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带两个 Carleman 位移的奇异积分方程 Noether 可解的充分必要条件

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带 Carleman 位移的奇异积分方程理论，近年来得到了很大发展．在[1]中建立了这种奇异积分方程的 Noether 理论，所用的基本方法是建立所谓的对应方程组(是不带位移的奇异积分方程组，它的理论是已知的，参看[2]，[3])．在[4]中讨论了带两个 Carleman 位移的奇异积分方程 Noether 可解的充分条件，并给出了计算指数的公式．本文目的是在文章[4]的基础上，利用不同的方法解决带两个 Carleman 位移的奇异积分方程 Noether 可解的充分必要条件问题，并把所得结果对带两个 Carleman 位移及未知函数复共轭值的奇异积分方程进行推广．

1．假设 $\Gamma=\Gamma_0+\Gamma_1+\cdots+\Gamma_m$ 是由 $m+1$ 条简单闭 Ляпунов 曲线组成，它围出一个连通区域 D^+．假设 $\alpha(t)$，$\beta(t)$ 是把 Γ 仍映射成它自身的两个不同的同胚，它们可以是正位移或反位移．在 Γ 上它们都满足 Carleman 条件 K_2，亦即满足条件 $\alpha[\alpha(t)]=t$，$\beta[\beta(t)]=t$，此外还假设 $\alpha[\beta(t)]=\beta[\alpha(t)]$．为了方便，我们将记 $\gamma(t)=\alpha[\beta(t)]=\beta[\alpha(t)]$，显然，$\gamma(t)$ 也是满足 Carleman 条件 K_2 的位移，关于位移 $\alpha(t)$，$\beta(t)$ 我们还要求 $\alpha'(t)$，$\beta'(t)\in H_\mu(\Gamma)$，而且 $\alpha'(t)\cdot\beta'(t)\neq 0$．

我们将讨论带两个 Carleman 位移的奇异积分方程

$$(\mathscr{K}\varphi)(t)\equiv(\mathscr{K}^0\varphi)(t)+\int_\Gamma K(t,\ \tau)\varphi(\tau)d\tau=g(t), \tag{1.1}$$

这里

$$\begin{aligned}(\mathscr{K}^0\varphi)(t)\equiv\ & a_0(t)\varphi(t)+a_1(t)\varphi[\alpha(t)]+a_2(t)\varphi[\beta(t)]+a_3(t)\varphi[\gamma(t)]\\ &+\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b_1(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\alpha(t)}d\tau\\ &+\frac{b_2(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\beta(t)}d\tau+\frac{b_3(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\gamma(t)}d\tau,\end{aligned} \tag{1.2}$$

其中 $a_k(t)$，$b_k(t)$，$k=0$，1，2，3，属于空间 $H_\mu(\Gamma)$，$g(t)$ 属于空间 $L_p(\Gamma)$，$p>1$，而 $K(t,\ \tau)$ 最多只具有弱奇异性，如果 $K(t,\ \tau)\equiv0$，我们将得到特征方程 $(\mathscr{K}^0\varphi)(t)=g(t)$．如果引入算子表示法，还可以把方程(1.1)写成如下形式

$$\begin{aligned}\mathscr{K}\equiv\ & a_0(t)\mathscr{I}+a_1(t)\mathscr{W}_1+a_2(t)\mathscr{W}_2+a_3(t)\mathscr{W}_3+b_0(t)\mathscr{S}\\ &+b_1(t)\mathscr{W}_1\mathscr{S}+b_2(t)\mathscr{W}_2\mathscr{S}+b_3(t)\mathscr{W}_3\mathscr{S}+\mathscr{D},\end{aligned} \tag{1.3}$$

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其中 $\mathscr{I}$ 是恒等算子，$\mathscr{W}_k(k=1, 2, 3)$ 是位移算子，$\mathscr{S}$ 是奇异积分算子，即

$$(\mathscr{W}_1\varphi)(t)\equiv\varphi[\alpha(t)],\ (\mathscr{W}_2\varphi)(t)\equiv\varphi[\beta(t)],\ (\mathscr{W}_3\varphi)(t)\equiv\varphi[\gamma(t)],$$

$$(\mathscr{S}\varphi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau,\qquad(\mathscr{W}_1\mathscr{S}\varphi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\alpha(t)}d\tau,$$

$$(\mathscr{W}_2\mathscr{S}\varphi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\beta(t)}d\tau,\quad(\mathscr{W}_3\mathscr{S}\varphi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\gamma(t)}d\tau,$$

而 $\mathscr{D}$ 是完全连续算子.

如果把奇异积分方程(1.1)中的 t 分别换成 $\alpha(t)$, $\beta(t)$ 和 $\gamma(t)$，并把这样得到的三个方程与方程(1.1)联立，再假设

$$\varphi(t)=\rho_1(t),\ \varphi[\alpha(t)]=\rho_2(t),\ \varphi[\beta(t)]=\rho_3(t),\ \varphi[\gamma(t)]=\rho_4(t),$$

