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A DIRECT SEARCH METHOD BY THE LOCAL POSITIVE BASIS FOR LINEARLY CONSTRAINED OPTIMIZATION

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We know that there exist some direct search methods (e.g. the complex method) developed for constrained optimization problems, but the theoretical aspect of these methods is very weak[1, 2]. In this paper, we consider the following linearly constrained optimization problem

$$(\text{LNP}): \min\{f(x)\,|\,x\in X\},\ X=\{x\,|\,x\in R^n,\ (a^i)^T x\geqslant \alpha_i,\ i\in I_m\},$$

where $I_m=\{1, 2, \cdots, m\}$. The local feasible cone and local positive basis are defined with any step size at any feasible point in X. Then we give a direct search method by the local positive basis, and prove its convergence. Some ideas here are motivated by the descent methods with fixed step size for unconstrained optimization[3] and [4] which treats constrained optimization. For the above purpose, first we give a discussion on the structure of a polyhedral convex cone.

§ 1. The Canonical Positive Basis of a Polyhedral Convex Cone

Assuming A to be a $n\times r$ matrix, we consider the following polyhedral convex cone:

$$C=\{z\,|\,z\in R^n,\ A^T z\geqslant 0\} \tag{1}$$

which is the positive normal cone with respect to the set of column vectors of A. Its general properties have been discussed by many authors, see, for example[5].

We shall illustrate its structure only in the case when rank $A=r$.

Definition 1. *Let*

$$V=\{z\,|\,A^T z=0\},$$
$$V^\perp=\{z\,|\,z=A\lambda,\ \lambda\in R^r\},\ W=C\cap V^\perp$$

and they are called the inner subspace of C, the normal subspace of C, and the normal cut cone respectively.

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Lemma 1. *The polyhedral convex cone C is the direct sum of its normal cut cone W and its inner subspace V.*

(Proof omitted).

Definition 2. *Suppose that C is a polyhedral convex cone, then a vector set $B=\{b^1, b^2, \cdots, b^q|, b^i\in C, i=1, \cdots, q\}$ is called a positive basis of C if the following two conditions are satisfied:*

(i) *Any Z in C is a positive combination of B, i.e., for any $z\in C$, we may have $z=\sum_{i=1}^{q}\lambda_i b^i$ with $\lambda^i\geqslant 0$;*

(ii) *Any b^i is not a positive combination of the other vectors in B, i.e. B is positive independent.*

Lemma 2. *Suppose that A is an $n\times r$ matrix with rank $A=r$, and $A=(A_1, A_2)^T$, where A_1 is an $r\times r$ matrix with det $A_1\neq 0$, then the vector set $B=\{b^1, b^2, \cdots, b^q\}$ ($q=n+1$ or n) built in the following way is a positive basis of the polyhedral convex cone $C=\{z|A^Tz\geqslant 0\}$ and is called a canonical positive basis of C.*

Case (i) $0<r<n$. In this case, take $B=\{b^1, \cdots, b^{n+1}\}$ $(q=n+1)$. For $1\leqslant p\leqslant r$, b^p is the corresponding column vector of $A(A^TA)^{-1}$. For $r+1\leqslant p\leqslant n$, b^p is the corresponding column vector $((-A_1^{-1}A_2)^T, E)^T$, where E is the unit matrix of order $n-r$, and $b^{n+1}=-\sum_{p=r+1}^{n} b^p$.

Case (ii) $r=n$. In this case, take $B=\{b^1, \cdots, b^n\}$ $(q=n)$, where every b^p is the corresponding column vector of $A(A^TA)^{-1}$.

Case (iii) $r=0$. In this case, we have $C=R^n$, and take $B=\{b^1, \cdots, b^{n+1}\}$ $(q=n+1)$, where $b^1, \cdots, b^n$ are coordinate vectors of R^n. and $b^{n+1}=-\sum_{p=1}^{n} b^p$.

