lj}(t,\ x,\ u,\ v)\left(\frac{\partial u_j}{\partial t}+\lambda_l(t,\ x,\ u,\ v,\ v_x)\frac{\partial u_j}{\partial x}\right)\\ \displaystyle\quad=\zeta_l(t,\ x,\ u,\ v)\left(\frac{\partial v}{\partial t}+\lambda_l(t,\ x,\ u,\ v,\ v_x)\frac{\partial v}{\partial x}\right)\\ \quad\quad+\mu_l(t,\ x,\ u,\ v,\ v_x),\quad (l=1,\ \cdots,\ n), & (1.2)\\ \displaystyle\frac{\partial v}{\partial t}-a(t,\ x,\ u,\ v,\ v_x)\frac{\partial^2 v}{\partial x^2}=b(t,\ x,\ u,\ v,\ v_x). & (1.3)\end{cases}$$

由[1]，通过适当的未知函数变换及自变量变换，总可设初始条件为

$t=0$:
$$u_j=0,\quad (j=1,\ \cdots,\ n); \tag{1.4}$$
$$v=0. \tag{1.5}$$

并可设成立

$$a(0,\ x,\ 0,\ 0,\ 0)\equiv 1. \tag{1.6}$$

此外，还可设成立

$$b(0,\ x,\ 0,\ 0,\ 0)=0, \tag{1.7}$$
$$\zeta_{lj}(0,\ x,\ 0,\ 0)=\delta_{lj}. \tag{1.8}$$

否则通过下列变换总可达到这一点

$$\bar{v}=v-tb(0,\ x,\ 0,\ 0,\ 0), \tag{1.9}$$

$$\bar{u}_l=\sum_{j=1}^{n}\zeta_{lj}(0,\ x,\ 0,\ 0)u_j,\quad (l=1,\ \cdots,\ n). \tag{1.10}$$

边值条件为

$x=1$:
$$u_{\bar{r}}=G_{\bar{r}}(t,\ u,\ v),\quad (\bar{r}=1,\ \cdots,\ h\quad h\leqslant n), \tag{1.11}$$

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$$\frac{\partial v}{\partial x}=F_+(t,\ u,\ v), \tag{1.12}$$

$$x=0:\qquad u_{\hat{s}}=\hat{G}_{\hat{s}}(t,\ u,\ v),\quad (\hat{s}=m+1,\ \cdots,\ n;\ m\geqslant 0), \tag{1.13}$$

$$\frac{\partial v}{\partial x}=F_-(t,\ u,\ v), \tag{1.14}$$

其中条件(1.11)及(1.13)的给法与双曲型方程组的特征分布相适应[2]，亦即假设成立如下的定向性条件

$$\begin{cases}\lambda_{\bar{r}}(0,\ 1,\ 0,\ 0,\ 0)<0,\quad \lambda_s(0,\ 1,\ 0,\ 0,\ 0)>0,\quad \begin{pmatrix}\bar{r}=1,\ \cdots,\ h\\ s=h+1,\ \cdots,\ n\end{pmatrix}, & (1.15)\\ \lambda_{\hat{r}}(0,\ 0,\ 0,\ 0,\ 0)<0,\quad \lambda_{\hat{s}}(0,\ 0,\ 0,\ 0,\ 0)>0,\quad \begin{pmatrix}\hat{r}=1,\ \cdots,\ m\\ \hat{s}=m+1,\ \cdots,\ n\end{pmatrix}, & (1.16)\end{cases}$$

为简洁起见，下文中凡在下标出现的 $\bar{r}$ 均表示 $\bar{r}=1,\ \cdots,\ h$，下标中出现的 $\hat{s}$ 均表示 $\hat{s}=m+1,\ \cdots,\ n$，不再一一列出.

此外，对上述边值问题还假设成立下述的相容性条件

$$\begin{cases}G_{\bar{r}}(0,\ 0,\ 0)=0, & (1.17)\\ \hat{G}_{\hat{s}}(0,\ 0,\ 0)=0, & (1.18)\\ \dfrac{\partial G_{\bar{r}}}{\partial t}(0,\ 0,\ 0)+\displaystyle\sum_{j=1}^{n}\frac{\partial G_{\bar{r}}}{\partial u_j}(0,\ 0,\ 0)\mu_j(0,\ 1,\ 0,\ 0,\ 0)=\mu_{\bar{r}}(0,\ 1,\ 0,\ 0,\ 0), & (1.19)\\ \dfrac{\partial \hat{G}_{\hat{s}}}{\partial t}(0,\ 0,\ 0)+\displaystyle\sum_{j=1}^{n}\frac{\partial \hat{G}_{\hat{s}}}{\partial n_j}(0,\ 0,\ 0)\mu_j(0,\ 0,\ 0,\ 0,\ 0)=\mu_{\hat{s}}(0,\ 0,\ 0,\ 0,\ 0), & (1.20)\\ F_{\pm}(0,\ 0,\ 0)=0 & (1.21)\end{cases}$$

以及示性数条件

$$\sum_{j=1}^{n}\left|\frac{\partial G_{\bar{r}}}{\partial u_j}(0,\ 0,\ 0)\right|<1, \tag{1.22}$$

$$\sum_{j=1}^{n}\left|\frac{\partial \hat{G}_{\hat{s}}}{\partial u_j}(0,\ 0,\ 0)\right|<1. \tag{1.23}$$

并记

$$\theta=\max_{\substack{1\leqslant\bar{r}\leqslant h\\ m+1\leqslant\hat{s}\leqslant n}}\left(\sum_{j=1}^{n}\left|\frac{\partial G_{\bar{r}}}{\partial u_j}(0,\ 0,\ 0)\right|,\ \sum_{j=1}^{n}\left|\frac{\partial \hat{G}_{\hat{s}}}{\partial u_j}(0,\ 0,\ 0)\right|\right)<1. \tag{1.24}$$

本文将证明对第二边值问题(1.2)—(1.5)，(1.11)—(1.14)，当其系数及边值函数适当光滑，且定向性条件(1.15)—(1.16)，相容性条件(1.17)—(1.21)及示性数条件(1.22)—(1.23)均成立时，对适当小的 $\delta>0$，在区域 $R(\delta)$ 上其古典解存在唯一. 其证明方法是先对热传导方程的第二边值问题建立较精细的先验估计(§2)，再对形如(1.2)的线性双曲型方程组的边值问题证明解的存在性并建立先验估计(§3)，然后用 Schander 不动点原理[3,4]证明解的存在性(§4)，并在§5中证明其唯一性，在§5的附注中还指出了当示性数条件(1.22)—(1.23)减弱为

$$\det\left(\delta_{\bar{r}r'}-\frac{\partial G_{\bar{r}}}{\partial u_r}(0,\ 0,\ 0)\right)\neq 0,\quad (\bar{r},\ r'=1,\ \cdots,\ h), \tag{1.25}$$

$$\det\left(\delta_{\hat{s}\hat{s}'}-\frac{\partial \hat{G}_{\hat{s}}}{\partial u_{\hat{s}}}(0,\ 0,\ 0)\right)\neq 0,\quad (\hat{s},\ \hat{s}'=m+1,\ \cdots,\ n) \tag{1.26}$$

时其结论仍成立.

§2. 热传导方程第二边值问题解的一些估计式

本节利用热传导方程基本解的性质，导出热传导方程第二边值问题解的表达式，并建立相应的估计式.

先在区域 $R(\delta_0)$ 上考察如下的热传导方程的第二边值问题

$$\begin{cases} \dfrac{\partial v}{\partial t}=\dfrac{\partial^2 v}{\partial x^2}+b(t,\ x), & (2.1)\\ t=0:\quad v=0, & (2.2)\\ x=0:\quad \dfrac{\partial v}{\partial x}=\varphi_1(t), & (2.3)\\ x=1:\quad \dfrac{\partial v}{\partial x}=\varphi_2(t). & (2.4)\end{cases}$$

因热传导方程的基本解为

$$G_0(t,\ x;\ \tau,\ \xi)=\frac{1}{2\sqrt{\pi(t-\tau)}}\,e^{-\frac{(x-\xi)^2}{4(t-\tau)}},\quad (t\geqslant\tau), \tag{2.5}$$

由反射法易知，热传导方程第一边值问题的格林函数及第二边值问题的牛孟函数分别为

$$G(t,\ x;\ \tau,\ \xi)=\sum_{n=-\infty}^{\infty}[G_0(t,\ x;\ \tau,\ 2n+\xi)-G_0(t,\ x;\ \tau,\ 2n-\xi)], \tag{2.6}$$

$$N(t,\ x;\ \tau,\ \xi)=\sum_{n=-\infty}^{\infty}[G_0(t,\ x;\ \tau,\ 2n+\xi)+G_0(t,\ x;\ \tau,\ 2n-\xi)]. \tag{2.7}$$

由定义，$N(t,\ x;\ \tau,\ \xi)$ 对变量 $(t,\ x)$ 而言，在 $t>\tau$ 时满足方程 $\dfrac{\partial N}{\partial t}=\dfrac{\partial^2 N}{\partial x^2}$，且当 $x=0$ 或 $x=1$ 时，成立 $\dfrac{\partial N}{\partial x}=0$；而对变量 $(\tau,\ \xi)$ 而言，在 $\tau<t$ 时满足共轭方程 $\dfrac{\partial N}{\partial \tau}=-\dfrac{\partial^2 N}{\partial \xi^2}$，且当 $\xi=0$ 或 $\xi=1$ 时成立 $\dfrac{\partial N}{\partial \xi}=0$. 而 $G(t,\ x;\ \tau,\ \xi)$ 对变量 $(t,\ x)$ 而言，在 $t>\tau$ 时满足方程 $\dfrac{\partial G}{\partial t}=\dfrac{\partial^2 G}{\partial x^2}$，且当 $x=0$ 或 $x=1$ 时成立 $G=0$；而对变量 $(\tau,\ \xi)$ 而言在 $\tau<t$ 时满足共轭方程 $\dfrac{\partial G}{\partial \tau}=-\dfrac{\partial^2 G}{\partial \xi^2}$，且当 $\xi=0$ 或 $\xi=1$ 时成立 $G=0$.

又易知成立

$$\frac{\partial G}{\partial x}=-\frac{\partial N}{\partial \xi},\quad \frac{\partial G}{\partial \xi}=-\frac{\partial N}{\partial x}. \tag{2.8}$$

令

$$V_\sigma(t,\ x)=t^{-\frac{\sigma}{2}}\exp\left\{-\frac{x^2}{16t}\right\}. \tag{2.9}$$

通过直接计算可得

$$\int_{-\infty}^{\infty}V_\sigma(t,\ \xi)d\xi=4\sqrt{\pi}\,t^{\frac{1-\sigma}{2}},\quad (t>0), \tag{2.10}$$

$$\int_0^t\int_{-\infty}^{\infty}V_\sigma(t-\tau,\ \xi)d\xi d\tau=\frac{8\sqrt{\pi}}{3-\sigma}t^{\frac{3-\sigma}{2}},\quad (\text{若 } \sigma<3), \tag{2.11}$$

$$\int_{t-\gamma}^t\int_{-\infty}^{\infty}V_\sigma(t-\tau,\ \xi)d\xi d\tau=\frac{8\sqrt{\pi}}{3-\sigma}\gamma^{\frac{3-\sigma}{2}},\quad (\text{若 } \sigma<3,\ \text{而 } 0\leqslant\gamma\leqslant t), \tag{2.12}$$

$$\int_0^{t_1-\gamma}\int_{-\infty}^{\infty}V_\sigma(t-\tau,\ \xi)d\xi\,d\tau\leqslant\frac{8\sqrt{\pi}}{\sigma-3}\gamma^{-\frac{\sigma-3}{2}},\quad (\text{若 } \sigma>3,\ \text{而 } 0<\gamma\leqslant t_1\leqslant t). \tag{2.13}$$

引理2.1 在区域 $R(\delta_0)$上，对热传导方程的格林函数及牛孟函数成立如下的估计式

$$\left|\frac{\partial^{i+j}G(t,\ x;\ \tau,\ \xi)}{\partial x^i\partial t^j}\right|,\ \left|\frac{\partial^{i+j}N(t,\ x;\ \tau,\ \xi)}{\partial x^i\partial t^j}\right|\leqslant P_{ij}V_{i+2j+1}(t-\tau,\ x-\xi),\quad (t\geqslant\tau), \tag{2.14}$$

$$\left|\frac{(x-\xi)^k\partial^{i+j}G}{(t-\tau)^s\partial x^i\partial t^j}\right|,\ \left|\frac{(x-\xi)^k\partial^{i+j}N}{(t-\tau)^s\partial x^i\partial t^j}\right|\leqslant P_{ijks}V_{i+2j+2s-k+1}(t-\tau,\ x-\xi),\quad (t\geqslant\tau), \tag{2.15}$$

其中 P_{ij}, P_{ijks} 均为常数.

证 以 $N(t,\ x;\ \tau,\ \xi)$为例，由于它对$(t,\ x)$满足热传导方程，故只需证明

$$\left|\frac{\partial^i N}{\partial x^i}\right|\leqslant P_iV_{i+1}(t-\tau,\ x-\xi). \tag{2.16}$$

易知在 $R(\delta_0)$上成立

$$(x-2n-\xi)^2\geqslant(x-\xi)^2+(2|n|-1)^2-1,$$
$$(x-2n+\xi)^2\geqslant(x-\xi)^2+(2|n|-2)^2-4.$$

故
$$\left|\frac{\partial^i G_0(t,\ x;\ \tau,\ 2n+\xi)}{\partial x^i}\right|\leqslant A_iV_{i+1}(t-\tau,\ x-2n-\xi)$$
$$\leqslant A_iV_{i+1}(t-\tau,\ x-\xi)\exp\left\{-\frac{(2|n|-1)^2-1}{16\delta_0}\right\}.$$

类似地有

$$\left|\frac{\partial^i G_0(t,\ x;\ \tau,\ 2n-\xi)}{\partial x^i}\right|\leqslant A_iV_{i+1}(t-\tau,\ x-\xi)\exp\left\{-\frac{(2|n|-2)^2-4}{16\delta_0}\right\}.$$

再注意到(2.6)就得到(2.16)式.

引理2.2 当$(t,\ x)\in R(\delta_0)$时，成立

$$\int_0^t\left|\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 0)\right|d\tau\leqslant C,\quad (x\neq 0), \tag{2.17}$$

$$\int_0^t\left|\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\right|d\tau\leqslant C,\quad (x\neq 1), \tag{2.18}$$

其中 C 为仅与 δ_0 有关的常数.