最后得到方程组

$$\begin{aligned}
&a_0(t)\rho_1(t)+a_1(t)\rho_2(t)+a_2(t)\rho_3(t)+a_3(t)\rho_4(t)\\
&\quad+\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\rho_1(\tau)}{\tau-t}d\tau+\frac{\nu_1b_1(t)}{\pi i}\int_\Gamma\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}\rho_2(\tau)d\tau\\
&\quad+\frac{\nu_2b_2(t)}{\pi i}\int_\Gamma\frac{\beta'(\tau)}{\beta(\tau)-\beta(t)}\rho_3(\tau)d\tau+\frac{\nu_1\nu_2b_3(t)}{\pi i}\int_\Gamma\frac{\gamma'(\tau)}{\gamma(\tau)-\gamma(t)}\rho_4(\tau)d\tau\\
&\quad+\int_\Gamma K(t,\ \tau)\rho_1(\tau)d\tau=g(t),\\
&a_1[\alpha(t)]\rho_1(t)+a_0[\alpha(t)]\rho_2(t)+a_3[\alpha(t)]\rho_3(t)+a_2[\alpha(t)]\rho_4(t)\\
&\quad+\frac{b_1[\alpha(t)]}{\pi i}\int_\Gamma\frac{\rho_1(\tau)}{\tau-t}d\tau+\frac{\nu_1b_0[\alpha(t)]}{\pi i}\int_\Gamma\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}\rho_2(\tau)d\tau\\
&\quad+\frac{\nu_2b_3[\alpha(t)]}{\pi i}\int_\Gamma\frac{\beta'(\tau)}{\beta(\tau)-\beta(t)}\rho_3(\tau)d\tau+\frac{\nu_1\nu_2b_2[\alpha(t)]}{\pi i}\int_\Gamma\frac{\gamma'(\tau)}{\gamma(\tau)-\gamma(t)}\rho_4(\tau)d\tau\\
&\quad+\int_\Gamma K[\alpha(t),\ \tau]\rho_1(\tau)d\tau=g[\alpha(t)],\qquad\qquad(1.4)\\
&a_2[\beta(t)]\rho_1(t)+a_3[\beta(t)]\rho_2(t)+a_0[\beta(t)]\rho_3(t)+a_1[\beta(t)]\rho_4(t)\\
&\quad+\frac{b_2[\beta(t)]}{\pi i}\int_\Gamma\frac{\rho_1(\tau)}{\tau-t}d\tau+\frac{\nu_1b_3[\beta(t)]}{\pi i}\int_\Gamma\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}\rho_2(\tau)d\tau\\
&\quad+\frac{\nu_2b_0[\beta(t)]}{\pi i}\int_\Gamma\frac{\beta'(\tau)}{\beta(\tau)-\beta(t)}\rho_3(\tau)d\tau+\frac{\nu_1\nu_2b_1[\beta(t)]}{\pi i}\int_\Gamma\frac{\gamma'(\tau)}{\gamma(\tau)-\gamma(t)}\rho_4(\tau)d\tau\\
&\quad+\int_\Gamma K[\beta(t),\ \tau]\rho_1(\tau)d\tau=g[\beta(t)],\\
&a_3[\gamma(t)]\rho_1(t)+a_2[\gamma(t)]\rho_2(t)+a_1[\gamma(t)]\rho_3(t)+a_0[\gamma(t)]\rho_4(t)\\
&\quad+\frac{b_3[\gamma(t)]}{\pi i}\int_\Gamma\frac{\rho_1(\tau)}{\tau-t}d\tau+\frac{\nu_1b_2[\gamma(t)]}{\pi i}\int_\Gamma\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}\rho_2(\tau)d\tau\\
&\quad+\frac{\nu_2b_1[\gamma(t)]}{\pi i}\int_\Gamma\frac{\beta'(\tau)}{\beta(\tau)-\beta(t)}\rho_3(\tau)d\tau+\frac{\nu_1\nu_2b_0[\gamma(t)]}{\pi i}\int_\Gamma\frac{\gamma'(\tau)}{\gamma(\tau)-\gamma(t)}\rho_4(\tau)d\tau\\
&\quad+\int_\Gamma K[\gamma(t),\ \tau]\rho_1(\tau)d\tau=g[\gamma(t)],
\end{aligned}$$

其中 $\nu_1=+1$ 或 -1，这依赖于 $\alpha(t)$ 是正位移或反位移，而 $\nu_2=+1$，或 -1 这依赖于 $\beta(t)$ 是正位移或反位移.

方程组(1.4)实际上是不带位移的奇异积分方程组，如果把它写成算子形式，将有

$$\mathscr{M}\equiv p(t)\mathscr{I}+q(t)\mathscr{S}+\mathscr{D}^*, \tag{1.5}$$

其中

$$p(t)=\begin{pmatrix} a_0(t) & a_1(t) & a_2(t) & a_3(t) \\ a_1[\alpha(t)] & a_0[\alpha(t)] & a_3[\alpha(t)] & a_2[\alpha(t)] \\ a_2[\beta(t)] & a_3[\beta(t)] & a_0[\beta(t)] & a_1[\beta(t)] \\ a_3[\gamma(t)] & a_2[\gamma(t)] & a_1[\gamma(t)] & a_0[\gamma(t)] \end{pmatrix}, \tag{1.6}$$

$$q(t)=\begin{pmatrix} b_0(t) & \nu_1 b_1(t) & \nu_2 b_2(t) & \nu_1\nu_2 b_3(t) \\ b_1[\alpha(t)] & \nu_1 b_0[\alpha(t)] & \nu_2 b_3[\alpha(t)] & \nu_1\nu_2 b_2[\alpha(t)] \\ b_2[\beta(t)] & \nu_1 b_3[\beta(t)] & \nu_2 b_0[\beta(t)] & \nu_1\nu_2 b_1[\beta(t)] \\ b_3[\gamma(t)] & \nu_1 b_2[\gamma(t)] & \nu_2 b_1[\gamma(t)] & \nu_1\nu_2 b_0[\gamma(t)] \end{pmatrix}, \tag{1.7}$$

$\mathscr{D}^*$ 是完全连续算子，或者

$$\mathscr{M}\equiv M(t,\ 1)\mathscr{P}+M(t,\ -1)\mathscr{Q}+\mathscr{D}^*,$$

其中 $M(t,\ j)=p(t)+jq(t),\ j=\pm1$，而

$$\mathscr{P}=\frac{1}{2}(\mathscr{I}+\mathscr{S}),\quad \mathscr{Q}=\frac{1}{2}(\mathscr{I}-\mathscr{S}).$$

以后我们把方程组(1.4)叫做方程(1.1)的对应方程组，$\mathscr{M}$ 叫做对应算子，而 $M(t,\ j)$ 叫做算子 $\mathscr{M}$ 的标符，也叫做算子 $\mathscr{K}$ 的标符.

为了方便，我们还引入方程(1.1)的三个伴随方程

$$\begin{aligned}(\mathscr{T}_1\chi)(t)\equiv\ & a_0(t)\chi(t)-a_1(t)\chi[\alpha(t)]-a_2(t)\chi[\beta(t)]+a_3(t)\chi[\gamma(t)] \\ & +\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-t}d\tau-\frac{b_1(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\alpha(t)}d\tau-\frac{b_2(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\beta(t)}d\tau \\ & +\frac{b_3(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\gamma(t)}d\tau+\int_\Gamma K(t,\ \tau)\chi(\tau)d\tau=0,\end{aligned} \tag{1.8}$$

$$\begin{aligned}(\mathscr{T}_2\chi)(t)\equiv\ & a_0(t)\chi(t)-a_1(t)\chi[\alpha(t)]+a_2(t)\chi[\beta(t)]-a_3(t)\chi[\gamma(t)] \\ & +\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-t}d\tau-\frac{b_1(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\alpha(t)}d\tau+\frac{b_2(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\beta(t)}d\tau \\ & -\frac{b_3(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\gamma(t)}d\tau+\int_\Gamma K(t,\ \tau)\chi(\tau)d\tau=0,\end{aligned} \tag{1.9}$$

$$\begin{aligned}(\mathscr{T}_3\chi)(t)\equiv\ & a_0(t)\chi(t)+a_1(t)\chi[\alpha(t)]-a_2(t)\chi[\beta(t)]-a_3(t)\chi[\gamma(t)] \\ & +\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-t}d\tau+\frac{b_1(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\alpha(t)}d\tau-\frac{b_2(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\beta(t)}d\tau \\ & -\frac{b_3(t)}{\pi i}\int_\Gamma\frac{\chi(\tau)}{\tau-\gamma(t)}d\tau+\int_\Gamma K(t,\ \tau)\chi(\tau)d\tau=0.\end{aligned} \tag{1.10}$$