Proof Here we give a proof for case (i). It is clear that $A^Tb^p\geqslant 0$ for $p=1, \cdots, n+1$, i.e. $B\subset C$. By lemma 1, for any $z\in C$, we have $z_1\in W\subset V^\perp$ and $z_2\in V$ such that $z=z_1+z_2$. From the definition of W, we may write $z_1=A\lambda=A(A^TA)^{-1}\mu$, where $\mu=A^TA\lambda=A^Tz_1\geqslant 0$, so z_1 is a positive combination of $b^1, \cdots, b^r$. Also because $b^{r+1}, \cdots, b^n$ constitute a basic solution set of V, we may get

$$z_2=\sum_{p=r+1}^{n}\gamma_p b^p=\sum_{p=r+1}^{n}(\gamma_p+\gamma_{n+1})b^p+\gamma_{n+1}b^{n+1},$$

where $\gamma_{n+1}=-\min(\gamma_{r+1}, \cdots, \gamma_n, 0)$ and hence the right side is a positive combination of $b^{r+1}, \cdots, b^{n+1}$. Therefore, z is a positive combination of B.

Now we prove that B is positive independent. If not, some b^p would be a positive combination of the other vectors in B except b^p. First if $1\leqslant p\leqslant r$, then applying the orthogonal projection operator from C onto the normal subspace $V^\perp$, we may get a representation of b^p as a positive combination of the other vectors in $\{b^1, \cdots, b^r\}$, which contradicts the linear independence of $\{b^1, \cdots, b^r\}$. Next if $r+1\leqslant p\leqslant n$ and b^p

were a positive combination of all other vectors in B, then applying the orthogonal projection operator from C onto the inner subspace, we may get a representation of b^p as a positive combination of the other vectors in $\{b^{r+1}, \cdots, b^{n+1}\}$. Substituting $-\sum_{j=r+1}^{n} b^j$ for b^{n+1} in the above representation, and transfering all the items to the left side, we may see that the coefficient of b^p is not zero, which contradicts the linear independence of $\{b^{r+1}, \cdots, b^n\}$. Finally if b^{n+1} were a positive combination of all the other vectors in B, we may similarly arrive at a contradiction.

The proof for case (ii) is similar and is omitted. As for case (iii), the conclusion is known[3]. Q.E.D.

Remark. In the above lemma, A_1 is called a basic submatrix of $A^T=(A_1, A_2)$, where A_1 is composed of the first r columns of A^T. In general cases, A_1 may be composed of any r columns of A^T and it is only required to satisfy $\det A_1 \neq 0$. The above proof has also shown that $\{b^1, \cdots, b^r\}$ is a positive basis of the normal cut cone W of C, and $\{b^{r+1}, \cdots, b^{n+1}\}$ is a positive basis of the inner subspace V of C.

Lemma 3. *Suppose that $f(x)$ is continuously differentiable at $x_0 \in R^n$, A is an $n\times r$ matrix and $\{b^1, \cdots, b^q\}$ is a positive basis of the cone $C=\{z \mid z\in R^n, A^Tz\geqslant 0\}$, then the following three conditions are equivalent to each other:*

(i) $\exists \nu \in R^r$, $\nu \geqslant 0$ *such that* $\nabla f(x_0)=A\nu$,

(ii) $\forall p\ (1\leqslant p\leqslant q)$, $\frac{\partial f}{\partial b^p}(x_0)\geqslant 0$,

(iii) $\forall z\in C$, $\frac{\partial f}{\partial z}(x_0)\geqslant 0$.

Proof Here $\frac{\partial f}{\partial z}(x_0)$ is defined to be $\lim_{\theta\to 0}(f(x_0+\theta z)-f(x_0))\theta^{-1}$ no matter whether z is a unit vector or not. So we always have $\frac{\partial f}{\partial z}(x_0)=\nabla^T f(x_0)z$. Thus, if we have (i), then from the fact that $\nu\geqslant 0$ and $A^Tb^p\geqslant 0$, we immediately get (ii) by the formula of $\frac{\partial f}{\partial z}(x_0)$. If we have (ii), then by the property of the positive basis we can prove (iii). And if we have (iii), i.e., $\forall z\in C$, we have $\nabla^T f(x_0)z\geqslant 0$, then by Farkas lemma we may get (i). Q.E.D.