证 由定义，易知有

$$\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 0)=C_0\sum_{n=-\infty}^{\infty}\frac{(x-2n)}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x-2n)^2}{4(t-\tau)}\right\}. \tag{2.19}$$

设 $\bar{x}\neq 0$, 令 $\rho=\dfrac{\bar{x}^2}{t-\tau}$ 有

$$\int_0^t\frac{|\bar{x}|}{(t-\tau)^{3/2}}\exp\left\{-\frac{\bar{x}^2}{4(t-\tau)}\right\}d\tau=\int_{\frac{\bar{x}^2}{t}}^{\infty}\rho^{-\frac{1}{2}}\exp\left\{-\frac{\rho}{4}\right\}d\rho\leqslant C_1\exp\left\{-\frac{\bar{x}^2}{8t}\right\},$$

其中
$$C_1=\int_0^\infty\rho^{-\frac{1}{2}}\exp\left\{-\frac{\rho}{8}\right\}d\rho.$$

于是设 $x\neq 0$, 在(2.19)式中逐项积分，利用上述估计式，并注意到在 $R(\delta_0)$ 上恒成立 $|x-2n|\geqslant 2|n|-1$, 就有

$$\int_0^t\left|\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 0)\right|d\tau\leqslant C_0C_1\sum_{n=-\infty}^{\infty}\exp\left\{-\frac{(2|n|-1)^2}{8\delta_0}\right\}\leqslant C.$$

类似地可得到(2.18)式.

又由(2.10)—(2.13)及引理2.1, 容易证明当$(t,\ x)\in R(\delta_0)$时, 成立

$$\int_0^t|N(t,\ x;\ \tau,\ 0)|d\tau,\ \int_0^t|N(t,\ x;\ \tau,\ 1)|d\tau\leqslant C_2t^{\frac{1}{2}},\tag{2.20}$$

其中C_2为仅与δ_0有关的常数.

引理2.3　假设在区域$R(\delta_0)$上$b(t,\ x)$连续, 对t及x分别属于$\mathrm{Lip}\frac{\alpha}{2}$及$\mathrm{Lip}\,\alpha(0<\alpha<1)$, 又$\varphi_i(t)\in C^1$, 且成立

$$\varphi_i(0)=0,\quad (i=1,\ 2),\tag{2.21}$$

则在区域$R(\delta_0)$上热传导方程的第二边值问题(2.1)—(2.4)必存在着唯一的解[注]

$$v(t,\ x)=\int_0^t\int_0^1 N(t,\ x;\ \tau,\ \xi)b(\tau,\ \xi)d\xi\,d\tau+\int_0^t N(t,\ x;\ \tau,\ 1)\varphi_2(\tau)d\tau$$
$$-\int_0^t N(t,\ x;\ \tau,\ 0)\varphi_1(\tau)d\tau,\tag{2.22}$$

且

$$\frac{\partial v}{\partial x}(t,\ x)=\int_0^t\int_0^1\frac{\partial N(t,\ x;\ \tau,\ \xi)}{\partial x}b(\tau,\ \xi)d\xi\,d\tau+\int_0^t\frac{\partial N(t,\ x;\ \tau,\ 1)}{\partial x}\varphi_2(\tau)d\tau$$
$$-\int_0^t\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 0)\varphi_1(\tau)d\tau,\tag{2.23}$$

$$\frac{\partial^2 v}{\partial x^2}(t,\ x)=\int_0^t\int_0^1\frac{\partial^2 N}{\partial x^2}(t,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)-b(\tau,\ x))d\xi\,d\tau$$
$$+\int_0^t N(t,\ x;\ \tau,\ 1)\dot\varphi_2(\tau)d\tau-\int_0^t N(t,\ x;\ \tau,\ 0)\dot\varphi_1(\tau)d\tau,\tag{2.24}$$

$$\frac{\partial v}{\partial t}(t,\ x)=\int_0^t\int_0^1\frac{\partial N}{\partial t}(t,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)-b(t,\ \xi))d\xi\,d\tau$$
$$+\int_0^1 N(t,\ x;\ 0,\ \xi)b(t,\ \xi)d\xi+\int_0^t N(t,\ x;\ \tau,\ 1)\dot\varphi_2(\tau)d\tau$$
$$-\int_0^t N(t,\ x;\ \tau,\ 0)\dot\varphi_1(\tau)d\tau,\tag{2.25}$$

其中$\dot\varphi_1$, $\dot\varphi_2$分别表示φ_1, φ_2对t的导数.

此外, 在$R(\delta_0)$上, v, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial t}$, $\frac{\partial^2 v}{\partial x^2}$连续, 且$\frac{\partial v}{\partial x}$对$t$属于$\mathrm{Lip}\frac{1+\alpha}{2}$, $\frac{\partial v}{\partial t}$及$\frac{\partial^2 v}{\partial x^2}$对$t$及对$x$分别属于$\mathrm{Lip}\frac{\alpha}{2}$及$\mathrm{Lip}\,\alpha$, 亦即$v\in\bar{C}^{2+\alpha}$.

此引理证明可利用牛孟函数及热势性质导出, 可参见[5], 有些结论也可从以下一些引理证明中推出.

引理2.4(第一估计式)　在引理2.3的假设下, 设$v=v(t,\ x)$是所考察的第二边值问题(2.1)—(2.4)在区域$R(\delta_0)$上的解, 则对任何$\delta(0\leqslant\delta\leqslant\delta_0)$, 在区域$R(\delta)=\{(t,\ x)|0\leqslant t\leqslant\delta,\ 0\leqslant x\leqslant1\}$上成立

注　由热势的性质, (2.22)—(2.25)中的最后二项当$x=1$或$x=0$时, 可分别按连续性来理解.

$$|v|\equiv\|v\|+\left\|\frac{\partial v}{\partial x}\right\|\leqslant C_3(\delta^{\frac{1}{2}}\|b\|+\|\varphi\|), \tag{2.26}$$

或

$$|v|\leqslant C_3(\delta^{\frac{1}{2}}\|b\|+\delta\|\dot{\varphi}\|), \tag{2.26$'$}$$

其中 C_3 为仅与 δ_0 有关的常数. 而 $\|v\|=\sup\limits_{(t,x)\in R(\delta)}|v|$.

证 利用(2.10)—(2.13), 引理 2.1 及引理 2.2, 并注意到(2.20)及(2.21)式就可得到

$$\|v\|\leqslant R_1\delta\|b\|+R_2\delta^{\frac{1}{2}}\|\varphi\|,$$

$$\left\|\frac{\partial v}{\partial x}\right\|\leqslant R_3\delta^{\frac{1}{2}}\|b\|+R_4\|\varphi\|,$$

其中 $R_i(i=1,\cdots,4)$为与 δ 无关的常数, 由此得到(2.26)及(2.26)$'$.

引理 2.5(第二估计式) 在引理 2.3 的假设下, 对任何 $\delta(0\leqslant\delta\leqslant\delta_0)$, 在区域 $R(\delta)$ 上成立

$$|v|_1\equiv|v|+\left\|\frac{\partial v}{\partial t}\right\|+\left\|\frac{\partial^2 v}{\partial x^2}\right\|+H_t^{\frac{1}{2}}\left[\frac{\partial v}{\partial x}\right]\leqslant C_4(\|b\|+\delta^{\frac{\alpha}{2}}H_x^\alpha[b]+\delta^{\frac{1}{2}}\|\varphi\|_1), \tag{2.27}$$

其中 C_4 为仅与 δ_0 有关的常数.

证 由(2.24)式, 利用(2.10)—(2.13)及引理 2.1, 并注意到(2.20)式, 就容易证明在 $R(\delta)$ 上

$$\left\|\frac{\partial^2 v}{\partial x^2}\right\|\leqslant d_1(\delta^{\frac{\alpha}{2}}H_x^\alpha[b]+\delta^{\frac{1}{2}}\|\varphi\|_1). \tag{2.28}$$

再利用 $v(t,x)$满足方程(2.1)就有

$$\left\|\frac{\partial v}{\partial t}\right\|\leqslant d_2(\|b\|+\delta^{\frac{\alpha}{2}}H_x^\alpha[b]+\delta^{\frac{1}{2}}\|\varphi\|_1). \tag{2.29}$$

于此及今后 $d_i(i=1,2,\cdots)$均表示适当的常数.

又由(2.23)式, 有

$$\frac{\partial v}{\partial x}(t,x)=\frac{\partial v_1}{\partial x}(t,x)+\frac{\partial v_2}{\partial x}(t,x)+\frac{\partial v_3}{\partial x}(t,x), \tag{2.30}$$

其中

$$\begin{aligned}\frac{\partial v_1}{\partial x}(t,x)&=\int_0^t\int_0^1\frac{\partial N}{\partial x}(t,x;\tau,\xi)b(\tau,\xi)d\xi\,d\tau\\&=\int_0^t\int_0^1\frac{\partial N}{\partial x}(t,x;\tau,\xi)(b(\tau,\xi)-b(\tau,x))d\xi\,d\tau\\&\quad+\int_0^t\int_0^1\frac{\partial N}{\partial x}(t,x;\tau,\xi)b(\tau,x)d\xi\,d\tau\\&=\int_0^t\int_0^1\frac{\partial N}{\partial x}(t,x;\tau,\xi)(b(\tau,\xi)-b(\tau,x))d\xi\,d\tau.\end{aligned} \tag{2.31}$$

这是因为

$$\begin{aligned}&\int_0^t\int_0^1\frac{\partial N}{\partial x}(t,x;\tau,\xi)b(\tau,x)d\xi\,d\tau\\&=-\int_0^t\int_0^1\frac{\partial G}{\partial\xi}(t,x;\tau,\xi)b(\tau,x)d\xi\,d\tau\\&=-\int_0^t[G(t,x;\tau,1)-G(t,x;\tau,0)]b(\tau,x)d\tau=0.\end{aligned} \tag{2.32}$$

又

$$\frac{\partial v_2}{\partial x}(t,\ x)=\int_0^t \frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\varphi_2(\tau)d\tau, \tag{2.33}$$

$$\frac{\partial v_3}{\partial x}(t,\ x)=-\int_0^t \frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 0)\varphi_1(\tau)d\tau. \tag{2.34}$$

由(2.31)式，利用(2.10)—(2.13)及引理 2.1，与[1]中方法类似地可以证明

$$H_t^{\frac{1+\alpha}{2}}\left[\frac{\partial v_1}{\partial x}\right]\leqslant d_3 H_x^\alpha[b]. \tag{2.35}$$

同理由(2.31)式可得

$$H_t^{\frac{1}{2}}\left[\frac{\partial v_1}{\partial x}\right]\leqslant d_4\|b\|. \tag{2.36}$$

又由(2.33)式，设 $x\neq 1$ 有

$$\begin{aligned}\frac{\partial^2 v_2}{\partial x\partial t}(t,\ x)&=\int_0^t \frac{\partial^2 N}{\partial x\partial t}(t,\ x;\ \tau,\ 1)\varphi_2(\tau)d\tau+\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\varphi_2(t)\\&=\int_0^t \frac{\partial^2 N}{\partial x\partial t}(t,\ x;\ \tau,\ 1)\varphi_2(\tau)d\tau\\&\equiv-\int_0^t \frac{\partial}{\partial \tau}\left(\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\right)\varphi_2(\tau)d\tau\\&=\int_0^t \frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\dot{\varphi}_2(\tau)d\tau-\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\varphi_2(\tau)\Big|_0^t\\&=\int_0^t \frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\dot{\varphi}_2(\tau)d\tau.\end{aligned} \tag{2.37}$$

再利用双层热势的性质，易知当 $x\to 1-0$ 时，上式 $\to\dot{\varphi}_2(t)$，故知在 $R(\delta_0)$ 上 $\frac{\partial^2 v_2}{\partial x\partial t}$ 保持连续，由引理 2.2 可得在 $R(\delta)$ 上成立

$$\left\|\frac{\partial^2 v_2}{\partial x\partial t}\right\|\leqslant d_4\|\varphi\|_1, \tag{2.38}$$

从而

$$H_t^{\frac{1+\alpha}{2}}\left[\frac{\partial v_2}{\partial x}\right]\leqslant d_5\delta^{\frac{1-\alpha}{2}}\|\varphi\|_1, \tag{2.39}$$

$$H_t^{\frac{1}{2}}\left[\frac{\partial v_2}{\partial x}\right]\leqslant d_6\delta^{\frac{1}{2}}\|\varphi\|_1, \tag{2.40}$$

对 $\frac{\partial v_3}{\partial x}$ 有完全类似的估计.

于是我们得到在 $R(\delta)$ 上成立

$$H_t^{\frac{1}{2}}\left[\frac{\partial v}{\partial x}\right]\leqslant d_7(\|b\|+\delta^{\frac{1}{2}}\|\varphi\|_1), \tag{2.41}$$

$$H_t^{\frac{1+\alpha}{2}}\left[\frac{\partial v}{\partial x}\right]\leqslant d_8(H_x^\alpha[b]+\delta^{\frac{1-\alpha}{2}}\|\varphi\|_1). \tag{2.42}$$

由(2.26)′，(2.28)，(2.29)及(2.41)就得到所需要的估计式(2.27).

引理 2.6(第三估计式)　在引理 2.3 的假设下，对任何 $\delta(0\leqslant\delta\leqslant\delta_0)$ 在区域 $R(\delta)$ 上成立

$$|v|_2 \equiv |v|_1 + H_t^{\frac{1+\alpha}{2}}\left[\frac{\partial v}{\partial x}\right] + H^\alpha\left[\frac{\partial v}{\partial t}\right] + H^\alpha\left[\frac{\partial^2 v}{\partial x^2}\right]$$

$$\leqslant C_5(\|b\| + H^\alpha[b] + \|\varphi\|_1), \tag{2.43}$$

其中 C_5 为仅与 δ_0 有关的常数.