如果分别对(1.8)，(1.9)，(1.10)令

$$\rho_1(t)=\chi(t),\ \rho_2(t)=-\chi[\alpha(t)],\ \rho_3(t)=-\chi[\beta(t)],\ \rho_4(t)=\chi[\gamma(t)],$$

$$\rho_1(t)=\chi(t),\ \rho_2(t)=-\chi[\alpha(t)],\ \rho_3(t)=\chi[\beta(t)],\quad \rho_4(t)=-\chi[\gamma(t)],$$

$$\rho_1(t)=\chi(t),\ \rho_2(t)=\chi[\alpha(t)],\quad \rho_3(t)=-\chi[\beta(t)],\ \rho_4(t)=-\chi[\gamma(t)],$$

可以得到以下结论：它们的对应方程组也是方程组(1.4).

2. Noether 可解的充分必要条件. 由于完全连续算子并不影响奇异积分算子的 Noether 性质,[1]不失一般性,可以认为算子 $\mathscr{K}$ 与 $\mathscr{T}_j(j=1, 2, 3)$ 中的 $K(t, \tau)\equiv 0$, 这时候,对应奇异积分算子可以写成

$$\mathscr{M}\equiv p(t)\mathscr{I}+q(t)\mathscr{S}+\mathscr{D}, \tag{2.1}$$

其中 $p(t)$, $q(t)$ 分别由(1.6), (1.7)规定,而

$$\mathscr{I}(\rho_1, \rho_2, \rho_3, \rho_4)=(\mathscr{I}\rho_1, \mathscr{I}\rho_2, \mathscr{I}\rho_3, \mathscr{I}\rho_4),$$
$$\mathscr{S}(\rho_1, \rho_2, \rho_3, \rho_4)=(\mathscr{S}\rho_1, \mathscr{S}\rho_2, \mathscr{S}\rho_3, \mathscr{S}\rho_4),$$
$$\mathscr{D}(\rho_1, \rho_2, \rho_3, \rho_4)=(\mu_1, \mu_2, \mu_3, \mu_4),$$

这里

$$\mu_1=b_1(t)[\mathscr{W}_1\mathscr{S}\mathscr{W}_1-\nu_1\mathscr{S}]\rho_2+b_2(t)[\mathscr{W}_2\mathscr{S}\mathscr{W}_2-\nu_2\mathscr{S}]\rho_3 +b_3(t)[\mathscr{W}_3\mathscr{S}\mathscr{W}_3-\nu_1\nu_2\mathscr{S}]\rho_4, \tag{2.2}$$

$$\mu_2=b_0[\alpha(t)][\mathscr{W}_1\mathscr{S}\mathscr{W}_1-\nu_1\mathscr{S}]\rho_2+b_3[\alpha(t)][\mathscr{W}_2\mathscr{S}\mathscr{W}_2-\nu_2\mathscr{S}]\rho_3 +b_2[\alpha(t)][\mathscr{W}_3\mathscr{S}\mathscr{W}_3-\nu_1\nu_2\mathscr{S}]\rho_4, \tag{2.3}$$

$$\mu_3=b_3[\beta(t)][\mathscr{W}_1\mathscr{S}\mathscr{W}_1-\nu_1\mathscr{S}]\rho_2+b_0[\beta(t)][\mathscr{W}_2\mathscr{S}\mathscr{W}_2-\nu_2\mathscr{S}]\rho_3 +b_1[\beta(t)][\mathscr{W}_3\mathscr{S}\mathscr{W}_3-\nu_1\nu_2\mathscr{S}]\rho_4, \tag{2.4}$$

$$\mu_4=b_2[\gamma(t)][\mathscr{W}_1\mathscr{S}\mathscr{W}_1-\nu_1\mathscr{S}]\rho_2+b_1[\gamma(t)][\mathscr{W}_2\mathscr{S}\mathscr{W}_2-\nu_2\mathscr{S}]\rho_3 +b_0[\gamma(t)][\mathscr{W}_3\mathscr{S}\mathscr{W}_3-\nu_1\nu_2\mathscr{S}]\rho_4. \tag{2.5}$$

我们将在四维向量空间 $L_p^4(\Gamma)$, $p>1$ 中来讨论对应奇异积分算子,我们把分别由四维向量

$$\{\rho(t),\ \ \rho[\alpha(t)],\ \ \rho[\beta(t)],\ \ \rho(\gamma(t)]\},$$
$$\{\rho(t),\ -\rho[\alpha(t)],\ -\rho[\beta(t)],\ \ \rho[\gamma(t)]\},$$
$$\{\rho(t),\ -\rho[\alpha(t)],\ \ \rho[\beta(t)],\ -\rho[\gamma(t)]\},$$
$$\{\rho(t),\ \ \rho[\alpha(t)],\ -\rho[\beta(t)],\ -\rho[\gamma(t)]\},$$

产生的子空间记做 $L_p^{4,0}(\Gamma)$, $L_p^{4,1}(\Gamma)$, $L_p^{4,2}(\Gamma)$, $L_p^{4,3}(\Gamma)$, 这里 $\rho(t)\in L_p(\Gamma)$, $p>1$.

引理1 四维向量空间 $L_p^4(\Gamma)$ 可以分解成子空间 $L_p^{4,0}(\Gamma)$, $L_p^{4,1}(\Gamma)$, $L_p^{4,2}(\Gamma)$ 和 $L_p^{4,3}(\Gamma)$ 的直接和.