§ 2. The Local Active Constraint and the Local Canonical Positive Basis

Now we study the constraint set $X=\{x \mid (a^i)^Tx\geqslant \alpha_i, i\in I_m\}$ of the problem (LNP). For any $I\subset I_m=\{1, 2, \cdots, m\}$ we introduce the following notations

$$A(I)=\{a^i \mid i\in I\}, \quad (2)$$

$$C(I)=\{z \mid (a^i)^Tz\geqslant 0,\ i\in I\}, \quad (3)$$

Sometimes we also use $A(I)$ to represent the $n\times|I|$ matrix composed of the vectors in $A(I)$ in their natural order. So x may be rewritten as $\{x|A^T(I_m)x\geqslant\alpha\}$, where $\alpha=(\alpha_1, \cdots, \alpha_m)^T$.

Definition 3. *Suppose* $x_0\in X$, $\delta\geqslant 0$ *and put*[4]

$$I(x_0, \delta)=\{p|\alpha_p\leqslant(a^p)^T x_0\leqslant\alpha_p+\delta\}, \tag{4}$$

$$C(I(x_0, \delta))=\{z|A^T(I(x_0, \delta))z\geqslant 0\}, \tag{5}$$

then $I(x_0, \delta)$ *is called the index set of the local active constraints with the pace size* δ *at the point* x_0, *and* $C(I(x_0, \delta))$ *is called the local feasible cone with the pace size* δ *at the point* x_0. *Furthemore, suppose* $B=\{b^1, \cdots, b^q\}$ $(q=n+1$ *or* $n)$ *is a canonical positive basis of* $C(I(x_0, \delta))$, *then* B *is called a local canonical positive basis with the pace size* δ *at the point* x_0, *or simply a local positive basis at the point* x_0.

Lemma 4. *Suppose* $x_i\in X$, $x_i\to x_0$, $\delta_i>0$ *and* $\delta_i\to 0$, *then there exists an infinite set* $J\subset\{1, 2, 3, \cdots\}$ *and an index set* I_0 *such that*

$$\forall i\in J,\ I(x_i, \delta_i)\equiv I_0\subset I(x_0, 0). \tag{6}$$

(Proof omitted).

Lemma 5. *Suppose that* $X=\{x|A^T(I_m)x\geqslant\alpha\}$ *is nonempty and bounded, and that* X *is non-degenerate, i.e., rank* $A(I(x_*, 0))=|I(x_*, 0)|=n$ *at any extreme point* x_* *of* X, *then there exists a positive number* $\delta_0>0$, *such that* $\forall\delta\in[0, \delta_0]$ *and* $\forall x\in X$, *we have rank* $A(I(x, \delta))=|I(x, \delta)|$.

Proof We may first deal with the case of $\delta=0$ by showing that its contrary is wrong, and then proceed to prove for the case of $\delta>0$. The details are omitted here[6].

§ 3. An Algorithm Model

Definition 4. (*the Algorithm Model of a Direct Search Method by the Local Positive Basis*) *For the linearly constrained optimization problem*

$$\text{(LNP)}: \min\{f(x)|x\in X\},\ X=\{x|(a^i)^T x\geqslant\alpha_i,\ i\in I_m\},$$

we design the following steps to produce an iterative point sequence $\{x_k\}$:

Step 1 *Choose* $\delta_l>0$ *and* $\varepsilon_l>0$ *such that* $\delta_l\to 0$ *and* $\varepsilon_l/\delta_l\to 0$. *Take* $x_1\in X$ *and set* $k:=i:=l:=1$.

Step 2 *Derive the index set of the local active constraints, i.e.* $I(x_k, \delta_l)=\{p|\alpha_p\leqslant(a^p)^T x_k\leqslant\alpha_p+\delta_l\}$, *and get the corresponding matrix*

$$A_*=A(I(x_k, \delta_l))=\{a^p|p\in I(x_k, \delta_l)\}.$$

If rank $A_*<r=|I(x_k, \delta_l)|$, *then set* $l:=l+1$ *and go back to this step.*

Step 3 *If rank* $A_*=r=|I(x_k, \delta_l)|$, *then determine the local canonical positive basis* $B^k=\{b^1, b^2, \cdots, b^r, b^{r+1}, \cdots, b^q\}$ $(q=n+1$ *or* $n)$ *with the pace size* δ_l *at the point* x_k.

Step 4 *For* $j=1, \cdots, q$, *examine the following inequality which is termed as the* *condition for making a move*

$$f(x_k+\lambda b^j)<f(x_k)-\varepsilon_l, \quad \lambda=\delta_l(\|b^j\|\cdot\max_{1\leqslant p\leqslant m}\|a^p\|)^{-1}. \tag{7}$$

If it is satisfied, then set $x_{k+1}=x_k+\lambda b^j$ *and* $k:=k+1$, *and go back to step* 2.