证　由(2.24)式，记

$$\frac{\partial^2 v}{\partial x^2}(t, x) = I_1(t, x) + I_2(t, x) + I_3(t, x), \tag{2.44}$$

其中

$$I_1(t, x) = \int_0^t \int_0^1 \frac{\partial^2 N}{\partial x^2}(t, x; \tau, \xi)[b(\tau, \xi) - b(\tau, x)]d\xi\, d\tau, \tag{2.45}$$

$$I_2(t, x) = \int_0^t N(t, x; \tau, 1)\dot{\varphi}_2(\tau)d\tau, \tag{2.46}$$

$$I_3(t, x) = -\int_0^t N(t, x; \tau, 0)\dot{\varphi}_1(\tau)d\tau. \tag{2.47}$$

我们有

$$I_1(t, x_1) - I_2(t, x_2) = \int_0^t \int_0^1 \frac{\partial^2 N}{\partial x^2}(t, x_1; \tau, \xi)(b(\tau, \xi) - b(\tau, x_1))d\xi\, d\tau$$

$$- \int_0^t \int_0^1 \frac{\partial^2 N}{\partial x^2}(t, x_2; \tau, \xi)(b(\tau, \xi) - b(\tau, x_2))d\xi\, d\tau. \tag{2.48}$$

记 $\gamma = (x_1 - x_2)^2 > 0$.

若 $t \geqslant \gamma$，则

$$I_1(t, x_1) - I_1(t, x_2) = \int_{t-\gamma}^t \int_0^1 \frac{\partial^2 N}{\partial x^2}(t, x_1; \tau, \xi)(b(\tau, \xi) - b(\tau, x_1))d\xi\, d\tau$$

$$- \int_{t-\gamma}^t \int_0^1 \frac{\partial^2 N}{\partial x^2}(t, x_2; \tau, \xi)(b(\tau, \xi) - b(\tau, x_2))d\xi\, d\tau$$

$$+ \int_0^{t-\gamma} \int_0^1 \left[\frac{\partial^2 N}{\partial x^2}(t, x_1; \tau, \xi) - \frac{\partial^2 N}{\partial x^2}(t, x_2; \tau, \xi)\right]$$

$$\cdot (b(\tau, \xi) - b(\tau, x_1))d\xi\, d\tau$$

$$+ \int_0^{t-\gamma} \int_0^1 \frac{\partial^2 N}{\partial x^2}(t, x_2; \tau, \xi)(b(\tau, x_2) - b(\tau, x_1))d\xi\, d\tau$$

$$= L_1 + L_2 + L_3 + L_4. \tag{2.49}$$

利用(2.10)—(2.13)及引理 2.1，易得

$$|L_1|,\ |L_2| \leqslant d_9 \gamma^{\frac{\alpha}{2}} H_x^\alpha[b]. \tag{2.50}$$

又　$$L_3 = \int_{x_2}^{x_1} \int_0^{t-\gamma} \int_0^1 \frac{\partial^3 N}{\partial x^3}(t, x; \tau, \xi)(b(\tau, \xi) - b(\tau, x_1))d\xi\, d\tau\, dx,$$

注意到　$$|x_1 - \xi|^\alpha \leqslant |\xi - x|^\alpha + |x - x_1|^\alpha \leqslant |\xi - x|^\alpha + \gamma^{\frac{\alpha}{2}}.$$

同理可得

$$|L_3| \leqslant d_{10} \gamma^{\frac{\alpha}{2}} H_x^\alpha[b]. \tag{2.51}$$

又　$$L_4 = \int_0^{t-\gamma} \int_0^1 \frac{\partial^2 N}{\partial \xi^2}(t, x_2; \tau, \xi)(b(\tau, x_2) - b(\tau, x_1))d\xi\, d\tau$$

$$= \int_0^{t-\gamma} \frac{\partial N}{\partial \xi}\Big|_{\xi=0}^{\xi=1}(b(\tau, x_2) - b(\tau, x_1))d\tau = 0,$$

故此时恒成立

$$|I_1(t,\ x_1)-I_1(t,\ x_2)|\leqslant d_{11}\gamma^{\frac{\alpha}{2}}H_x^\alpha[b].\tag{2.52}$$

又当 $t\leqslant\gamma$ 时，由(2.48)式直接进行计算，同样可得(2.52)式.

此外在 $x\neq1$ 时，由(2.46)式有

$$\frac{\partial I_2(t,\ x)}{\partial x}=\int_0^t\frac{\partial N}{\partial x}(t,\ x;\ \tau,\ 1)\dot{\varphi}_2(\tau)d\tau,\tag{2.53}$$

且由双层热势性质在 $x=1$ 时，上式仍可按连续性来理解，由引理 2.2 可得在 $R(\delta)$ 上成立

$$\left\|\frac{\partial I_2(t,\ x)}{\partial x}\right\|\leqslant d_{12}\|\varphi\|_1,\tag{2.54}$$

从而必成立

$$H_x^\alpha[I_2(t,\ x)]\leqslant d_{13}\|\varphi\|_1.\tag{2.55}$$

对 $I_3(t,\ x)$ 有完全类似的估计

这样，在 $R(\delta)$ 上成立

$$H_x^\alpha\left[\frac{\partial^2 v}{\partial x^2}\right]\leqslant d_{14}(H_x^\alpha[b]+\|\varphi\|_1).\tag{2.56}$$

此外

$$\begin{aligned}I_1(t_1,\ x)-I_1(t_2,\ x)=&\int_0^{t_1}\int_0^1\frac{\partial^2 N}{\partial x^2}(t_1,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)-b(\tau,\ x))d\xi\,d\tau\\&-\int_0^{t_2}\int_0^1\frac{\partial^2 N}{\partial x^2}(t_2,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)-b(\tau,\ x))d\xi\,d\tau.\end{aligned}\tag{2.57}$$

设 $\delta\geqslant t_1\geqslant t_2\geqslant0$, 并记 $\gamma=t_1-t_2$.

若 $t_1-2\gamma=t_2-\gamma\geqslant0$, 有

$$\begin{aligned}I_1(t_1,\ x)-I_1(t_2,\ x)=&\int_{t_1-2\gamma}^{t_1}\int_0^1\frac{\partial^2 N}{\partial x^2}(t_1,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)-b(\tau,\ x))d\xi\,d\tau\\&-\int_{t_2-\gamma}^{t_2}\int_0^1\frac{\partial^2 N}{\partial x^2}(t_2,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)-b(\tau,\ x))d\xi\,d\tau\\&+\int_0^{t_2-\gamma}\int_0^1\frac{\partial^2 N}{\partial x^2}(t_1,\ x;\ \tau,\ \xi)-\frac{\partial^2 N}{\partial x^2}(t_2,\ x;\ \tau,\ \xi)(b(\tau,\ \xi)\\&-b(\tau,\ x))d\xi\,d\tau.\end{aligned}$$

采用类似的方法可得

$$|I_1(t_1,\ x)-I_1(t_2,\ x)|\leqslant d_{15}\gamma^{\frac{\alpha}{2}}H_x^\alpha[b],\tag{2.58}$$

而在 $t_1-2\gamma=t_2-\gamma<0$ 时，直接由(2.57)式进行估计也可以得到(2.58)式.

又

$$\begin{aligned}I_2(t_1,\ x)-I_2(t_2,\ x)=&\int_0^{t_1}N(t_1,\ x;\ \tau,\ 1)\dot{\varphi}_2(\tau)d\tau\\&-\int_0^{t_2}N(t_2,\ x;\ \tau,\ 1)\dot{\varphi}_2(\tau)d\tau.\end{aligned}\tag{2.59}$$

若 $t_1-2\gamma=t_2-\gamma\geqslant0$ 有

$$I_2(t_1, x)-I_2(t_2, x)=\int_{t_1-2\gamma}^{t_1} N(t_1, x; \tau, 1)\dot{\varphi}_2(\tau)d\tau$$
$$-\int_{t_2-\gamma}^{t_2} N(t_2, x; \tau, 1)\dot{\varphi}_2(\tau)d\tau$$
$$+\int_0^{t_2-\gamma}(N(t_1, x; \tau, 1)-N(t_2, x; \tau, 1))\dot{\varphi}_2(\tau)d\tau$$
$$=K_1+K_2+K_3. \tag{2.60}$$

由(2.10)—(2.13)易知

$$|N(t, x; \tau, \xi)|\leqslant d_{15}(t-\tau)^{-\frac{1}{2}},$$
$$\left|\frac{\partial N}{\partial t}(t, x; \tau, \xi)\right|\leqslant d_{16}(t-\tau)^{-\frac{3}{2}}.$$

于是

$$|K_1|, \ |K_2|\leqslant d_{17}\gamma^{\frac{1}{2}}\|\varphi\|_1, \tag{2.61}$$

而

$$|K_3|\leqslant\left|\int_{t_2}^{t_1}\int_0^{t_2-\gamma}\frac{\partial N}{\partial t}\dot{\varphi}_2(\tau)d\tau dt\right|$$
$$\leqslant d_{18}\int_{t_2}^{t_1}\int_0^{t_2-\gamma}(t-\tau)^{-\frac{3}{2}}d\tau dt\cdot\|\varphi\|_1$$
$$\leqslant d_{19}(t_1-t_2)\left(\gamma^{-\frac{1}{2}}-t_2^{-\frac{1}{2}}\right)\|\varphi\|_1\leqslant d_{19}\gamma^{\frac{1}{2}}\|\varphi\|_1. \tag{2.62}$$

于是

$$|I_2(t_1, x)-I_2(t_2, x)|\leqslant d_{20}\gamma^{\frac{1}{2}}\|\varphi\|_1. \tag{2.63}$$

在 $t_1-2\gamma=t_2-\gamma<0$ 时，直接由(2.59)式进行估计，同样可得上式.

对 I_3 有完全同样的估计.

这样，在 $R(\delta)$ 上应成立

$$H_t^{\frac{\alpha}{2}}\left[\frac{\partial^2 v}{\partial x^2}\right]\leqslant d_{21}\left(H_x^\alpha[b]+\delta^{\frac{1-\alpha}{2}}\|\varphi\|_1\right). \tag{2.64}$$

合并(2.56)，(2.64)式就有

$$H^\alpha\left[\frac{\partial^2 v}{\partial x^2}\right]\leqslant d_{22}(H_x^\alpha[b]+\|\varphi\|_1). \tag{2.65}$$

再利用 v 满足方程(2.1)，就有

$$H^\alpha\left[\frac{\partial v}{\partial t}\right]\leqslant d_{23}(H^\alpha[b]+\|\varphi\|_1). \tag{2.66}$$

合并(2.27)，(2.42)，(2.65)，(2.66)就得到所需要的估计式(2.43).

§3. 线性双曲型方程组混合问题解的一些估计式

本节中我们将在 $R(\delta_0)$ 上讨论形如(1.2)的线性双曲型方程组的混合问题，并建立其相应的先验估计. 其基本的方法是通过沿特征线的积分，将原方程组化为等价的积分方程组，然后通过迭代得到解的存在性及唯一性，再由相应的积分关系式导出解的估计式.

考察

$$\begin{cases}\sum_{j=1}^{n}\zeta_{lj}(t,\ x)\left(\dfrac{\partial u_j}{\partial t}+\lambda_l(t,\ x)\dfrac{\partial u_j}{\partial x}\right)=\zeta_l(t,\ x)\left(\dfrac{\partial v}{\partial t}+\lambda_l(t,\ x)\dfrac{\partial v}{\partial x}\right) \\ \qquad\qquad +\mu_l(t,\ x),\quad (l=1,\ \cdots,\ n), & (3.1)\\ t=0,\ u_j=0,\quad (1\leqslant j\leqslant n), & (3.2)\\ x=1,\ \sum_{j=1}^{n}\zeta_{\bar{r}j}(t,\ 1)u_j=\psi_{\bar{r}}(t),\quad (\bar{r}=1,\ \cdots,\ h,\ h\leqslant n), & (3.3)\\ x=0,\ \sum_{j=1}^{n}\zeta_{\hat{s}j}(t,\ 0)u_j=\hat{\psi}_{\hat{s}}(t),\quad (\hat{s}=m+1,\ \cdots,\ n,\ m\geqslant 0), & (3.4)\end{cases}$$

其中 $v=v(t,\ x)$ 为已知函数.

不失一般性，设在区域 $R(\delta_0)$ 上成立

$$|\det\zeta_{lj}(t,\ x)|\geqslant K>0,$$
$$\zeta_{lj}(0,\ x)=\delta_{lj}, \tag{3.5}$$

且设成立定向性条件

$$\begin{cases}x=1\text{ 上:}\ \lambda_{\bar{r}}(t,\ 1)<0,\ \lambda_s(t,\ 1)\geqslant 0\ \begin{pmatrix}\bar{r}=1,\ \cdots,\ h\\ s=h+1,\ \cdots,\ n\end{pmatrix},\\ x=0\text{ 上:}\ \lambda_{\hat{r}}(t,\ 0)\leqslant 0,\ \lambda_{\hat{s}}(t,\ 0)>0\ \begin{pmatrix}\hat{r}=1,\ \cdots,\ m\\ \hat{s}=m+1,\ \cdots,\ n\end{pmatrix},\end{cases} \tag{3.6}$$

及相容性条件

$$\begin{cases}\psi_{\bar{r}}(0)=0,\\ \hat{\psi}_{\hat{s}}(0)=0,\\ \zeta_{\bar{r}}(0,\ 1)\left(\dfrac{\partial v}{\partial t}(0,\ 1)+\lambda_{\bar{r}}(0,\ 1)\dfrac{\partial v}{\partial x}(0,\ 1)\right)+\mu_{\bar{r}}(0,\ 1)=\dot{\psi}_{\bar{r}}(0),\\ \zeta_{\hat{s}}(0,\ 0)\left(\dfrac{\partial v}{\partial t}(0,\ 0)+\lambda_{\hat{s}}(0,\ 0)\dfrac{\partial v}{\partial x}(0,\ 0)\right)+\mu_{\hat{s}}(0,\ 0)=\dot{\hat{\psi}}_{\hat{s}}(0),\end{cases} \tag{3.7}$$

其中 $\dot{\psi}_{\bar{r}}$, $\dot{\hat{\psi}}_{\hat{s}}$ 分别表示 $\psi_{\bar{r}}$, $\hat{\psi}_{\hat{s}}$ 对 t 的一阶导数.

引理3.1 设在 $R(\delta_0)$ 上 ζ_{lj}, ζ_l, v 为连续可微函数，λ_l, μ_l, $\dfrac{\partial\lambda_l}{\partial x}$, $\dfrac{\partial\mu_l}{\partial x}$ 为连续函数，$\psi_{\bar{r}}$, $\hat{\psi}_{\hat{s}}$ 为连续可微函数，且定向性条件(3.6)，相容性条件(3.7)成立，则定解问题(3.1)—(3.4)在 $R(\delta_0)$ 上存在着唯一的连续可微解[注].