证 只要证明 $L_p^{4,0}(\Gamma)\cap L_p^{4,1}(\Gamma)\cap L_p^{4,2}(\Gamma)\cap L_p^{4,3}(\Gamma)=\{0\}$, 而且任意四维向量 $\varphi(t)=\{\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t)\}\in L_p^4(\Gamma)$ 都可以表成 $\varphi(t)=\psi_0(t)+\psi_1(t)+\psi_2(t)+\psi_3(t)$, 其中 $\psi_j(t)\in L_p^{4,j}(\Gamma)$, $j=0, 1, 2, 3$ 就可以了.

如果假设

$$\psi_0(t)=\left\{\frac{\varphi_1(t)+\varphi_2[\alpha(t)]+\varphi_3[\beta(t)]+\varphi_4[\gamma(t)]}{4},\right.$$
$$\frac{\varphi_2(t)+\varphi_1[\alpha(t)]+\varphi_4[\beta(t)]+\varphi_3[\gamma(t)]}{4},$$
$$\frac{\varphi_3(t)+\varphi_4[\alpha(t)]+\varphi_1[\beta(t)]+\varphi_2[\gamma(t)]}{4},$$
$$\left.\frac{\varphi_4(t)+\varphi_3[\alpha(t)]+\varphi_2[\beta(t)]+\varphi_1[\gamma(t)]}{4}\right\},$$

$$\psi_1(t)=\left\{\frac{\varphi_1(t)-\varphi_2[\alpha(t)]-\varphi_3[\beta(t)]+\varphi_4[\gamma(t)]}{4},\right.$$
$$\frac{\varphi_2(t)-\varphi_1[\alpha(t)]-\varphi_4[\beta(t)]+\varphi_3[\gamma(t)]}{4},$$
$$\frac{\varphi_3(t)-\varphi_4[\alpha(t)]-\varphi_1[\beta(t)]+\varphi_2[\gamma(t)]}{4},$$
$$\left.\frac{\varphi_4(t)-\varphi_3[\alpha(t)]-\varphi_2[\beta(t)]+\varphi_1[\gamma(t)]}{4}\right\},$$
$$\psi_2(t)=\left\{\frac{\varphi_1(t)-\varphi_2[\alpha(t)]+\varphi_3[\beta(t)]-\varphi_4[\gamma(t)]}{4},\right.$$
$$\frac{\varphi_2(t)-\varphi_1[\alpha(t)]+\varphi_4[\beta(t)]-\varphi_3[\gamma(t)]}{4},$$
$$\frac{\varphi_3(t)-\varphi_4[\alpha(t)]+\varphi_1[\beta(t)]-\varphi_2[\gamma(t)]}{4},$$
$$\left.\frac{\varphi_4(t)-\varphi_3[\alpha(t)]+\varphi_2[\beta(t)]-\varphi_1[\gamma(t)]}{4}\right\},$$
$$\psi_3(t)=\left\{\frac{\varphi_1(t)+\varphi_2[\alpha(t)]-\varphi_3[\beta(t)]-\varphi_4[\gamma(t)]}{4},\right.$$
$$\frac{\varphi_2(t)+\varphi_1[\alpha(t)]-\varphi_4[\beta(t)]-\varphi_3[\gamma(t)]}{4},$$
$$\frac{\varphi_3(t)+\varphi_4[\alpha(t)]-\varphi_1[\beta(t)]-\varphi_2[\gamma(t)]}{4},$$
$$\left.\frac{\varphi_4(t)+\varphi_3[\alpha(t)]-\varphi_2[\beta(t)]-\varphi_1[\gamma(t)]}{4}\right\},$$

不难验证，向量 $\psi_j(t)\in L_p^{4,j}(\Gamma)\,(j=0,\ 1,\ 2,\ 3)$，而且 $\varphi(t)=\psi_0(t)+\psi_1(t)+\psi_2(t)+\psi_3(t)$；此外，如果 $\rho(t)=\{\rho_1(t),\ \rho_2(t),\ \rho_3(t),\ \rho_4(t)\}\in L_p^{4,0}(\Gamma)\cap L_p^{4,1}(\Gamma)\cap L_p^{4,2}(\Gamma)\cap L_p^{4,3}(\Gamma)$，则有

$$\rho_2(t)=\rho_1[\alpha(t)] \quad 和 \quad \rho_2(t)=-\rho_1[\alpha(t)],$$
$$\rho_3(t)=\rho_1[\beta(t)] \quad 和 \quad \rho_3(t)=-\rho_1[\beta(t)],$$
$$\rho_4(t)=\rho_1[\gamma(t)] \quad 和 \quad \rho_4(t)=-\rho_1[\gamma(t)],$$

从而有 $\rho_2(t)=\rho_3(t)=\rho_4(t)\equiv0$，又 $\rho_1(t)=\rho_2[\alpha(t)]\equiv0$，于是 $\rho(t)\equiv0$. 引理 1 得证.

引理 2　子空间 $L_p^{4,j}(\Gamma)\,(j=0,\ 1,\ 2,\ 3)$ 关于算子 $\mathscr{M}$ 是不变的，算子 $\mathscr{M}$ 到子空间 $L_p^{4,0}(\Gamma)$ 的收缩 $\mathscr{M}_0=\mathscr{M}|_{L_p^{4,0}(\Gamma)}$ 在 Noether 意义下与算子 $\mathscr{K}$ 等价，而算子 $\mathscr{M}$ 到子空间 $L_p^{4,j}(\Gamma)$ 的收缩 $\mathscr{M}_j=\mathscr{M}|_{L_p^{4,j}(\Gamma)}$ 在 Noether 意义下与伴随算子 $\mathscr{T}_j$ 等价，$j=1,\ 2,\ 3$.