Step 5 *If the conditions for making a move are satisfied for none of the* j's ($j=1, 2, \cdots, q$), *then set* $x_{k+1}:=x_k$, $k(i):=k$, $y_i:=x_{k(i)}$, $l(i):=l$, $k:=k+1$, $i:=i+1$ *and* $l:=l+1$, *and go back to step* 2.

In the above algorithm model, δ_l is called the step size for a move, and ε_l the descent threshold with their meanings shown in step 4. Besides, k, i and l are refferred to as the indices of the iterative point x_k, the leading point y_i and the step size δ_1 respectively.

Lemma 6. *Suppose that the linearly constraint set* X *is nonempty, bounded and non-degenerate, and that the objective function is continuous on* X, *then the direct search method by the local positive basis possesses the following properties:*

(i) *There can only be a finite number of cases which call for an increase in* l *or a decrease in the step size* δ_l *in step* 2.

(ii) *All* $x_k\in X$.

(iii) *For a certain* l *in step* 4, *there can only be a finite number of cases, in which the conditions for making a move are satisfied.*

(iv) *The leading point sequence* $\{y_i\}$ *produced in step* 5 *is infinite.*

Proof The property (i) may be proved by means of lemma 5. Now we verify that in step 4, $x_{k+1}\in X$ when $x_k\in X$. For any $p\in I(x_k, \delta_l)$, since b^j is a vector in the local positive basis, we have $(a^p)^T b^j\geqslant 0$ and since $(a^p)^T x_k\geqslant \alpha_p$ and $x_{k+1}=x_k+\lambda b^j$ with $\lambda>0$, we get $(a^p)^T x_{k+1}\geqslant\alpha_p$. As for any $p\overline{\in} I(x_k, \delta_l)$, we have $(a^p)^T x_k>\alpha_p+\delta_l$. Noticing that $|(a^p)^T b^j|\leqslant\|a^p\|\cdot\|b^j\|$ and $x_{k+1}=x_k+\lambda b^j$ with $\lambda=\delta_l(\|b^j\|\cdot\max_{1\leqslant p\leqslant m}\|a^p\|)^{-1}$, we may obtain the following inequality

$$(a^p)^T x_{k+1}>\alpha_p+\delta_l-|(a^p)^T b^j|\cdot\delta_l(\|b^j\|\cdot\max_{1\leqslant p\leqslant m}\|a^p\|)^{-1}\geqslant\alpha_p. \tag{8}$$

Therefore, we are sure to have $x_{k+1}\in X$. In addition we have taken $x_1\in X$ in step 1, so we get the property (ii) by induction. As to the property (iii), because there is a fixed descent threshold ε_l in the condition for making a move for fixed l, and because $\min\{f(x)\,|\,x\in X\}$ is attainable, this property may be proved by showing that its contrary is wrong. Finally the property (iv) may be obtained from the properties (i) and (iii). Q.E.D.

§ 4. The Convergence of the Algorithm Model

Theorem. *Suppose that the constraint set* $X=\{x\,|\,A^T(I_m)x\geqslant\alpha\}$ *is nonempty, bounded and nondegenerate, the objective function* $f(x)$ *is continuosly differentiable, and* *that $\{y_i\}$ is a leading point sequence produced with the direct search method by the local positive basis, then any accumulation point x_* of $\{y_i\}$ is a Kuhn-Tucker point of the problem (LNP).*

Proof We have seen from lemma 6 that the leading point sequence $\{y_i\}$ is infinite. According to step 4 and step 5 in the algorithm model, the i-th leading point $y_i=x_{k(i)}$ satisfies the following system of inequalities

$$f(x_k+\lambda b^j)\geqslant f(x_k)-\varepsilon_l \quad (j=1,\ 2,\ \cdots,\ q), \tag{9}$$

where $q=n+1$ or n, $\lambda=\delta_l\cdot(\|b^j\|\cdot\max_{1\leqslant p\leqslant m}\|a^p\|)^{-1}$, $k=k(i)$ is the iterative point index, and $l=l(i)$ is step size index. Hence we may rewrite (9) as follows

$$f(y_i+\lambda_{ij}b^j)-f(y_i)\geqslant-\varepsilon_{l(i)} \quad (j=1,\ 2,\ \cdots,\ q), \tag{10}$$

$$\lambda_{ij}=\delta_{l(i)}(\|b^j\|\max_{1\leqslant p\leqslant m}\|a^p\|)^{-1}. \tag{11}$$