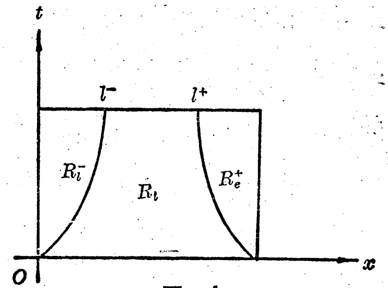

图 1

证　过(0, 0)沿 t 增加方向引第 l 族特征线，记为 l^-，过(0, 1)点沿 t 增加方向引第 l 族特征线，记为 l^+. 原区域 $R(\delta_0)$ 分成三个区域 R_l^-, R_l, R_l^+, (如图1)，这里视 l 的取值不同，可能仅出现二个区域或一个区域.

过点 $(t,\ x)$ 沿 t 减少方向引第 l 族特征线 $\xi=f_l(\tau;\ t,\ x)$，它满足

注　易见，当 v 具有二阶连续偏导数时，此引理是[2]中结果的推论，而目前对 v 仅要求连续可微，但(3.1)的右端只包括 v 沿着特征方向的导数，故这引理本身可作为[2]中有关结果的推广.

$$\begin{cases} \dfrac{df_l(\tau;\ t,\ x)}{d\tau} = \lambda_l(\tau,\ f_l(\tau;\ t,\ x)), \\ \tau = t;\ \ \xi = x. \end{cases} \tag{3.8}$$

设此特征线与 $R(\delta_0)$ 的边界 $(x=0,\ x=1$ 及 $t=0)$ 的交点的 t 坐标为 $\tau_l(t,\ x)$，x 坐标为 $\xi_l(t,\ x)$，则显然 $\xi_l(t,\ x)=f_l(\tau_l(t,\ x);\ t,\ x)$ 且有

当 $(t,\ x)\in R_l$ 时：$\tau_l(t,\ x)=0,\ \xi_l(t,\ x)=f_l(0;\ t,\ x),\quad (l=1,\ \cdots,\ n)$；　(3.9)

当 $(t,\ x)\in R_l^+$ 时：$\xi_l(t,\ x)=1,\ f_l(\tau_l(t,\ x);\ t,\ x)=1,\quad (l=1,\ \cdots,\ h)$；　(3.10)

当 $(t,\ x)\in R_l^-$ 时：$\xi_l(t,\ x)=0,\ f_l(\tau_l(t,\ x);\ t,\ x)=0,\quad (l=m+1,\ \cdots,\ n)$.　(3.11)

类似于[2]，可将上述问题化为积分方程组来进行讨论.

将(3.1)的第 l 个方程沿着第 l 族特征线积分，可得积分关系式

$$\sum_{j=1}^{n}\zeta_{lj}(t,\ x)u_j(t,\ x)=\tilde{\psi}_l(\tau_l(t,\ x))+\int_{\tau_l(t,x)}^{t}\left(\sum_{j=1}^{n}\frac{D\zeta_{lj}}{D_l\tau}u_j+\zeta_l\frac{Dv}{D_l\tau}+\mu_l\right)d\tau,\quad (l=1,\ \cdots,\ n), \tag{3.12}$$

其中被积函数的自变量均为 $(\tau,\ \xi)=(\tau;\ f_l(\tau;\ t,\ x))$，且

$$\frac{D}{D_l\tau}=\frac{\partial}{\partial\tau}+\lambda_l\frac{\partial}{\partial\xi} \tag{3.13}$$

而

$$\tilde{\psi}_l(\tau_l(t,\ x))=\begin{cases}\psi_{\bar r}(\tau_{\bar r}(t,\ x)), & \text{当}(t,\ x)\in R_{\bar r}^+,\ (l=\bar r=1,\ \cdots,\ h);\\ \hat\psi_{\hat s}(\tau_{\hat s}(t,\ x)), & \text{当}(t,\ x)\in R_{\hat s}^-,\ (l=\hat s=m+1,\ \cdots,\ n);\\ 0, & \text{当}(t,\ x)\in R_l,\ (l=1,\ \cdots,\ n),\end{cases} \tag{3.14}$$

称(3.12)为解的第一积分关系式.

我们用迭代法来证明积分方程组(3.12)连续可微解的存在唯一性. 对 $R(\delta_0)$ 上满足 $y(0,\ x)=0$ 的 C^1 函数 $y(t,\ x)$ 记

$$Q_l[y](t,\ x)=\tilde{\psi}_l(\tau_l(t,\ x))+\int_{\tau_l(t,x)}^{t}\left(\sum_{j=1}^{n}\frac{D\zeta_{lj}}{D_l\tau}y_j+\zeta_l\frac{Dv}{D_l\tau}+\mu_l\right)d\tau \tag{3.15}$$

并用

$$\sum_{j=1}^{n}\zeta_{lj}(t,\ x)u_j(t,\ x)=Q_l[y]. \tag{3.16}$$

定义从 $y(t,\ x)$ 到 $u(t,\ x)$ 的映照 S：$u=S(y)$. 显然，当 $y\in C^1(R(\delta_0))$ 时，$Q_l[y]$ 在 R_l^+，R_l^-，R_l 上分别为 C^1. 因此，为证明 S 亦是 $R(\delta_0)$ 上 C^1 到 C^1 的映照，只需证明在 l^+ 及 l^- 上 $Q_l[y]$ 及其对 x 的一阶偏导数均连续. 以 l^+ 为例说明之，由相容性条件(3.7)，$Q_l[y]$ 本身在 $R(\delta_0)$ 上的连续性是显然的. 其一阶偏导数 $\frac{\partial}{\partial x}Q_l[y]$ 的连续性可由其所满足的积分关系式来证明. 对 $Q_l[y]$ 关于 x 求一次导数[注]，通过分部积分得到

$$\begin{aligned}\frac{\partial Q_l[y]}{\partial x}=&\dot{\tilde{\psi}}_l(\tau_l(t,\ x))\frac{\partial\tau_l(t,\ x)}{\partial x}+\int_{\tau_l(t,x)}^{t}\left[\sum_{j=1}^{n}\frac{D\zeta_{lj}}{D_l\tau}\frac{\partial y_j}{\partial\xi}+\frac{\partial\zeta_l}{\partial\xi}\frac{Dv}{D_l\tau}+\frac{\partial\mu_l}{\partial\xi}\right.\\&\left.-\sum_{j=1}^{n}\frac{Dy_j}{D_l\tau}\frac{\partial\zeta_{lj}}{\partial\xi}-\frac{D\zeta_l}{D_l\tau}\frac{\partial v}{\partial\xi}\right]\frac{\partial f_l(\tau;\ t,\ x)}{\partial x}d\tau+\sum_{j=1}^{n}\frac{\partial\zeta_{lj}}{\partial x}(t,\ x)y_j(t,\ x)\\&+\frac{\partial v}{\partial x}(t,\ x)\zeta_l(t,\ x)+\theta_l(t,\ x).\end{aligned} \tag{3.17}$$

注　这里可先暂假设 ζ_{lj}，v 等二次连续可微推导 $\frac{\partial Q_l[y]}{\partial x}$ 的积分表达式，由于所得的积分表达式中仅出现有关量的一阶导数，故可知对 ζ_{lj}，v 为一阶连续可微时，所得的积分表达式仍成立.

而 θ_l 表达式如下

$$\begin{cases} 当 (t, x)\in R_l, & \theta_l(t, x) = -\dfrac{\partial v}{\partial \xi}(0, \xi_l(t, x))\zeta_l(0, \xi_l(t, x))\dfrac{\partial \xi_l(t, x)}{\partial x}, \\ & (l=1, \cdots, n); \\ 当 (t, x)\in R_{\bar r}^+, & \theta_{\bar r}(t, x) = -\left(\displaystyle\sum_{j=1}^{n}\frac{\partial \zeta_{\bar r j}}{\partial \tau} y_j + \zeta_{\bar r}\frac{\partial v}{\partial \tau} + \mu_{\bar r}\right)\Bigg|_{(\tau_l(t,x),1)}\dfrac{\partial \tau_{\bar r}}{\partial x}(t, x), \\ & (l=\bar r=1, \cdots, h); \\ 当 (t, x)\in R_{\hat s}^-, & \theta_{\hat s}(t, x) = -\left(\displaystyle\sum_{j=1}^{n}\frac{\partial \zeta_{\hat s j}}{\partial \tau} y_j + \zeta_{\hat s}\frac{\partial v}{\partial \tau} + \mu_{\hat s}\right)\Bigg|_{(\tau_{\hat s}(t,x),0)}\dfrac{\partial \tau_{\hat s}}{\partial x}(t, x), \\ & (l=\hat s=m+1, \cdots, n). \end{cases} \tag{3.18}$$

在 l^+ 上，由于 $\tau_l(t, x)=0$，$\xi_l(t, x)=1$，易得 $\dfrac{\partial Q_l[y]}{\partial x}$ 的跃度为

$$\begin{aligned}\left[\frac{\partial Q_{\bar r}[y]}{\partial x}\right] &\equiv \frac{\partial Q_{\bar r}[y]}{\partial x}\Big|_+ - \frac{\partial Q_{\bar r}[y]}{\partial x}\Big|_- \\ &= \frac{\partial \tau_{\bar r}}{\partial x}(t, x)\left[\dot{\psi}_{\bar r}(0) - \zeta_{\bar r}(0, 1)\frac{\partial v}{\partial t}(0, 1) - \mu_{\bar r}(0, 1)\right] \\ &\quad + \frac{\partial v}{\partial x}(0, 1)\zeta_{\bar r}(0, 1)\frac{\partial \xi_{\bar r}}{\partial x}(t, x).\end{aligned} \tag{3.19}$$

但注意到(3.9)及(3.10)式，可得当 $(t, x)\in l^+$ 时成立

$$\frac{\partial \xi_{\bar r}}{\partial x}(t, x) = -\lambda_{\bar r}(0, 1)\frac{\partial \tau_{\bar r}}{\partial x}(t, x), \quad (\bar r=1, \cdots, h), \tag{3.20}$$

将(3.20)代入(3.19)式，并利用相容性条件(3.7)的后二式可得

$$\left[\frac{\partial Q_{\bar r}[y]}{\partial x}\right] = 0, \quad (\bar r=1, \cdots, h), \tag{3.21}$$

对 l^- 亦有类似结论，故 $\dfrac{\partial Q[y]}{\partial x}$ 在 $R(\delta_0)$ 上也连续.

由此得 S 是 $C^1(R(\delta_0))$ 到 $C^1(R(\delta_0))$ 的映照，在 $C^1(R(\delta_0))$ 上取范数

$$\|u\|_* = \sup|e^{-at}u(t, x)| + b\left(\sup\left|e^{-at}\frac{\partial u}{\partial x}(t, x)\right| + \sup\left|e^{-at}\frac{\partial u}{\partial t}(t, x)\right|\right). \tag{3.22}$$

可以证明，当 a, b 充分大时，S 为 $C^1(R(\delta_0))$ 到 $C^1(R(\delta_0))$ 的压缩算子，故存在不动点 u，使 $u=Su$，此 u 即积分方程组(3.12)的 C^1 解，亦即问题(3.1)—(3.4)的 C^1 解，引理 3.1 得证.

以下设法推导解的一些估计式，为此引入下列函数集合．记

$$\begin{cases} \Gamma_0 = \left\{\lambda_l, \zeta_l, \mu_l, \dfrac{1}{\lambda_{\bar r}(t, 1)}, \dfrac{1}{\lambda_{\hat s}(t, 0)}\right\}; \\ \Gamma_1 = \left\{\zeta_{lj}, \dfrac{\partial \zeta_{lj}}{\partial t}, \dfrac{\partial \zeta_{lj}}{\partial x}, \zeta_l, \dfrac{\partial \zeta_l}{\partial t}, \dfrac{\partial \zeta_l}{\partial x}, \lambda_l, \dfrac{1}{\det(\zeta_{lj})}\right\}, \\ \Gamma_2 = \Gamma_1 \cup \left\{\dfrac{\partial \lambda_l}{\partial x}, \mu_l, \dfrac{\partial \mu_l}{\partial x}, \dfrac{1}{\lambda_{\bar r}(t, 1)}, \dfrac{1}{\lambda_{\hat s}(t, 0)}\right\}. \end{cases} \quad \begin{pmatrix} l=1, \cdots, n \\ \bar r=1, \cdots, h \\ \hat s=m+1, \cdots, n \end{pmatrix}; \tag{3.23}$$

引理 3.2(第一估计式)　设 $u(t, x)$ 是定解问题(3.1)—(3.4)在 $R(\delta_0)$ 上的 C^1 解，且定向性条件(3.6)成立，则对任何 $\delta\,(0\leqslant\delta\leqslant\delta_0)$，在 $R(\delta)$ 上成立估计式

$$\|u\| \leqslant (1+K_1\delta)\|\psi\| + (H_0+K_1\delta)\|v\| + K_1\delta\|\mu\|, \tag{3.24}$$

其中 K_1 是仅与 $R(\delta_0)$ 上的模 $\|\Gamma_1\|$ 有关的常数. 而 $H_0=2\sup\limits_{\substack{l=1,\cdots,n\\(t,x)\in R(\delta_0)}}|\zeta_l(t,x)|$.

证　将第一积分关系式(3.12)改写为

$$\sum_{j=1}^{n}\zeta_{lj}u_j=\widetilde{\psi}_l(\tau_l(t,x))+\int_{\tau_l(t,x)}^{t}\Big(\sum_{j=1}^{n}\frac{D\zeta_{lj}}{D_l\tau}u_j-\frac{D\zeta_l}{D_l\tau}v+\mu_l\Big)d\tau$$
$$+\zeta_l(t,x)v(t,x)-\zeta_l(\tau_l(t,x),\xi_l(t,x))v(\tau_l(t,x),\xi_l(t,x)).\quad(3.25)$$

采用通常的 Haar 估计(参见[5]),可由此得到对 $\sum\limits_{j=1}^{n}\zeta_{lj}u_j$ 的估计, 再由(3.5)式, 即可得第一估计式(3.24).