证　只要证明

$$\begin{aligned}&\mathscr{M}_0(\rho(t),\ \rho[\alpha(t)],\ \rho[\beta(t)],\ \rho[\gamma(t)])\\&=\mathscr{M}(\rho(t),\ \rho[\alpha(t)],\ \rho[\beta(t)],\ \rho[\gamma(t)])=(e_1^{(0)},\ e_2^{(0)},\ e_3^{(0)},\ e_4^{(0)})\\&=(\mathscr{K}\rho,\ \mathscr{W}_1\mathscr{K}\rho,\ \mathscr{W}_2\mathscr{K}\rho,\ \mathscr{W}_3\mathscr{K}\rho),\end{aligned}\tag{2.6}$$

$$\begin{aligned}&\mathscr{M}_1(\rho(t),\ -\rho[\alpha(t)],\ -\rho[\beta(t)],\ \rho[\gamma(t)])\\&=\mathscr{M}(\rho(t),\ -\rho[\alpha(t)],\ -\rho[\beta(t)],\ \rho[\gamma(t)])=(e_1^{(1)},\ e_2^{(1)},\ e_3^{(1)},\ e_4^{(1)})\\&=(\mathscr{T}_1\rho,\ -\mathscr{W}_1\mathscr{T}_1\rho,\ -\mathscr{W}_2\mathscr{T}_1\rho,\ \mathscr{W}_3\mathscr{T}_1\rho),\end{aligned}\tag{2.7}$$

$$\begin{aligned}&\mathscr{M}_2(\rho(t),\ -\rho[\alpha(t)],\ \rho[\beta(t)],\ -\rho[\gamma(t)])\\&\quad=\mathscr{M}(\rho(t),\ -\rho[\alpha(t)],\ \rho[\beta(t)],\ -\rho[\gamma(t)])=(e_1^{(2)},\ e_2^{(2)},\ e_3^{(2)},\ e_4^{(2)})\\&\quad=(\mathscr{T}_2\rho,\ -\mathscr{W}_1\mathscr{T}_2\rho,\ \mathscr{W}_2\mathscr{T}_2\rho,\ -\mathscr{W}_3\mathscr{T}_2\rho),\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\mathscr{M}_3(\rho(t),\ \rho[\alpha(t)],\ -\rho[\beta(t)],\ -\rho[\gamma(t)])\\&\quad=\mathscr{M}(\rho(t),\ \rho[\alpha(t)],\ -\rho[\beta(t)],\ -\rho[\gamma(t)])=(e_1^{(3)},\ e_2^{(3)},\ e_3^{(3)},\ e_4^{(3)})\\&\quad=(\mathscr{T}_3\rho,\ \mathscr{W}_1\mathscr{T}_3\rho,\ -\mathscr{W}_2\mathscr{T}_3\rho,\ -\mathscr{W}_3\mathscr{T}_3\rho),\end{aligned}\tag{2.9}$$

就可以了，首先证明等式(2.6)

$$\begin{aligned}e_1^{(0)}&=a_0(t)\rho(t)+a_1(t)\rho[\alpha(t)]+a_2(t)\rho[\beta(t)]+a_3(t)\rho[\gamma(t)]+b_0(t)\mathscr{S}\rho(t)\\&\quad+\nu_1b_1(t)\mathscr{S}\rho[\alpha(t)]+\nu_2b_2(t)\mathscr{S}\rho[\beta(t)]+\nu_1\nu_2b_3(t)\mathscr{S}\rho[\gamma(t)]+\mu_1{}^*\\&=a_0(t)\rho(t)+a_1(t)\rho[\alpha(t)]+a_2(t)\rho[\beta(t)]+a_3(t)\rho[\gamma(t)]+b_0(t)\mathscr{S}\rho(t)\\&\quad+b_1(t)\mathscr{W}_1\mathscr{S}\rho(t)+b_2(t)\mathscr{W}_2\mathscr{S}\rho(t)+b_3(t)\mathscr{W}_3\mathscr{S}\rho(t)=\mathscr{K}\rho,\\e_2^{(0)}&=a_1[\alpha(t)]\rho(t)+a_0[\alpha(t)]\rho[\alpha(t)]+a_3[\alpha(t)]\rho[\beta(t)]+a_2[\alpha(t)]\rho[\gamma(t)]\\&\quad+b_1[\alpha(t)]\mathscr{S}\rho(t)+\nu_1b_0[\alpha(t)]\mathscr{S}\rho[\alpha(t)]+\nu_2b_3[\alpha(t)]\mathscr{S}\rho[\beta(t)]\\&\quad+\nu_1\nu_2b_2[\alpha(t)]\mathscr{S}\rho[\gamma(t)]+\mu_2\\&=\mathscr{W}_1[a_0(t)\rho(t)+a_1(t)\rho[\alpha(t)]+a_2(t)\rho[\beta(t)]+a_3(t)\rho[\gamma(t)]+b_0(t)\mathscr{S}\rho(t)\\&\quad+b_1(t)\mathscr{W}_1\mathscr{S}\rho(t)+b_2(t)\mathscr{W}_2\mathscr{S}\rho(t)+b_3(t)\mathscr{W}_3\mathscr{S}\rho(t)]=\mathscr{W}_1\mathscr{K}\rho,\\e_3^{(0)}&=a_2[\beta(t)]\rho(t)+a_3[\beta(t)]\rho[\alpha(t)]+a_0[\beta(t)]\rho[\beta(t)]+a_1[\beta(t)]\rho[\gamma(t)]\\&\quad+b_2[\beta(t)]\mathscr{S}\rho(t)+\nu_1b_3[\beta(t)]\mathscr{S}\rho[\alpha(t)]+\nu_2b_0[\beta(t)]\mathscr{S}\rho[\beta(t)]\\&\quad+\nu_1\nu_2b_1[\beta(t)]\mathscr{S}\rho[\gamma(t)]+\mu_3\\&=a_2[\beta(t)]\rho(t)+a_3[\beta(t)]\rho[\alpha(t)]+a_0[\beta(t)]\rho[\beta(t)]+a_1[\beta(t)]\rho[\gamma(t)]\\&\quad+b_2[\beta(t)]\mathscr{S}\rho(t)+b_3[\beta(t)]\mathscr{W}_1\mathscr{S}\rho(t)+b_0[\beta(t)]\mathscr{W}_2\mathscr{S}\rho(t)\\&\quad+b_1[\beta(t)]\mathscr{W}_3\mathscr{S}\rho(t)=\mathscr{W}_2\mathscr{K}\rho,\\e_4^{(0)}&=a_3[\gamma(t)]\rho(t)+a_2[\gamma[t)]\rho[\alpha(t)]+a_1[\gamma(t)]\rho[\beta(t)]+a_0[\gamma(t)]\rho[\gamma(t)]\\&\quad+b_3[\gamma(t)]\mathscr{S}\rho(t)+\nu_1b_2[\gamma(t)]\mathscr{S}\rho[\alpha(t)]+\nu_2b_1[\gamma(t)]\mathscr{S}\rho[\beta(t)]\\&\quad+\nu_1\nu_2b_0[\gamma(t)]\mathscr{S}\rho[\gamma(t)]+\mu_4\\&=a_3[\gamma(t)]\rho(t)+a_2[\gamma(t)]\rho[\alpha(t)]+a_1[\gamma(t)]\rho[\beta(t)]+a_0[\gamma(t)]\rho[\gamma(t)]\\&\quad+b_3[\gamma(t)]\mathscr{S}\rho(t)+b_2[\gamma(t)]\mathscr{W}_1\mathscr{S}\rho(t)+b_1[\gamma(t)]\mathscr{W}_2\mathscr{S}\rho(t)\\&\quad+b_0[\gamma(t)]\mathscr{W}_3\mathscr{S}\rho(t)=\mathscr{W}_3\mathscr{K}\rho.\end{aligned}$$