Now let us assume that $J\subset\{1,\ 2,\ 3,\ \cdots\}$ and $\lim_{i\in J}y_i=x_*$. We consider the index set $I(y_i,\ \delta_{l(i)})$ of the local active constraints with pace size $\delta_{l(i)}$ at $y_i=x_{k(i)}$. By lemma 4, there exists an infinite set $K\subset J$ such that

$$\forall i\in K,\ I(y_i,\ \delta_{l(i)})\equiv I_0\subset I(x_*,\ 0). \tag{12}$$

Because the way for taking local canonical positive basis in relation to the same I_0 is determined by the corresponding basic submatrix, so we only have a finite number of choices. Hence we may also pick out a subsequence from $\{B^{k(i)}\,|\,i\in K\}$ such that all the local canonical positive basis in this subsequence are one and the same. With no loss of generality, we may assume that for any $i\in K$, the corresponding local canomical positive basis is $B^{k(i)}\equiv B^0=\{b^1,\ b^2,\ \cdots,\ b^q\}$ $(q=n+1$ or $n)$, which is independent of i. Dividing (10) by (11), applying the mean value theorem, taking the limit for $i\in K$, and noticing that $\varepsilon_{l(i)}/\delta_{l(i)}\to 0$, we may get

$$\frac{\partial f}{\partial b^j}(x_*)\geqslant 0 \quad (j=1,\ 2,\ \cdots,\ q). \tag{13}$$

By lemma 3, we have

$$\frac{\partial f}{\partial z}(x_*)\geqslant 0 \tag{14}$$

for any $z\in C(I(y_i,\ \delta_{l(i)}))\equiv C(I_0)$ which is a local feasible cone. Also from (12), we know $C(I(x_*,\ 0))\subset C(I_0)$, where $C(I(x_*,\ 0))$ is the feasible cone at x_*. Thus we still have (14) for any $z\in C(I(x_*,\ 0))=\{z\,|\,(a^p)^Tz\geqslant 0,\ p\in I(x_*,\ 0)\}$. Again by lemma 3, $\nabla f(x_*)$ is a positive combination of $A(I(x_*,\ 0))$, hence the Kuhn-Tucker conditions at x_* are satisfied. Q.E.D.

Remark. On the ground of the above convergence theorem, we may further prove that if $f(x)$ is both continuously differentiable and strictly convex, then the whole iterative point sequence $\{x_k\}$ produced by the algorithm model converges to the unique optimum point of the problem (LNP). In considering the steps in the algorithm model, if we also take some pattern moves like those in the Hooke-Jeeves technique, then the convergence properties may be retained, and these pattern moves may be helpful to practical calculations. In addition, when the constraint set have degenerate extreme points, the basic idea in the algorithm model may still be useful, although the structure of the local feasible cone in the general case turns out to be rather complicated.

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线性约束最优化的局部正基方向搜索法

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摘　要

讨论下列线性约束最优化问题

$$(\mathrm{LNP}):\ \min_{x\in X} f(x),\quad X=\{x\,|\,x\in R^n,\ (a^i)^T x\geqslant\alpha_i,\ i\in I_m\},$$

其中 $I_m=\{1, 2, \cdots, m\}$. 对于 X 中的能行点，定义了局部能行锥与相应的局部正基——即生成该锥的一组正独立的向量，给出了沿着局部正基方向进行目标函数值比较与迭代点移动的算法模型，简称为局部正基方向搜索法，本文并证明了这算法的收敛性定理：

定理　设约束集合 $X=\{x\,|\,(a^i)^T x\geqslant\alpha_i,\ i\in I_m\}$ 非空有界且非退化，目标函数 $f(x)$ 连续可微，$\{y_i\}$ 是局部正基方向搜索法产生的某个点列，那末 $\{y_i\}$ 的任意极限点 x_* 必是问题(LNP)的 Kuhn-Tucker 点。