记

$$\|u\|_1=\|u\|+\left\|\frac{\partial u}{\partial t}\right\|+\left\|\frac{\partial u}{\partial x}\right\|,\quad(3.26)$$

$$\|u\|_1^*=\|u\|+\left\|\frac{\partial u}{\partial t}\right\|+\varepsilon\left\|\frac{\partial u}{\partial x}\right\|,\quad(3.27)$$

$$\|u\|_{1+\beta}=\|u\|_1+H_*^\beta\Big[\frac{\partial u}{\partial t}\Big]+H_*^\beta\Big[\frac{\partial u}{\partial x}\Big],\quad(3.28)$$

$$\|u\|_{1+\beta}^*=\|u\|_1^*+H_t^\beta\Big[\frac{\partial u}{\partial t}\Big]+\varepsilon\Big(H_x^\beta\Big[\frac{\partial u}{\partial t}\Big]+H_*^\beta\Big[\frac{\partial u}{\partial x}\Big]\Big),\quad(0<\beta\leqslant1),\quad(3.29)$$

其中 ε 为适当小的正常数,其值根据需要在下文§4中确定,而

$$H_*^\beta[f]=H_x^\beta[f]+H_t^\beta[f].\quad(3.30)$$

引理3.3(第二估计式)　设 $u(t,x)$ 是定解问题(3.1)—(3.4)在 $R(\delta_0)$ 上 C^1 解, 且(3.6)成立, 并设 λ_l, μ_l 对 t 属于 Lipβ, $(0<\beta\leqslant1)$ 则对任何 $\delta(0\leqslant\delta\leqslant\delta_0)$ 在 $R(\delta)$ 上成立估计式

$$\|u\|_1^*\leqslant(1+d_0^{-1}\varepsilon+K_2\delta^\beta)\|\dot{\psi}\|+(K_0+K_2\delta)(1+\|v\|_1),\quad(3.31)$$

其中 $d_0=\min\limits_{\substack{1\leqslant\bar{r}\leqslant h\\m+1\leqslant\hat{s}\leqslant n}}\{-\lambda_{\bar{r}}(0,1),\lambda_{\hat{s}}(0,0)\}$, K_0 仅依赖于 $R(\delta_0)$ 上的模 $\|\Gamma_0\|$, K_2 仅依赖于 $R(\delta_0)$ 上的模 $\|\Gamma_2\|$ 及 $H_t^\beta[\Gamma_0]$.

证　先推导解的一阶偏导数所满足的积分关系式, 在推导过程中, 暂设 u, v 为 C^2 函数.

记

$$p_i=\frac{\partial u_i}{\partial t},\quad w_i=\frac{\partial u_i}{\partial x},\quad q=\frac{\partial v}{\partial t},\quad r=\frac{\partial v}{\partial x}.\quad(3.32)$$

将(3.1)关于 x 求导一次得

$$\sum_{j=1}^{n}\zeta_{lj}\Big(\frac{\partial w_j}{\partial t}+\lambda_l\frac{\partial w_j}{\partial x}\Big)=\zeta_l\Big(\frac{\partial r}{\partial t}+\lambda_l\frac{\partial r}{\partial x}\Big)+\bar{\mu}_l,\quad(3.33)$$

其中

$$\bar{\mu}_l=-\sum_{j=1}^{n}\frac{\partial\zeta_{lj}}{\partial x}p_j-\sum_{j=1}^{n}\frac{\partial(\zeta_{lj}\lambda_l)}{\partial x}w_j+\frac{\partial\zeta_l}{\partial x}q+\frac{\partial(\zeta_l\lambda_l)}{\partial x}r+\frac{\partial\mu_l}{\partial x}.\quad(3.34)$$

将边值条件(3.3)关于 t 求导一次得

$$\sum_{j=1}^{n}\zeta_{\bar{r}j}(t,1)p_j=\dot{\psi}_{\bar{r}}(t)-\sum_{j=1}^{n}\frac{\partial\zeta_{\bar{r}j}(t,1)}{\partial t}u_j(t,1),\quad(\bar{r}=1,\cdots,h,\ h\leqslant n),\quad(3.35)$$

再利用原方程(3.1)即得在 $x=1$ 上成立

$$\sum_{j=1}^{n}\zeta_{\bar{r}j}(t,\ 1)w_j=\frac{1}{\lambda_{\bar{r}}(t,\ 1)}\Big\{-\dot{\psi}_{\bar{r}}(t)+\sum_{j=1}^{n}\frac{\partial\zeta_{\bar{r}j}(t,\ 1)}{\partial t}u_j(t,\ 1)$$
$$+(\zeta_{\bar{r}}(t,\ 1)(q(t,\ 1)+\lambda_{\bar{r}}(t,\ 1)r(t,\ 1))+\mu_{\bar{r}}(t,\ 1))\Big\}\equiv\bar{\psi}_{\bar{r}}(t). \quad (3.36)$$

同理，在 $x=0$ 上成立

$$\sum_{j=1}^{n}\zeta_{\hat{s}j}(t,\ 0)w_j=\frac{1}{\lambda_{\hat{s}}(t,\ 0)}\Big\{-\dot{\hat{\psi}}_{\hat{s}}(t)+\sum_{j=1}^{n}\frac{\partial\zeta_{\hat{s}j}}{\partial t}(t,\ 0)u_j(t,\ 0)+\zeta_{\hat{s}}(t,\ 0)(q(t,\ 0)$$
$$+\lambda_{\hat{s}}(t,\ 0)r(t,\ 0))+\mu_{\hat{s}}(t,\ 0)\Big\}\equiv\bar{\hat{\psi}}_{\hat{s}}(t). \quad (3.37)$$

又初始条件为

$$t=0,\quad w_j=0. \quad (3.38)$$

由于相容性条件(3.7)，显然有

$$\begin{cases}\bar{\psi}_{\bar{r}}(0)=0, & (\bar{r}=1,\ \cdots,\ h);\\ \bar{\hat{\psi}}_{\hat{s}}(0)=0, & (\hat{s}=m+1,\ \cdots,\ n).\end{cases} \quad (3.39)$$

类似于(3.26)的导出，可得 w_j 满足下述积分方程组

$$\sum_{j=1}^{n}\zeta_{lj}(t,\ x)w_j(t,\ x)=\tilde{\bar{\psi}}_l(\tau_l(t,\ x))+\int_{\tau_l(t,x)}^{t}\Big(\sum_{j=1}^{n}\frac{D\zeta_{lj}}{D_l\tau}w_j-\frac{D\zeta_l}{D_l\tau}r+\bar{\mu}_l\Big)d\tau$$
$$+\zeta_l(t,\ x)r(t,\ x)-\zeta_l(\tau_l(t,\ x),\ \xi_l(t,\ x))r(\tau_l(t,\ x),\ \xi_l(t,\ x)), \quad (3.40)$$

其中

$$\tilde{\bar{\psi}}_l(\tau_l(t,\ x))=\begin{cases}\bar{\psi}_{\bar{r}}(\tau_{\bar{r}}(t,\ x)), & 当(t,\ x)\in R_{\bar{r}}^{+},\quad l=\bar{r}\ 时;\\ \bar{\hat{\psi}}_{\hat{s}}(\tau_{\hat{s}}(t,\ x)), & 当(t,\ x)\in R_{\hat{s}}^{-},\quad l=\hat{s}\ 时;\\ 0, & 当(t,\ x)\in R_l 时.\end{cases} \quad (3.41)$$

由(3.1)及(3.40)可知 p_j 满足的积分方程组为

$$\sum_{j=1}^{n}\zeta_{lj}(t,\ x)p_j(t,\ x)=-\lambda_l(t,\ x)\tilde{\bar{\psi}}_l(\tau_l(t,\ x))-\lambda_l(t,\ x)$$
$$\int_{\tau_l(t,x)}^{t}\Big(\sum_{j=1}^{n}\frac{D\zeta_{lj}}{D_l\tau}w_j-\frac{D\zeta_l}{D_l\tau}r+\bar{\mu}_l\Big)d\tau$$
$$+\zeta_l(t,\ x)q(t,\ x)+\lambda_l(t,\ x)\zeta_l(\tau_l(t,\ x),\ \xi_l(t,\ x))r(\tau_l,\ \xi_l)$$
$$+\mu_l(t,\ x). \quad (3.42)$$

由于(3.40)及(3.42)中仅出现 u, v 的一阶导数，故对 u, v 为 C^1 时此积分关系式仍成立．称(3.40)及(3.42)为解的第二积分关系式．

利用 Haar 估计，由(3.40)及(3.42)可得在 $R(\delta)$ 上成立较粗略的估计

$$\|w\|,\ \|p\|\leqslant K_2(1+\|\dot{\psi}\|+\|v\|_1), \quad (3.43)$$

其中

$$\|\dot{\psi}\|=\max_{\substack{1\leqslant\bar{r}\leqslant h\\ m+1\leqslant\hat{s}\leqslant n}}(\|\dot{\psi}_{\bar{r}}\|,\ \|\dot{\hat{\psi}}_{\hat{s}}\|). \quad (3.44)$$

由表达式(3.36)、(3.37)并注意到 $u(0,\ 0)=0$, $u(0,\ 1)=0$ 及(3.43)可得

$$|\bar{\psi}_{\bar{r}}(t)|,\ |\bar{\hat{\psi}}_{\hat{s}}(t)|\leqslant(d_0^{-1}+K_2\delta^{\beta})\|\dot{\psi}\|+K_2\delta(1+\|\dot{\psi}\|$$
$$+\|v\|_1)+K_0(1+\|v\|_1), \quad (3.45)$$

于是对 $\tilde{\bar{\psi}}_l(\tau_l(t,\ x))$ 可得类似的估计．

又注意到在 $R(\delta)$ 上使 $\tau_l(t,\ x)\neq 0$ 的 $(t,\ x)$ 满足 $|x-1|+t\leqslant K_0\delta$ 或 $x+t\leqslant K_0\delta$，可得

$$|\lambda_l(t,\ x)\tilde{\tilde{\psi}}_l(\tau_l(t,\ x))|\leqslant(1+K_2\delta^\beta)\|\dot{\psi}\|+K_2\delta(1+\|\dot{\psi}\|+\|v\|_1)+K_0(1+\|v\|_1). \tag{3.46}$$

对第二积分关系式中的其它非积分项再进行估计可得

$$|\zeta_l(t,\ x)r(t,\ x)-\zeta_l(\tau_l(t,\ x),\ \xi_l(t,\ x))r(\tau_l,\ \xi_l)|\leqslant K_0\|v\|_1, \tag{3.47}$$

$$|\zeta_l(t,\ x)q(t,\ x)+\lambda_l(t,\ x)\zeta_l(\tau_l,\ \xi_l)r(\tau_l,\ \xi_l)+\mu_l(t,\ x)|\leqslant K_0(1+\|v\|_1). \tag{3.48}$$

又利用 (3.43) 式对被积函数项成立

$$\left|\sum_{j=1}^n\frac{D\zeta_{lj}}{D_l\tau}w_j-\frac{D\zeta_l}{D_l\tau}r+\bar{\mu}_l\right|\leqslant K_2(1+\|\dot{\psi}\|+\|v\|_1). \tag{3.49}$$

将 (3.45)—(3.49) 代入 (3.40) 及 (3.42)，即可得第二估计式 (3.31)。

在叙述第三估计式前，先证明下述引理：

引理 3.4　对任何 $\delta(0\leqslant\delta\leqslant\delta_0)$ 在 $R(\delta)$ 上成立

$$\left\|\frac{\partial\tau_l}{\partial t}\right\|\leqslant 1+D_*\delta^\beta,\qquad\left\|\frac{\partial\tau_l}{\partial x}\right\|\leqslant d_0^{-1}+D_*\delta^\beta, \tag{3.50}$$

$$\left\|\frac{\partial\xi_l}{\partial t}\right\|\leqslant\|\lambda\|+D_*\delta,\quad\left\|\frac{\partial\xi_l}{\partial x}\right\|\leqslant 1+D_*\delta, \tag{3.51}$$

其中 D_* 仅依赖于 d_0，$\|\lambda\|$，$\left\|\frac{\partial\lambda}{\partial x}\right\|$ 及 $H_t^\beta[\lambda_l]$。

证　由 (3.8) 易知

$$\frac{\partial f_l}{\partial x}=e^{-\int_\tau^t\frac{\partial\lambda_l}{\partial\xi}(\tau_1,f_l(\tau_1;t,x))d\tau_1}, \tag{3.52}$$

$$\frac{\partial f_l}{\partial t}=-\lambda_l(t,\ x)e^{-\int_\tau^t\frac{\partial\lambda_l}{\partial\xi}(\tau_1,f_l(\tau_1;t,x))d\tau_1}, \tag{3.53}$$

由此及 ξ_l 之定义即得 (3.51)。

再注意到当 $(t,\ x)\in R_l(l=1,\ \cdots,\ n)$ 时，$\tau_l=0$，此时 (3.50) 恒成立。而当 $(t,\ x)\in R_{\bar{r}}^+$ $(\bar{r}=1,\ \cdots,\ h)$ 时，成立 $f_{\bar{r}}(\tau_{\bar{r}}(t,\ x),\ t,\ x)=1$。将其两端分别关于 t 及 x 求导一次，并将 (3.52)，(3.53) 代入即得

$$\frac{\partial\tau_{\bar{r}}}{\partial x}=-\frac{1}{\lambda_{\bar{r}}(\tau_{\bar{r}},\ 1)}e^{\int_t^{\tau_{\bar{r}}}\frac{\partial\lambda_{\bar{r}}}{\partial\xi}(\tau_1,f_{\bar{r}}(\tau_1;t,x))d\tau_1}, \tag{3.54}$$

$$\frac{\partial\tau_{\bar{r}}}{\partial t}=-\frac{\lambda_{\bar{r}}(t,\ x)}{\lambda_{\bar{r}}(\tau_{\bar{r}},\ 1)}e^{\int_t^{\tau_{\bar{r}}}\frac{\partial\lambda_{\bar{r}}}{\partial\xi}(\tau_1,f_{\bar{r}}(\tau_1;t,x))d\tau_1}. \tag{3.55}$$

由此式，并注意到当 $(t,\ x)\in R_{\bar{r}}^+$ 时，$|x-1|+t\leqslant K_0\delta$，就可得对 $\bar{r}=1,\ \cdots,\ h$，(3.50) 成立。对 $(t,\ x)\in R_{\hat{s}}^-(\hat{s}=m+1,\ \cdots,\ n)$ 亦有同样结论，故 (3.50) 成立。

引理 3.5（第三估计式）　进一步假设 Γ_2 中的一切函数对 t，x 均属于 $\mathrm{Lip}\,\beta\ (0<\beta\leqslant 1)$，且 $\frac{\partial v}{\partial t}$，$\frac{\partial v}{\partial x}$ 关于 t，x 均属于 $\mathrm{Lip}\,\beta$，$\dot{\psi}_{\bar{r}}$ 关于 t 属于 $\mathrm{Lip}\,\beta$，则定解问题 (3.1)—(3.4) 在 $R(\delta_0)$ 上的解 $u(t,\ x)$ 的一阶偏导数 $\frac{\partial u}{\partial t}$，$\frac{\partial u}{\partial x}$ 关于 t，x 亦属于 $\mathrm{Lip}\,\beta$（记为 $u\in C^{1+\beta}$）且

$$\|u\|_{1+\beta}^*\leqslant(1+2d_0^{-1}\varepsilon+d_0^{-2}\varepsilon+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+(K_2+K_3\delta)(1+\|\dot{\psi}\|+\|v\|_{1+\beta}), \tag{3.56}$$

其中 K_2 仅依赖于 $R(\delta_0)$ 上的模 $\|\Gamma_2\|$ 及 $H_t^\beta[\Gamma_0]$，而正常数 K_3 还依赖于 $R(\delta_0)$ 上的 $H_*^\beta[\Gamma_2]$.