类似地可以证明等式(2.7)、(2.8)和(2.9). 例如

$$\begin{aligned}e_1^{(1)}&=a_0(t)\rho(t)-a_1(t)\rho[\alpha(t)]-a_2(t)\rho[\beta(t)]+a_3(t)\rho[\gamma(t)]+b_0(t)\mathscr{S}\rho(t)\\&\quad-\nu_1b_1(t)\mathscr{S}\rho[\alpha(t)]-\nu_2b_2(t)\mathscr{S}\rho[\beta(t)]+\nu_1\nu_2b_3(t)\mathscr{S}\rho[\gamma(t)]+\tilde{\mu}_1{}^{**}\\&=a_0(t)\rho(t)-a_1(t)\rho[\alpha(t)]-a_2(t)\rho[\beta(t)]+a_3(t)\rho[\gamma(t)]+b_0(t)\mathscr{S}\rho(t)\\&\quad-b_1(t)\mathscr{W}_1\mathscr{S}\rho(t)-b_2(t)\mathscr{W}_2\mathscr{S}\rho(t)+b_3(t)\mathscr{W}_3\mathscr{S}\rho(t)\\&=\mathscr{T}_1\rho,\end{aligned}$$

---

* 这里 μ_1 由(2.2)式规定，只是取 $\rho_2=\rho[\alpha(t)]$, $\rho_3=\rho[\beta(t)]$, $\rho_4=\rho[\gamma(t)]$. 下面用到的 μ_2, μ_3, μ_4 也仿此利用(2.3), (2.4), (2.5)规定.

** 这里 $\tilde{\mu}_1$ 由(2.2)式规定，只是取 $\rho_2=-\rho[\alpha(t)]$, $\rho_3=-\rho[\beta(t)]$, $\rho_4=\rho(\gamma(t)]$, 下面用到的 $\tilde{\mu}_2$, $\tilde{\mu}_3$, $\tilde{\mu}_4$ 也仿此利用(2.3), (2.4), (2.5)规定.

$$\begin{aligned}e_2^{(1)}=&a_1[\alpha(t)]\rho(t)-a_0[\alpha(t)]\rho[\alpha(t)]-a_3[\alpha(t)]\rho[\beta(t)]+a_2[\alpha(t)]\rho[\gamma(t)]\\&+b_1[\alpha(t)]\mathscr{S}\rho(t)-\nu_1b_0[\alpha(t)]\mathscr{S}\rho[\alpha(t)]-\nu_2b_3[\alpha(t)]\mathscr{S}\rho[\beta(t)]\\&+\nu_1\nu_2b_2[\alpha(t)]\mathscr{S}\rho[\gamma(t)]+\tilde{\mu}_2\\=&a_1[\alpha(t)]\rho(t)-a_0[\alpha(t)]\rho[\alpha(t)]-a_3[\alpha(t)]\rho[\beta(t)]+a_2[\alpha(t)]\rho[\gamma(t)]\\&+b_1[\alpha(t)]\mathscr{S}\rho(t)-b_0[\alpha(t)]\mathscr{W}_1\mathscr{S}\rho(t)-b_3[\alpha(t)]\mathscr{W}_2\mathscr{S}\rho(t)\\&+b_2[\alpha(t)]\mathscr{W}_3\mathscr{S}\rho(t)=-\mathscr{W}_1\mathscr{T}_1\rho,\\e_3^{(1)}=&a_2[\beta(t)]\rho(t)-a_3[\beta(t)]\rho[\alpha(t)]-a_0[\beta(t)]\rho[\beta(t)]+a_1[\beta(t)]\rho[\gamma(t)]\\&+b_2[\beta(t)]\mathscr{S}\rho(t)-\nu_1b_3[\beta(t)]\mathscr{S}\rho[\alpha(t)]-\nu_2b_0[\beta(t)]\mathscr{S}\rho[\beta(t)]\\&+\nu_1\nu_2b_1[\beta(t)]\mathscr{S}\rho[\gamma(t)]+\tilde{\mu}_3\\=&a_2[\beta(t)]\rho(t)-a_3[\beta(t)]\rho[\alpha(t)]-a_0[\beta(t)]\rho[\beta(t)]+a_1[\beta(t)]\rho[\gamma(t)]\\&+b_2[\beta(t)]\mathscr{S}\rho(t)-b_3[\beta(t)]\mathscr{W}_1\mathscr{S}\rho(t)-b_0[\beta(t)]\mathscr{W}_2\mathscr{S}\rho(t)\\&+b_3[\beta(t)]\mathscr{W}_3\mathscr{S}\rho(t)=-\mathscr{W}_2\mathscr{T}_1\rho,\\e_4^{(1)}=&a_3[\gamma(t)]\rho(t)-a_2[\gamma(t)]\rho[\alpha(t)]-a_1[\gamma(t)]\rho[\beta(t)]+a_0[\gamma(t)]\rho[\gamma(t)]\\&+b_3[\gamma(t)]\mathscr{S}\rho(t)-\nu_1b_2[\gamma(t)]\mathscr{S}\rho[\alpha(t)]-\nu_2b_1[\gamma(t)]\mathscr{S}\rho[\beta(t)]\\&+\nu_1\nu_2b_0[\gamma(t)]\mathscr{S}\rho[\gamma(t)]+\tilde{\mu}_4\\=&a_3[\gamma(t)]\rho(t)-a_2[\gamma(t)]\rho[\alpha(t)]-a_1[\gamma(t)]\rho[\beta(t)]+a_0[\gamma(t)]\rho[\gamma(t)]\\&+b_3[\gamma(t)]\mathscr{S}\rho(t)-b_2[\gamma(t)]\mathscr{W}_1\mathscr{S}\rho(t)-b_1[\gamma(t)]\mathscr{W}_2\mathscr{S}\rho(t)\\&+b_0[\gamma(t)]\mathscr{W}_3\mathscr{S}\rho(t)=\mathscr{W}_3\mathscr{T}_1\rho.\end{aligned}$$

从而引理 2 得证.