证　利用 Haar 估计，由第二积分关系式及(3.40)，(3.42)可得粗的估计

$$H_t^\beta[w],\ H_t^\beta[p],\ H_x^\beta[w],\ H_x^\beta[p]\leqslant K_3(1+\|v\|_{1+\beta}+\|\dot{\psi}\|)+K_2H_t^\beta[\dot{\psi}],\tag{3.57}$$

其中

$$H_t^\beta[\dot{\psi}]=\max_{\substack{1\leqslant\bar{r}\leqslant h\\ m+1\leqslant\hat{s}\leqslant n}}(H_t^\beta[\dot{\psi}_{\bar{r}}],\ H_t^\beta[\dot{\hat{\psi}}_{\hat{s}}]).\tag{3.58}$$

对第二积分关系式中非积分项估计其 Hölder 常数，由(3.36)及(3.37)，有

$$H_t^\beta[\bar{\psi}_{\bar{r}}(t)],\ H_t^\beta[\bar{\hat{\psi}}_{\hat{s}}(t)]\leqslant(d_0^{-1}+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+K_2(1+\|\dot{\psi}\|+\|v\|_{1+\beta})+(K_2+K_3\delta)\|u\|_1^*.\tag{3.59}$$

利用引理 3.4，即得在 $R(\delta)$ 上成立

$$H_t^\beta[\bar{\bar{\psi}}_l(\tau_l(t,\ x))]\leqslant(d_0^{-1}+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+K_2(1+\|\dot{\psi}\|+\|v\|_{1+\beta})+(K_2+K_3\delta)\|u\|_1^*,\tag{3.60}$$

$$H_x^\beta[\bar{\bar{\psi}}_l(\tau_l(t,\ x))]\leqslant(d_0^{-2}+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+K_2(1+\|\dot{\psi}\|+\|v\|_{1+\beta})+(K_2+K_3\delta)\|u\|_1^*,\tag{3.61}$$

与引理 3.3 中的证明类似，由此可得

$$H_t^\beta[\lambda_l(t,\ x)\bar{\bar{\psi}}_l(\tau_l(t,\ x))]\leqslant(1+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+K_2(1+\|\dot{\psi}\|+\|v\|_{1+\beta})+(K_2+K_3\delta)\|u\|_1^*,\tag{3.62}$$

$$H_x^\beta[\lambda_l(t,\ x)\bar{\bar{\psi}}_l(\tau_l(t,\ x))]\leqslant(d_0^{-1}+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+K_2(1+\|\dot{\psi}\|+\|v\|_{1+\beta})+(K_2+K_3\delta)\|u\|_1^*,\tag{3.63}$$

由(3.60)—(3.63)，并利用第二估计式(3.31)可得

$$H_t^\beta[\lambda_l(t,\ x)\bar{\bar{\psi}}_l(\tau_l)]+\varepsilon\{H_x^\beta[\lambda_l(t,\ x)\bar{\bar{\psi}}_l(\tau_l)]+H_*^\beta[\bar{\bar{\psi}}_l(\tau_l)]\}\leqslant(1+2d_0^{-1}\varepsilon+d_0^{-2}\varepsilon+K_2\delta^\beta)H_t^\beta[\dot{\psi}]+(K_2+K_3\delta)(1+\|\dot{\psi}\|+\|v\|_{1+\beta}).\tag{3.64}$$

对第二积分关系式其它非积分项有估计式

$$H_*^\beta[\zeta_l(t,\ x)r(t,\ x)-\zeta_l(\tau_l,\ \xi_l)r(\tau_l,\ \xi_l)]\leqslant K_2(1+\|v\|_{1+\beta}),\tag{3.65}$$

$$H_*^\beta[\zeta_l(t,\ x)q(t,\ x)-\zeta_l(\tau_l,\ \xi_l)r(\tau_l,\ \xi_l)\lambda_l(t,\ x)+\mu_l(t,\ x)]\leqslant K_2(1+\|v\|_{1+\beta}).\tag{3.66}$$

利用(3.57)对被积函数项成立

$$H_*^\beta\Big[\sum_{j=1}^n\frac{D\zeta_{lj}}{D_l\tau}w_j-\frac{D\zeta_l}{D_l\tau}r+\bar{\mu}_l\Big]\leqslant K_3(1+\|v\|_{1+\beta}+\|\dot{\psi}\|)+K_2H_t^\beta[\dot{\psi}].\tag{3.67}$$

将(3.64)—(3.67)代入(3.40)及(3.42)即得第三估计式(3.56).

§4. 解的存在性

利用§2，§3中对热传导方程第二边值问题及线性双曲型方程组混合问题所建立的估计式，本节中证明拟线性双曲抛物耦合方程组第二边值问题解的存在性.

假设方程组(1.2)，(1.3)的系数及边值函数在所考察的区域上满足下述光滑性条件(简记为条件(A))：

(i) $\zeta_{lj}(t, x, u, v)$及其一阶偏导数连续，且其一阶偏导数$\frac{\partial\zeta_{lj}}{\partial t}$, $\frac{\partial\zeta_{lj}}{\partial x}$, $\frac{\partial\zeta_{lj}}{\partial u_k}(k=1, \cdots, n)$, $\frac{\partial\zeta_{lj}}{\partial v}$对$t$, x, u, v均属于Lip $\frac{\alpha}{2}$. 对$\zeta_l(t, x, u, v)$条件相同. 又假设成立$|\det|\zeta_{lj}||\geqslant D_0>0(D_0=$常数).

(ii) $\lambda_l(t, x, u, v, r)\left(r=\frac{\partial v}{\partial x}\right)$及$\frac{\partial\lambda_l}{\partial x}$, $\frac{\partial\lambda_l}{\partial u_k}$, $\frac{\partial\lambda_l}{\partial v}$, $\frac{\partial\lambda_l}{\partial r}$连续，且$\lambda_l$关于$t$属于Lip $\frac{\alpha}{2}$, $\frac{\partial\lambda_l}{\partial x}$, $\frac{\partial\lambda_l}{\partial u_k}$, $\frac{\partial\lambda_l}{\partial v}$关于$t$, x, u, v, r属于Lip $\frac{\alpha}{2}$, $\frac{\partial\lambda_l}{\partial r}$关于$t$, x, u, v属于Lip $\frac{\alpha}{2}$, 而关于r属于$\frac{1}{2}$. 对μ_l假设相同.

(iii) $a(t, x, u, v, r)$连续，对t属于Lip $\frac{\alpha}{2}$, 对x属于Lip α, 对u, v, r属于Lip 1. 对b有同样假设.

(iv) $G_{\bar{r}}(t, u, v)(\bar{r}=1, \cdots, h)$及$\hat{G}_{\hat{s}}(t, u, v)(\hat{s}=m+1, \cdots, n)$连续，且其一切一阶偏导数关于$t$, u, v属于Lip $\frac{\alpha}{2}$.

(v) $F_{\pm}$关于t, u, v连续可微.

定理 4.1(存在性定理) 假设条件(A)成立，且$a(t, x, u, v, v_x)\geqslant a_0>0$, 并假设成立定向性条件(1.15)—(1.16), 相容性条件(1.17)—(1.21)及示性数条件(1.22)—(1.23), 则必存在适当小的正数$\delta_*\leqslant\delta_0$, 使在区域$R(\delta_*)$上拟线性双曲抛物耦合方程组的第二边值问题(1.2)—(1.3), (1.4)—(1.5), (1.11)—(1.14)存在解$u\in C^{1+\frac{\alpha}{2}}$, $v\in\bar{C}^{2+\alpha}$.

证 由示性数条件(1.22)—(1.24), 定向性条件(1.15)—(1.16)知

$$\theta=\max_{\substack{1\leqslant\bar{r}\leqslant h\\ m+1\leqslant\hat{s}\leqslant n}}\left(\sum_{j=1}^{n}\left|\frac{\partial G_{\bar{r}}}{\partial u_j}(0, 0, 0)\right|, \sum_{j=1}^{n}\left|\frac{\partial\hat{G}_{\hat{s}}}{\partial u_j}(0, 0, 0)\right|\right)<1, \tag{4.1}$$

$$d_0=\min_{\substack{1\leqslant\bar{r}\leqslant h\\ m+1\leqslant\hat{s}\leqslant n}}\{-\lambda_{\bar{r}}(0, 1, 0, 0, 0), \lambda_{\hat{s}}(0, 0, 0, 0, 0)\}>0, \tag{4.2}$$

取$\varepsilon>0$充分小使

$$\theta_1=(1+d_0^{-1}\varepsilon)\theta<1, \tag{4.3}$$

$$\theta_2=(1+2d_0^{-1}\varepsilon+d_0^{-2}\varepsilon)\theta<1. \tag{4.4}$$

在$R(\delta)$上引入函数集合

$$\Sigma_*(\delta)=\left\{(u, v)\,\middle|\,u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\in C^0, u(0, x)=v(0, x)=0\right\}, \tag{4.5}$$

$$\Sigma_1(\delta)=\left\{(u, v)\,\middle|\,\begin{array}{l}u\in C^{1+\frac{\alpha}{2}}, v\in\bar{C}^{2+\alpha}, u(0, x)=v(0, x)=0\\ \frac{\partial u_j}{\partial t}(0, x)=\mu_j(0, x, 0, 0, 0), \frac{\partial v}{\partial t}(0, x)=0\end{array}\right\}, \tag{4.6}$$

$$\Sigma(\delta)=\left\{(u, v)\,\middle|\,\begin{array}{l}(u, v)\in\Sigma_1(\delta), \|u\|\leqslant A_0, \|u\|_1^*\leqslant A_1, \|u\|^*_{1+\frac{\alpha}{2}}\leqslant A_2\\ |v|\leqslant B_0, |v|_1\leqslant B_1, |v|_2\leqslant B_2\end{array}\right\}, \tag{4.7}$$

其中A_i, $B_i(i=0, 1, 2)$均为正常数(其值待定), 且$A_0\leqslant A_1\leqslant A_2$, $B_0\leqslant B_1\leqslant B_2$.

对$(\tilde{u}, \tilde{v})\in\Sigma_1(\delta)$, 记

$$\begin{cases}\tilde{\zeta}_{lj}(t,\ x)=\zeta_{lj}(t,\ x,\ \tilde{u}(t,\ x),\ \tilde{v}(t,\ x));\\ \tilde{\zeta}_{l}(t,\ x)=\zeta_{l}(t,\ x,\ \tilde{u}(t,\ x),\ \tilde{v}(t,\ x));\\ \tilde{\lambda}_{l}(t,\ x)=\lambda_{l}\left(t,\ x,\ \tilde{u}(t,\ x),\ \tilde{v}(t,\ x),\ \dfrac{\partial\tilde{v}}{\partial x}(t,\ x)\right);\\ \tilde{\mu}_{l}(t,\ x)=\mu_{l}\left(t,\ x,\ \tilde{u}(t,\ x),\ \tilde{v}(t,\ x),\ \dfrac{\partial\tilde{v}}{\partial x}(t,\ x)\right);\\ \tilde{b}(t,\ x)=b\left(t,\ x,\ \tilde{u}(t,\ x),\ \tilde{v}(t,\ x),\ \dfrac{\partial\tilde{v}}{\partial x}(t,\ x)\right)\\ \qquad +\left[a\left(t,\ x,\ \tilde{u},\ \tilde{v},\ \dfrac{\partial\tilde{v}}{\partial x}\right)-1\right]\dfrac{\partial^2\tilde{v}}{\partial x^2}.\end{cases} \tag{4.8}$$

在 $R(\delta)$ 上求解如下线性方程组的定解问题

$$\sum_{j=1}^{n}\tilde{\zeta}_{lj}(t,\ x)\left(\frac{\partial u_j}{\partial t}+\tilde{\lambda}_l(t,\ x)\frac{\partial u_j}{\partial x}\right)=\tilde{\zeta}_l(t,\ x)\left(\frac{\partial\tilde{v}}{\partial t}+\tilde{\lambda}_l(t,\ x)\frac{\partial\tilde{v}}{\partial x}\right)+\tilde{\mu}_l(t,\ x),\quad (l=1,\ \cdots,\ n); \tag{4.9}$$

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=\tilde{b}(t,\ x); \tag{4.10}$$

$$t=0:\quad u_j=0; \tag{4.11}$$

$$v=0; \tag{4.12}$$

$$x=1:\quad \sum_{j=1}^{n}\tilde{\zeta}_{\bar{r}j}(t,\ 1)u_j=G_{\bar{r}}(t,\ \tilde{u}(t,\ 1),\ \tilde{v}(t,\ 1))+\sum_{j=1}^{n}(\tilde{\zeta}_{\bar{r}j}(t,\ 1)-\delta_{\bar{r}j})\tilde{u}_j(t,\ 1)\equiv\psi_{\bar{r}}(t),\quad (\bar{r}=1,\ \cdots,\ h); \tag{4.13}$$

$$\frac{\partial v}{\partial x}(t,\ 1)=F_+(t,\ \tilde{u}(t,\ 1),\ \tilde{v}(t,\ 1))\equiv\varphi_+(t); \tag{4.14}$$

$$x=0:\quad \sum_{j=1}^{n}\tilde{\zeta}_{\hat{s}j}(t,\ 0)u_j=\hat{G}_{\hat{s}}(t,\ \tilde{u}(t,\ 0),\ \tilde{v}(t,\ 0))+\sum_{j=1}^{n}(\tilde{\zeta}_{\hat{s}j}(t,\ 0)-\delta_{\hat{s}j})\tilde{u}_j(t,\ 0)\equiv\psi_{\hat{s}}(t),\quad (\hat{s}=m+1,\ \cdots,\ n) \tag{4.15}$$

$$\frac{\partial v}{\partial x}(t,\ 0)=F_-(t,\ \tilde{u}(t,\ 0),\ \tilde{v}(t,\ 0))\equiv\varphi_-(t), \tag{4.16}$$

其中(4.10), (4.12), (4.14), (4.16)构成§2中所述的热传导方程的第二边值问题，易证其满足引理2.3所需的一切条件，故在 $R(\delta)$ 上存在唯一解 $v(t,\ x)\in\bar{C}^{2+\alpha}$，且成立估计式(2.26), (2.27)及(2.43). 而(4.9), (4.11), (4.13), (4.15)构成§3中所讨论的线性双曲型方程组混合问题，易证其满足引理3.1及引理3.4的一切条件，故在 $R(\delta)$ 上存在唯一解 $u\in C^{1+\frac{\alpha}{2}}$，且成立估计式(3.24), (3.31)及(3.56). 记由 $(\tilde{u},\ \tilde{v})$ 通过求解上述定解问题而得到 (u,v) 的算子为 T，则 $T(\tilde{u},\ \tilde{v})=(u,\ v)$ 是 $\Sigma_1(\delta)$ 到其自身的映照. 下面说明利用§2, §3中解的估计式可适当选取常数 A_0, A_1, A_2, B_0, B_1, B_2 及 δ_*，使 T 成为 $\Sigma(\delta_*)$ 到其自身的映照.