定理 1 为了使算子

$$\begin{aligned}\mathscr{K}\equiv&a_0(t)\mathscr{I}+a_1(t)\mathscr{W}_1+a_2(t)\mathscr{W}_2+a_3(t)\mathscr{W}_3+b_1(t)\mathscr{S}\\&+b_1(t)\mathscr{W}_1\mathscr{S}+b_2(t)\mathscr{W}_2\mathscr{S}+b_3(t)\mathscr{W}_3\mathscr{S}+\mathscr{D}\end{aligned}\tag{1.3}$$

是 Noether 算子的充分必要条件是它的标符是非退化的, 也就是

$$\det M(t,\ j)=\det(p(t)+jq(t))\neq 0\quad (j=\pm 1),\tag{2.10}$$

Noether 算子 $\mathscr{K}$ 的指数是

$$\operatorname{Ind}\mathscr{K}=\frac{1}{8\pi}\left\{\arg\frac{\det M(t,\ -1)}{\det M(t,\ +1)}\right\}_\Gamma.\tag{2.11}$$

证　首先证明条件的充分性, 根据定义我们知道算子 $\mathscr{K}$ 与对应算子 $\mathscr{M}$ 的标符是完全一致的, 因此只要条件(2.10)满足, 那么对应算子 $\mathscr{M}$ 就是 Noether 算子[1]. 又根据引理 1 知道算子 $\mathscr{M}$ 是算子 $\mathscr{M}_j$($j=0,\ 1,\ 2,\ 3$)的直接和, 从而算子 $\mathscr{M}_0=\mathscr{M}|_{L_p^{(0)}(\Gamma)}$ 也就是 Noether 算子, 再根据引理 2 知道算子 $\mathscr{K}$ 也一定是 Noether 算子.

反过来, 再证明条件的必要性. 这时候, 假设算子 $\mathscr{K}$ 是 Noether 算子, 于是伴随算子 $\mathscr{T}_j$($j=1,\ 2,\ 3$)也必定都是Noether算子(参看[4]定理 7), 再根据引理 2 知道算子 $\mathscr{M}_j$ ($j=0,\ 1,\ 2,\ 3$)都是 Noether 算子, 从而算子 $\mathscr{M}$ 也是 Noether 算子, 从而条件(2.10)必定满足[1].

除此以外, 我们知道[4] $\operatorname{Ind}\mathscr{K}=\operatorname{Ind}\mathscr{T}_j$($j=1,\ 2,\ 3$)而且 $\operatorname{Ind}\mathscr{M}=\operatorname{Ind}\mathscr{K}+\operatorname{Ind}\mathscr{T}_1+\operatorname{Ind}\mathscr{T}_2+\operatorname{Ind}\mathscr{T}_3$, 从而有

$$\operatorname{Ind}\mathscr{K}=\frac{1}{8\pi}\left\{\arg\frac{\det M(t,\ -1)}{\det(M(t,\ +1)}\right\}_{\Gamma}.$$

3. 对于带两个 Carleman 位移的奇异积分方程组来说只不过是把方程(1.1)中的系数 $a_k(t)$, $b_k(t)$ $(k=0, 1, 2, 3)$理解为$(m\times m)$阶方阵，它们的元素属于空间 $H_\mu(\Gamma)$, $g(t)$是 m 维向量，它的元素属于 $L_p(\Gamma)$, $p>1$. 而 $K(t, \tau)$也应理解为$(m\times m)$阶方阵，它的元素最多只具有弱奇异性，这样一来，上一段中所有讨论不需要做任何本质上的改变都可以搬到方程组的情形上来. 只是对应方程组(1.4)中方程与未知函数的个数由 4 增加到 $4m$.

对于带两个 Carleman 位移的奇异积分方程组来说，如果引用算子记号将有以下结论：

定理 2 为了使算子

$$\mathscr{K}\equiv a_0(t)\mathscr{J}+a_1(t)\mathscr{W}_1+a_2(t)\mathscr{W}_2+a_3(t)\mathscr{W}_3+b_0(t)\mathscr{S}+b_1(t)\mathscr{W}_1\mathscr{S}+b_2(t)\mathscr{W}_2\mathscr{S}+b_3(t)\mathscr{W}_3\mathscr{S}+\mathscr{D}, \tag{3.1}$$

是 Noether 算子的充分必要条件是它的标符是非退化的，也就是说

$$\det M(t,\ j)=\det(p(t)+jq(t))\neq 0,\quad j=\pm 1. \tag{3.2}$$

这里 $a_k(t)$, $b_k(t)$ $(k=0, 1, 2, 3)$是$(m\times m)$阶方阵，而 $p(t)$, $q(t)$是$(4m\times 4m)$阶方阵，它的具体形式与(1.6), (1.7)完全类似，只不过代替 $a_k(t)$, $b_k(t)$的位置，在这里应该换成相应的$(m\times m)$阶方阵.

另外算子 $\mathscr{K}$ 的指数公式(2.11)也是成立的.

4. 我们考虑以下更为一般的奇异积分方程

$$\begin{aligned}(\mathscr{K}\varphi)(t)\equiv{}&a_0(t)\varphi(t)+a_1(t)\varphi[\alpha(t)]+a_2(t)\varphi[\beta(t)]\\&+a_3(t)\varphi[\gamma(t)]+c_0(t)\overline{\varphi(t)}+c_1(t)\overline{\varphi[\alpha(t)]}\\&+c_2(t)\overline{\varphi[\beta(t)]}+c_3(t)\overline{\varphi[\gamma(t)]}\\&+\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b_1(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\alpha(t)}d\tau+\frac{b_2(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\beta(t)}d\tau\\&+\frac{b_3(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\gamma(t)}d\tau+d_0(t)\overline{\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau}\\&+d_1(t)\overline{\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\alpha(t)}d\tau}+d_2(t)\overline{\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\beta(t)}d\tau}\\&+d_3(t)\overline{\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-\gamma(t)}d\tau}+\int_\Gamma K_1(t,\ \tau)\varphi(\tau)d\tau\\&+\overline{\int_\Gamma K_2(t,\ \tau)\varphi(\tau)d\tau}=g(t).\end{aligned} \tag{4.1}$$