由 $\tilde{b}(t,\ x)$ 的定义知当 $(\tilde{u},\ \tilde{v})\in\Sigma(\delta)$ 时成立

$$\|\tilde{b}\|\leqslant D_1(A_0,\ B_0)+D_2(A_1,\ B_1)\delta^{\frac{\alpha}{2}}, \tag{4.17}$$

$$H^{\alpha}[\tilde{b}]\leqslant D_3(A_1,\ B_1)+D_4(A_1,\ B_2)\delta^{\frac{\alpha}{2}}, \tag{4.18}$$

其中 $D_1(A_0,\ B_0)$ 表示与 A_0, B_0 有关的常数，其余类同.

由 $\varphi_\pm(t)$ 的定义(4.14)及(4.16)知

$$\|\dot{\varphi}\|\leqslant D_5(A_1,\ B_1). \tag{4.19}$$

将(4.17)—(4.19)代入§2中的估计式(2.26), (2.27)及(2.43)可得

$$|v|\leqslant D_6(A_1,\ B_1)\delta^{\frac{1}{2}}, \tag{4.20}$$

$$|v|_1\leqslant D_7(A_0,\ B_0)+D_8(A_1,\ B_2)\delta^{\frac{\alpha}{2}}, \tag{4.21}$$

$$|v|_2\leqslant D_9(A_1,\ B_1)+D_{10}(A_1,\ B_2)\delta^{\frac{\alpha}{2}}. \tag{4.22}$$

又对由(4.9), (4.11), (4.13), (4.15)构成的线性双曲型方程组的混合问题采用类似方法讨论,引入集合

$$\begin{cases}\widetilde{\Gamma}_0=\left\{\widetilde{\lambda}_l,\ \widetilde{\zeta}_l,\ \widetilde{\mu}_l,\ \dfrac{1}{\widetilde{\lambda}_{\bar{r}}(t,\ 1)},\ \dfrac{1}{\widetilde{\lambda}_{\hat{s}}(t,\ 0)}\right\};\\ \widetilde{\Gamma}_1=\left\{\widetilde{\zeta}_{lj},\ \dfrac{\partial\widetilde{\zeta}_{lj}}{\partial t},\ \dfrac{\partial\widetilde{\zeta}_{lj}}{\partial x},\ \widetilde{\zeta}_l,\ \dfrac{\partial\widetilde{\zeta}_l}{\partial t},\ \dfrac{\partial\widetilde{\zeta}_l}{\partial x},\ \widetilde{\lambda}_l,\ \dfrac{1}{\det(\widetilde{\zeta}_{lj})}\right\};\\ \widetilde{\Gamma}_2=\widetilde{\Gamma}_1\cup\left\{\dfrac{\partial\widetilde{\lambda}_l}{\partial x},\ \widetilde{\mu}_l,\ \dfrac{\partial\widetilde{\mu}_l}{\partial x},\ \dfrac{1}{\widetilde{\lambda}_{\bar{r}}(t,\ 1)},\ \dfrac{1}{\widetilde{\lambda}_{\hat{s}}(t,\ 0)}\right\},\end{cases} \tag{4.23}$$

则有

$$\|\widetilde{\Gamma}_0|\leqslant D_{11}(A_0,\ B_0),$$

$$\|\widetilde{\Gamma}_1\|,\ \|\widetilde{\Gamma}_2\|\leqslant D_{12}(A_1,\ B_1),$$

$$H_t^{\frac{\alpha}{2}}[\widetilde{\Gamma}_0]\leqslant D_{13}(A_1,\ B_1),$$

$$H_*^{\frac{\alpha}{2}}[\widetilde{\Gamma}_2]\leqslant D_{14}(A_2,\ B_2),$$

且由(4.13)及(4.1)知

$$\|\dot{\psi}_r(t)\|\leqslant\theta A_1+D_{15}(A_0,\ B_1)+D_{16}(A_1,\ B_1)\delta^{\frac{\alpha}{2}}. \tag{4.24}$$

类似地得

$$H_t^{\frac{\alpha}{2}}[\dot{\psi}_{\bar{r}}]\leqslant\theta A_2+D_{17}(A_1,\ B_2)+D_{18}(A_2,\ B_2)\delta^{\frac{\alpha}{2}}; \tag{4.25}$$

对 $\dot{\psi}_{\hat{s}}(t)$ 亦有同样结论.

对(4.9), (4.11), (4.13), (4.15)应用§3中的估计式(3.24), (3.31), (3.56), 并注意到(4.24), (4.25)及常数 $K_i(i=1,\ 2,\ 3)$ 的含义可知

$$\|u\|\leqslant D_{19}(A_1,\ B_1)\delta, \tag{4.26}$$

$$\|u\|_1^*\leqslant\theta_1A_1+D_{20}(A_0,\ B_1)+D_{21}(A_1,\ B_2)\delta^{\frac{\alpha}{2}}, \tag{4.27}$$

$$\|u\|^*_{1+\frac{\alpha}{2}}\leqslant\theta_2A_2+D_{22}(A_1,\ B_2)+D_{23}(A_2,\ B_2)\delta^{\frac{\alpha}{2}}. \tag{4.28}$$

现选取

$$\begin{cases}A_0>0,\ B_0>0;\\ B_1=D_7(A_0,\ B_0)+1;\\ A_1=\dfrac{D_{20}(A_0,\ B_1)+1}{1-\theta_1};\\ B_2=D_9(A_1,\ B_1)+1;\\ A_2=\dfrac{D_{22}(A_1,\ B_2)+1}{1-\theta_2}.\end{cases} \tag{4.29}$$

由(4.20)—(4.22)及(4.26)—(4.28)，易知当 $\delta_*>0$ 适当小时，算子 T 是由 $\Sigma(\delta_*)$ 到其自身的映照，类似由[1]，对函数集合 $\Sigma_*(\delta_*)$ 中元素 $(u(t,\ x),\ v(t,\ x))$ 引入范数

$$\|(u,\ v)\|_*=\|u\|_1+\|v\|_1+\left\|\frac{\partial^2 v}{\partial x^2}\right\|. \tag{4.30}$$

显然，$\Sigma_*(\delta_*)$ 关于上述范数构成 Banach 空间，而 $\Sigma(\delta_*)$ 是 $\Sigma_*(\delta_*)$ 中凸、闭、致密集．且 T 是 $\Sigma(\delta_*)$ 到自身的连续映照，利用 Schander 不动点原理[3,4]，即知存在 $(u(t,\ x),\ v(t,\ x))\in\Sigma(\delta_*)$ 使 $(u,\ v)=T(u,\ v)$，这就证明了所考察的拟线性双曲抛物耦合方程组第二边值问题解的存在性．

§5. 解的唯一性

定理 5.1(唯一性定理)　在定理 4.1 的假设下，当 δ 适当小时，第二边值问题的解 $u\in C^{1+\frac{\alpha}{2}}$，$v\in\bar{C}^{2+\alpha}$ 在 $R(\delta)$ 上是唯一的．

证　设上述第二边值问题在区域 $R(\delta)$ 上有两个解 $(u^{(1)},\ v^{(1)})$ 及 $(u^{(2)},\ v^{(2)})$，记

$$\breve{u}=u^{(1)}-u^{(2)},\quad \breve{v}=v^{(1)}-v^{(2)}. \tag{5.1}$$

易知它们满足

$$\begin{cases}\displaystyle\sum_{j=1}^{n}\breve{\zeta}_{lj}(t,\ x)\left(\frac{\partial\breve{u}_j}{\partial t}+\breve{\lambda}_l(t,\ x)\frac{\partial\breve{u}_j}{\partial x}\right)=\breve{\zeta}_l(t,\ x)\left(\frac{\partial\breve{v}}{\partial t}+\breve{\lambda}_l(t,\ x)\frac{\partial\breve{v}}{\partial x}\right)\\ \qquad\qquad +\breve{\mu}_l(t,\ x),\quad (l=1,\ \cdots,\ n), & (5.2)\\ \dfrac{\partial\breve{v}}{\partial t}-\breve{a}(t,\ x)\dfrac{\partial^2\breve{v}}{\partial x^2}=\breve{b}(t,\ x); & (5.3)\\ t=0:\ \breve{u}=\breve{v}=0; & (5.4)\\ x=1:\ \begin{cases}\displaystyle\sum_{j=1}^{n}\breve{\zeta}_{\bar{r}j}\breve{u}_j=\breve{\psi}_{\bar{r}}(t),\quad(\bar{r}=1,\ \cdots,\ h); & (5.5)\\ \dfrac{\partial\breve{v}}{\partial x}=\breve{\varphi}_+(t); & (5.6)\end{cases}\\ x=0:\ \begin{cases}\displaystyle\sum_{j=1}^{n}\breve{\zeta}_{\hat{s}j}\breve{u}_j=\breve{\psi}_{\hat{s}}(t),\quad(\hat{s}=m+1,\ \cdots,\ n); & (5.7)\\ \dfrac{\partial\breve{v}}{\partial x}=\breve{\varphi}_-(t), & (5.8)\end{cases}\end{cases}$$

其中

$$\begin{cases}\breve{\zeta}_{lj}(t,\ x)=\zeta_{lj}(t,\ x,\ u^{(1)}(t,\ x),\ v^{(1)}(t,\ x));\\ \breve{\lambda}_l(t,\ x)=\lambda_l\left(t,\ x,\ u^{(1)}(t,\ x),\ v^{(1)}(t,\ x),\ \dfrac{\partial v^{(1)}}{\partial x}(t,\ x)\right);\\ \breve{\zeta}_l(t,\ x)=\zeta_l(t,\ x,\ u^{(1)}(t,\ x),\ v^{(1)}(t,\ x));\\ \breve{\mu}_l(t,\ x)=\mu_l\left(t,\ x,\ u^{(1)},\ v^{(1)},\ \dfrac{\partial v^{(1)}}{\partial x}\right)-\mu_l\left(t,\ x,\ u^{(2)},\ v^{(2)},\ \dfrac{\partial v^{(2)}}{\partial x}\right)\\ \qquad -\displaystyle\sum_{j=1}^{n}[\zeta_{lj}(t,\ x,\ u^{(1)},\ v^{(1)})-\zeta_{lj}(t,\ x,\ u^{(2)},\ v^{(2)})]\frac{\partial u_j^{(2)}}{\partial t}\\ \qquad -\displaystyle\sum_{j=1}^{n}[\zeta_{lj}(t,\ x,\ u^{(1)},\ v^{(1)})\lambda_l\left(t,\ x,\ u^{(1)},\ v^{(1)},\ \frac{\partial v^{(1)}}{\partial x}\right) & (5.9)\end{cases}$$

$$-\zeta_{lj}(t,x,u^{(2)},v^{(2)})\lambda_l\left(t,x,u^{(2)},v^{(2)},\frac{\partial v^{(2)}}{\partial x}\right)\Big]\frac{\partial u_j^{(2)}}{\partial x}$$

$$+(\zeta_l(t,x,u^{(1)},v^{(1)})-\zeta_l(t,x,u^{(2)},v^{(2)}))\frac{\partial v^2}{\partial t}$$

$$-\Big[\zeta_l(t,x,u^{(1)},v^{(1)})\lambda_l\left(t,x,u^{(1)},v^{(1)},\frac{\partial v^{(1)}}{\partial x}\right)$$

$$-\zeta_l(t,x,u^{(2)},v^{(2)})\lambda_l\left(t,x,u^{(2)},v^{(2)},\frac{\partial v^{(2)}}{\partial x}\right)\Big]\frac{\partial v^{(2)}}{\partial x},$$

$$\begin{cases}\breve{a}(t,x)=a\left(t,x,u^{(1)}(t,x),v^{(1)}(t,x),\dfrac{\partial v^{(1)}}{\partial x}(t,x)\right);\\ \breve{b}(t,x)=\left[b\left(t,x,u^{(1)},v^{(1)},\dfrac{\partial v^{(1)}}{\partial x}\right)-b\left(t,x,u^{(2)},v^{(2)},\dfrac{\partial v^{(2)}}{\partial x}\right)\right]\\ \qquad+\left[a\left(t,x,u^{(1)},v^{(1)},\dfrac{\partial v^{(1)}}{\partial x}\right)-a\left(t,x,u^{(2)},v^{(2)},\dfrac{\partial v^{(2)}}{\partial x}\right)\right]\dfrac{\partial^2 v^{(2)}}{\partial x^2},\end{cases}\tag{5.10}$$

$$\breve{\psi}_{\bar r}(t)=\sum_{j=1}^{n}[g_{\bar rj}(t,1)+\breve{\zeta}_{\bar rj}(t,1)-\delta_{\bar rj}]\breve{u}_j+g_{\bar r}(t,1)\breve{v}.\tag{5.11}$$

而

$$g_{\bar rj}(t,x)=\int_0^1\frac{\partial G_{\bar r}}{\partial u_j}(t,\sigma u^{(1)}(t,x)+(1-\sigma)u^{(2)}(t,x),\sigma v^{(1)}(t,x)+(1-\sigma)v^{(2)}(t,x))d\sigma,\tag{5.12}$$

$$g_{\bar r}(t,x)=\int_0^1\frac{\partial G_{\bar r}}{\partial v}(t,\sigma u^{(1)}(t,x)+(1-\sigma)u^{(2)}(t,x),\sigma v^{(1)}(t,x)+(1-\sigma)v^{(2)}(t,x))d\sigma,\tag{5.13}$$

$$\begin{aligned}\breve{\varphi}_+(t)=&\sum_{j=1}^{n}\int_0^1\frac{\partial F_+}{\partial u_j}(t,\sigma u^{(1)}(t,1)+(1-\sigma)u^{(2)}(t,1),\sigma v^{(1)}(t,1)\\&+(1-\sigma)v^{(2)}(t,1))d\sigma\cdot\breve{u}_j(t,1)\\&+\int_0^1\frac{\partial F_+}{\partial v}(t,\sigma u^{(1)}(t,1)+(1-\sigma)u^{(2)}(t,1),\sigma v^{(1)}(t,1)\\&+(1-\sigma)v^{(2)}(t,1))d\sigma\cdot\breve{v}(t,1).\end{aligned}\tag{5.14}$$

对 $\breve{\psi}_{\bar s}(t)$ 及 $\breve{\varphi}_-(t)$ 类同.