如果把由方程(4.1)取复共轭值所得到的方程与方程(4.1)联立，并引入新的未知函数

$$\varphi_1(t)=\varphi(t),\quad \varphi_2(t)=\overline{\varphi(t)},$$

就可以得到以下方程组

$$
\begin{aligned}
&a_0(t)\varphi_1(t)+a_1(t)\varphi_1[\alpha(t)]+a_2(t)\varphi_1[\beta(t)]+a_3(t)\varphi_1[\gamma(t)]+c_0(t)\varphi_2(t)\\
&\quad+c_1(t)\varphi_2[\alpha(t)]+c_2(t)\varphi_2[\beta(t)]+c_3(t)\varphi_2[\gamma(t)]+\frac{b_0(t)}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-t}d\tau\\
&\quad+\frac{b_1(t)}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-\alpha(t)}d\tau+\frac{b_2(t)}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-\beta(t)}d\tau+\frac{b_3(t)}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-\gamma(t)}d\tau\\
&\quad-\frac{d_0(t)}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\bar t}d\tau-\frac{d_1(t)}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\overline{\alpha(t)}}d\tau\\
&\quad-\frac{d_2(t)}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\overline{\beta(t)}}d\tau-\frac{d_3(t)}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\overline{\gamma(t)}}d\tau\\
&\quad+\int_\Gamma K_1(t,\tau)\varphi_1(\tau)d\tau+\int_\Gamma\overline{K_2(t,\tau)}\,\overline{\tau'^2(\sigma)}\varphi_2(\tau)d\tau=g(t),
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&\overline{c_0(t)}\varphi_1(t)+\overline{c_1(t)}\varphi_1[\alpha(t)]+\overline{c_2(t)}\varphi_1[\beta(t)]+\overline{c_3(t)}\varphi_1[\gamma(t)]\\
&\quad+\overline{a_0(t)}\varphi_2(t)+\overline{a_1(t)}\varphi_2[\alpha(t)]+\overline{a_2(t)}\varphi_2[\beta(t)]+\overline{a_3(t)}\varphi_2[\gamma(t)]\\
&\quad+\frac{\overline{d_0(t)}}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-t}d\tau+\frac{\overline{d_1(t)}}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-\alpha(t)}d\tau+\frac{\overline{d_2(t)}}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-\beta(t)}d\tau\\
&\quad+\frac{\overline{d_3(t)}}{\pi i}\int_\Gamma\frac{\varphi_1(\tau)}{\tau-\gamma(t)}d\tau-\frac{\overline{b_0(t)}}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\bar t}d\tau-\frac{\overline{b_1(t)}}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\overline{\alpha(t)}}d\tau\\
&\quad-\frac{\overline{b_2(t)}}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\overline{\beta(t)}}d\tau-\frac{\overline{b_3(t)}}{\pi i}\int_\Gamma\frac{\varphi_2(\tau)\overline{\tau'^2(\sigma)}}{\bar\tau-\overline{\gamma(t)}}d\tau\\
&\quad+\int_\Gamma K_2(t,\tau)\varphi_1(\tau)d\tau+\int_\Gamma\overline{K_1(t,\tau)}\,\overline{\tau'^2(\sigma)}\varphi_2(\tau)d\tau=\overline{g(t)}.
\end{aligned}
$$

这个方程组也叫做方程(4.1)的对应方程组，由于这里要利用取复共轭值的运算，它在复空间 $L_p(\Gamma)$ 中是反线性有界算子．但是，只要在实数域上来讨论这种算子的话，那么它在指定空间中就是线性的．我们把这样的实线性空间用 $\widetilde{L}_p(\Gamma)$ 来表示．以后我们假设算子 $\mathscr{K}$ 是作用在线性空间 $\widetilde{L}_p(\Gamma)$ 上的，也就是说，我们讨论方程(4.1)解的线性无关性时是指带实系数的线性无关性．而讨论方程组(4.2)解的线性无关性时，仍然是指带复系数的线性无关性．这样一来，利用[1]中的结果，可以知道带有两个 Carleman 位移和未知函数复共轭值的奇异积分方程(4.1)和对应奇异积分方程组(4.2)，同时是，或者不是 Noether 方程，而且方程(4.1)和方程组(4.2)的指数将是相等的．也就是说，我们已经把方程(4.1)转化为第 3 段中讨论的方程组情形了．从而不难得到相应的结论．

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THE SUFFICIENT AND NECESSARY CONDITIONS FOR NOETHER'S SOLVABILITY OF SINGULAR INTEGRAL EQUATIONS WITH TWO CARLEMAN'S SHIFTS

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ABSTRACT

In this paper we consider the problem of solvability of singular integral equtions with two Carleman's shifts

$$(\mathscr{K}\varphi)(t) \equiv a_0(t)\varphi(t)+a_1(t)\varphi[\alpha(t)]+a_2(t)\varphi[\beta(t)]+a_3(t)\varphi[\gamma(t)]$$

$$+\frac{b_0(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b_1(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\alpha(t)}d\tau+\frac{b_2(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\beta(t)}d\tau$$

$$+\frac{b_3(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\gamma(t)}d\tau+\int_\Gamma K(t,\tau)\varphi(\tau)d\tau=g(t). \qquad (1.1)$$

Suppose that Γ is a closed simple Lyapunoff's curve and $\alpha(t)$, $\beta(t)$ which satisfy Carleman's conditions and $\alpha[\beta(t)]=\beta[\alpha(t)]$ are two different homeomorphisms of Γ onto itself, and that $a_k(t)$, $b_k(t)$, $k=0, 1, 2, 3$ belong to the space $H_\mu(\Gamma)$, $g(t)$ belongs to the space $L_p(\Gamma)$, $p>1$ and $K(t,\tau)$ has only weak singularity.

The following main results are obtained:

1. Singular integral eqution (1.1) is solvable if and only if the Noether's conditions

$$\det(p(t)\pm q(t))\neq 0$$

are satisfied.

2. Index of sigular integral eqution (1.1) is calculated by the formula

$$\operatorname{Ind}\mathscr{K}=\frac{1}{8\pi}\left\{\arg\frac{\det(p(t)-q(t))}{\det(p(t)+q(t))}\right\}_\Gamma,$$

where $p(t)$ and $q(t)$ are matrices of coefficents of so-called corresponding system of equtions.

All these results have been generalized for systems of singular integral equtions with two Carleman's shifts and complex conjugate of unknown functions.