注意到系数和边值函数的光滑性条件以及初始条件可知在 $R(\delta)$ 上成立

$$\|\breve{\mu}\|\leqslant Q_1\left(\|\breve{u}\|+\|\breve{v}\|+\left\|\frac{\partial\breve{v}}{\partial x}\right\|\right),\tag{5.15}$$

$$\|\breve{b}\|\leqslant Q_2\left(\|\breve{u}\|+\|\breve{v}\|+\left\|\frac{\partial\breve{v}}{\partial x}\right\|\right),\tag{5.16}$$

$$\|\breve{\psi}\|=\max_{\substack{1\leqslant\bar r\leqslant h\\ m+1\leqslant\bar s\leqslant n}}(\|\breve{\psi}_{\bar s}(t)\|,\ \|\breve{\psi}_{\bar r}(t)\|)\leqslant(\theta+Q_3\delta^{\frac{\alpha}{2}})\|\breve{u}\|+Q_4\|\breve{v}\|,\tag{5.17}$$

$$\|\breve{\varphi}_\pm(t)\|\leqslant Q_5(\|\breve{u}\|+\|\breve{v}\|),\tag{5.18}$$

其中 $\theta<1$ 由(1.24)式定义，而 $Q_i(i=1,2,\cdots)$ 是与 δ 无关的常数.

对线性双曲型方程组的边值问题(5.2), (5.4), (5.5), (5.7)运用§3中的第一估计式(3.24)，得到

$$\|\check{u}\| \leqslant (1+Q_6\delta)\|\check{\psi}\| + Q_7\|\check{v}\| + Q_8\delta\|\check{\mu}\|$$

$$\leqslant (\theta + Q_9\delta^{\frac{\alpha}{2}})\|\check{u}\| + Q_{10}\|\check{v}\| + Q_{11}\delta\left\|\frac{\partial\check{v}}{\partial x}\right\|. \tag{5.19}$$

对线性抛物型方程的第二边值问题(5.3), (5.4), (5.6), (5.8), 易知§2中所建立的第一估计式此时仍能成立(参见[7]), 故有

$$\|\check{v}\| \leqslant Q_{12}\delta\|\check{b}\| + Q_{13}\delta^{\frac{1}{2}}\|\check{\varphi}\| \leqslant Q_{14}\delta^{\frac{1}{2}}(\|\check{u}\| + \|\check{v}\|) + Q_{14}\delta\left\|\frac{\partial\check{v}}{\partial x}\right\|, \tag{5.20}$$

$$\left\|\frac{\partial\check{v}}{\partial x}\right\| \leqslant Q_{15}\delta^{\frac{1}{2}}\|\check{b}\| + Q_{16}\|\check{\varphi}\| \leqslant Q_{17}(\|\check{u}\| + \|\check{v}\|) + Q_{17}\delta^{\frac{1}{2}}\left\|\frac{\partial\check{v}}{\partial x}\right\|. \tag{5.21}$$

由(5.18)—(5.20)易知当 $\delta>0$ 适当小时在 $R(\delta)$ 上成立

$$\|\check{u}\| + \|\check{v}\| + \left\|\frac{\partial\check{v}}{\partial x}\right\| = 0.$$

由此得到解的唯一性.

注　在上述存在唯一性定理证明中,我们要求成立示性数条件(1.22)—(1.24), 如对拟线性双曲型方程组的边值问题一样, 这只是保证解存在唯一的一个有效的充分条件,为了保证解的存在唯一性, 代替(1.22)—(1.23)只需对边值函数加上条件

$$\det\left(\delta_{\bar{r}r'} - \frac{\partial G_{\bar{r}}}{\partial u_{r'}}(0,\ 0,\ 0)\right) \neq 0, \quad (\bar{r},\ r'=1,\ \cdots,\ h), \tag{5.22}$$

$$\det\left(\delta_{\hat{s}\hat{s}'} - \frac{\partial G_{\hat{s}}}{\partial u_{\hat{s}'}}(0,\ 0,\ 0)\right) \neq 0, \quad (\hat{s},\ \hat{s}'=m+1,\ \cdots,\ n) \tag{5.23}$$

即可. 事实上,此时由隐函数存在定理,可将边值条件(1.11)—(1.14)改写为

当 $x=1$ 时,
$$u_{\bar{r}} = H_{\bar{r}}(t,\ u_s,\ v), \quad \begin{pmatrix} \bar{r}=1,\ \cdots,\ h \\ s=h+1,\ \cdots,\ n \end{pmatrix}, \tag{5.24}$$

$$\frac{\partial v}{\partial x} = F_+(t,\ u,\ v); \tag{5.25}$$

当 $x=0$ 时,
$$u_{\hat{s}} = H_{\hat{s}}(t,\ u_{\hat{r}},\ v), \quad \begin{pmatrix} \hat{r}=1,\ \cdots,\ m \\ \hat{s}=m+1,\ \cdots,\ n \end{pmatrix}, \tag{5.26}$$

$$\frac{\partial v}{\partial x} = F_-(t,\ u,\ v). \tag{5.27}$$

即把 u 的边值条件写为关于一部分变量解出的形式. 再引入未知函数的可逆变换

$$\bar{u}_i = [a_i x + b_i(1-x)]u_i \equiv I_i(x)u_i\ (i=1,\ \cdots,\ n), \tag{5.28}$$

并取 $a_s = b_{\hat{r}} = 1$, $a_{\bar{r}} = b_{\hat{s}}$ 为充分小正数, 以 $u_i = \frac{\bar{u}_i}{I_i}$ 代入方程(1.2), (1.3), 并将第 l 个方程两端均乘以 $f_l(x)$, 则易证所得的关于 $\bar{u}_i$ 及 v 的方程组及边值函数满足定理 4.1 及定理 5.1 所需的一切条件, 且相应的示性数条件亦成立, 于是我们得到.

定理 5.2　将定理 4.1 中的示性数条件(1.22)—(1.23)减弱为条件(5.22)—(5.23), 定理 4.1 及定理 5.1 的结论仍然成立.

特别,对第二边值问题(1.2)—(1.5), (5.24)—(5.27)在定理 4.1 的条件下(除去示性数条件)成立定理 4.1 及定理 5.1 的结论.

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SECOND INITIAL-BOUNDARY VALUE PROBLEMS FOR QUASI-LINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEMS

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ABSTRACT

On a rectangular domain

$$R(\delta)=\{0\leqslant t\leqslant\delta,\ 0\leqslant x\leqslant 1\}. \tag{1}$$

We consider the second initial-boundary value problem for the quasi-linear hyperbolic-parabolic coupled system

$$\begin{cases} \sum_{j=1}^{n}\zeta_{lj}(t,\ x,\ u,\ v)\left(\frac{\partial u_j}{\partial t}+\lambda_l(t,\ x,\ u,\ v,\ v_x)\frac{\partial u_j}{\partial x}\right) \\ \quad =\zeta_l(t,\ x,\ u,\ v)\left(\frac{\partial v}{\partial t}+\lambda_l(t,\ x,\ u,\ v,\ v_x)\frac{\partial v}{\partial x}\right) \\ \quad +\mu_l(t,\ x,\ u,\ v,\ v_x),\quad (l=1,\ \cdots,\ n), & (2) \\ \frac{\partial v}{\partial t}-a(t,\ x,\ u,\ v,\ v_x)\frac{\partial^2 v}{\partial x^2}=b(t,\ x,\ u,\ v,\ v_x). & (3) \end{cases}$$

Without loss of generality, the initial conditions may be written as

$$t=0:\qquad u_j=0,\ (j=1,\ \cdots,\ n),\ v=0, \tag{4}$$

and we can suppose that

$$\begin{cases} a(0,\ x,\ 0,\ 0,\ 0)\equiv 1; \\ b(0,\ x,\ 0,\ 0,\ 0)\equiv 0; \\ \zeta_{lj}(0,\ x,\ 0,\ 0)\equiv\delta_{lj}=\begin{cases}1, & \text{if } l=j, \\ 0, & \text{if } l\neq j.\end{cases} \end{cases} \tag{5}$$

The boundary conditions are as follows:

$$\text{on } x=1,\quad \begin{cases} u_{\bar r}=G_{\bar r}(t,\ u,\ v), & (\bar r=1,\ \cdots,\ h;\ h\leqslant n), \\ \frac{\partial v}{\partial x}=F_+(t,\ u,\ v); \end{cases} \tag{6}$$

$$\text{on } x=0,\quad \begin{cases} u_{\hat s}=\hat G_{\hat s}(t,\ u,\ v), & (\hat s=m+1,\ \cdots,\ n;\ m\geqslant 0), \\ \frac{\partial v}{\partial x}=F_-(t,\ u,\ v). \end{cases} \tag{7}$$

We assume that the following conditions are satisfied:

(1) the orientability condition

$$\lambda_{\bar r}(0,\ 1,\ 0,\ 0,\ 0)<0,\ \ \lambda_s(0,\ 1,\ 0,\ 0,\ 0)>0,\ \ \begin{pmatrix}\bar r=1,\ \cdots,\ h \\ s=h+1,\ \cdots,\ n\end{pmatrix}, \tag{8}$$

$$\lambda_{\hat{r}}(0,\ 0,\ 0,\ 0,\ 0)<0,\quad \lambda_{\hat{s}}(0,\ 0,\ 0,\ 0,\ 0)>0,\quad \begin{pmatrix}\hat{r}=1,\ \cdots,\ m\\ \hat{s}=m+1,\ \cdots,\ n\end{pmatrix}; \tag{9}$$

(2) the compatibility condition

$$\begin{cases}G_{\bar{r}}(0,\ 0,\ 0)=0,\\ \hat{G}_{\hat{s}}(0,\ 0,\ 0)=0;\end{cases}\quad (\bar{r}=1,\ \cdots,\ h,\ \hat{s}=m+1,\ \cdots,\ n), \tag{10}$$

$$\begin{cases}\dfrac{\partial G_{\bar{r}}}{\partial t}(0,\ 0,\ 0)+\sum\limits_{j=1}^{n}\dfrac{\partial G_{\bar{r}}}{\partial u_j}(0,\ 0,\ 0)\mu_j(0,\ 1,\ 0,\ 0,\ 0)=\mu_{\bar{r}}(0,\ 1,\ 0,\ 0,\ 0),\\ \dfrac{\partial \hat{G}_{\hat{s}}}{\partial t}(0,\ 0,\ 0)+\sum\limits_{j=1}^{n}\dfrac{\partial \hat{G}_{\hat{s}}}{\partial u_j}(0,\ 0,\ 0)\mu_j(0,\ 0,\ 0,\ 0,\ 0)=\mu_{\hat{s}}(0,\ 0,\ 0,\ 0,\ 0),\\ \qquad (\bar{r}=1,\ \cdots,\ h;\ \hat{s}=m+1,\ \cdots,\ n);\end{cases} \tag{11}$$

$$F_{\pm}(0,\ 0,\ 0)=0; \tag{12}$$

(3) the condition of characterizing number

$$\begin{matrix}\sum\limits_{j=1}^{n}\left|\dfrac{\partial G_{\bar{r}}}{\partial u_j}(0,\ 0,\ 0)\right|<1,\\ \sum\limits_{j=1}^{n}\left|\dfrac{\partial \hat{G}_{\hat{s}}}{\partial u_j}(0,\ 0,\ 0)\right|<1;\end{matrix}\quad (\bar{r}=1,\ \cdots,\ h;\ \hat{s}=m+1,\ \cdots,\ n), \tag{13}$$

(4) The smoothness condition: the coefficients of the system and the boundary conditions are suitably smooth.

By means of certain a priori estimations for the solution of the heat equation and the linear hyperbolic system, using an iteration method and Leray-Schauder fixed point theorem, we have proved

Theorem 1. Under the preceding hypotheses, for the second initial-boundary value problem (2)—(4), (6), (7), there exists uniquely a classical solution on $R(\delta)$ where $\delta>0$ is suitably small.

Theorem 2. In theorem 1, the condition of characterizing number (13) may be ameliorated as the following solvable condition:

$$\begin{cases}\det\left|\delta_{\bar{r}r'}-\dfrac{\partial G_{\bar{r}}}{\partial u_{r'}}(0,\ 0,\ 0)\right|\neq 0, & (\bar{r},\ r'=1,\ \cdots,\ h);\\ \det\left|\delta_{\hat{s}\hat{s}'}-\dfrac{\partial G_{\hat{s}}}{\partial u_{\hat{s}'}}(0,\ 0,\ 0)\right|\neq 0, & (\hat{s},\ \hat{s}'=m+1,\ \cdots,\ n);\end{cases} \tag{14}$$

i. e., the boundary conditions (6), (7) may be written as

$$\text{on } x=1,\quad \begin{cases}u_{\bar{r}}=H_{\bar{r}}(t,\ u_s,\ v),\\ \dfrac{\partial v}{\partial x}=F_+(t,\ u,\ v),\end{cases}\quad \begin{pmatrix}\bar{r}=1,\ \cdots,\ h\\ s=h+1,\ \cdots,\ n\end{pmatrix}; \tag{6$'$}$$

$$\text{on } x=0,\quad \begin{cases}u_{\hat{s}}=H_{\hat{s}}(t,\ u_{\hat{r}},\ v),\\ \dfrac{\partial v}{\partial x}=F_-(t,\ u,\ v),\end{cases}\quad \begin{pmatrix}\hat{r}=1,\ \cdots,\ m\\ \hat{s}=m+1,\ \cdots,\ n\end{pmatrix}. \tag{7$'$}$$