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# ONE SPECIAL INVERSE PROBLEM OF THE SECOND ORDER DIFFERENTIAL EQUATION ON THE WHOLE REAL AXIS

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## Introduction

To solve an axial symmetric KDV equation, we must consider the following eigenvalue problem[1, 6]

$$-\varphi''(x, \lambda)-Q(x)\varphi(x, \lambda)=\lambda\varphi(x, \lambda), \quad -\infty\leqslant x\leqslant\infty, \tag{0.1}$$

where

$$Q(x)=x+q(x). \tag{0.2}$$

The inverse problem of the second order differential equation with two singular points has been considered by Bloch[2]. He pointed out that the potential function can be determined by certain $2\times2$ spectral matrix. But we shall point out that when $Q(x)=x-q(x)$ with $q(x)$ satisfying the following conditions

$$q(x)\in C^1(-\infty, \infty), \quad \int_{-\infty}^{\infty}|s^i q(s)|\,ds\leqslant\infty, \quad i=0, 1, \tag{I}$$

then the coefficient function $q(x)$ can be determined by one spectral function. In § 1 we introduce the corresponding Riemann function, with which we establish a transformation between the function $\varphi_0(x, \lambda)$ and $\varphi(x, \lambda)$, where $\varphi_0(x, \lambda)=-\sqrt{\pi}\ \mathrm{Ai}(x-\lambda)$ is a solution of equation (0.1) when $Q(x)=x$, and $\varphi(x, \lambda)$ is a solution of equation (0.1) when $Q(x)=x+q(x)$. In § 2 we prove the completeness of one spectral function by Titchmarch-Kodaira's theory. Finally we derive an integral equation which is analogous to Gel'fand-Levitan equation.

## 1. The existence of the transformation.

In proving the theorem, the following lemmas will be required.

**Lemma 1.1.** *Let*

$$x=\left[\frac{1}{2}(\eta_0^2-\eta^2)(\xi-\xi_0)\right]^{\frac{1}{2}}, \tag{1.1}$$

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$$V(\xi_0,\ \eta_0;\ \xi,\ \eta)=J_0(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \tag{1.2}$$

*then the function* $V(\xi_0,\ \eta_0;\ \xi,\ \eta)$ *satisfies the equation*

$$\frac{\partial^2 V}{\partial\eta\,\partial\xi}-\frac{1}{4}\eta V=0 \tag{1.3}$$

*and*

$$V(\xi_0,\ \eta_0;\ \xi_0,\ \eta)=V(\xi_0,\ \eta_0;\ \xi,\ \eta_0)=1. \tag{1.4}$$

*In other words,* $V(\xi_0,\ \eta_0;\ \xi,\ \eta)$ *is the Riemann function of the equation* (1.3) *and has the symmetric property*

$$V(\xi_0,\ \eta_0;\ \xi,\ \eta)=V(\xi,\ \eta;\ \xi_0,\ \eta_0). \tag{1.5}$$

*Proof* Because

$$\frac{\partial V}{\partial\xi}=J_0'(x)x^{-1}\cdot\frac{1}{4}(\eta_0^2-\eta^2),$$

$$\frac{\partial V}{\partial\eta\,\partial\xi}-\frac{1}{4}\eta V=-\frac{1}{4}\eta\left[J_0''(x)+\frac{1}{x}J_0'(x)+J_0(x)\right]=0.$$

From (1.1) and (1.2), we have (1.4) and (1.5).

From the relation $J_{n+1}(x)+J_{n-1}(x)=\frac{2n}{x}J_n(x)$, $J_n'(x)=-J_n(x)+\frac{nJ_n(x)}{x}$ and $|J_n(x)|\leqslant 1$, we have $\left|\frac{J_1(x)}{x}\right|\leqslant 1$, $\left|\frac{J_3(x)}{x}\right|\leqslant\frac{1}{3}$, $\left|\frac{J_2(x)}{x}\right|\leqslant\frac{1}{3}$, and differentiating (1.2), we obtain the following corollary.

**Corollary.** *We have the inequalities*

$$|V|\leqslant 1,\quad\left|\frac{\partial V}{\partial\xi_0}\right|\leqslant\frac{1}{4}(\eta_0^2-\eta^2),\quad\left|\frac{\partial V}{\partial\eta_0}\right|\leqslant\frac{1}{2}\eta_0(\xi-\xi_0),$$

$$\left|\frac{\partial V}{\partial\eta\,\partial\xi}\right|\leqslant\frac{1}{4}\eta_0,\quad\left|\frac{\partial^2 V}{\partial\xi_0^2}\right|\leqslant\frac{1}{48}(\eta_0^2-\eta^2)^2,\quad\left|\frac{\partial^2 V}{\partial\eta_0^2}\right|\leqslant\frac{1}{3}\eta_0^2(\xi-\xi_0)^2+\frac{1}{2}(\xi-\xi_0). \tag{1.6}$$

Let

$$\sigma(x)=\int_{-\infty}^{\infty}|q(t)|\,dt,\quad\sigma_2(x)=\int_x^{\infty}\sigma(t)\,dt=\int_x^{\infty}(s-x)\,|q(s)|\,ds. \tag{1.7}$$

where the right hand side is integrable, when $q(x)$ satisfies condition (I).

**Lemma 1.2.** *If* $q(x)$ *satisfies condition (I), then the integral equation*

$$\tilde{K}(\xi_0,\ \eta_0)=\frac{1}{4}\int_{\xi_0}^{\infty}V(\xi_0,\ \eta_0;\ \xi,\ 0)\,q\left(\frac{\xi}{2}\right)d\xi$$
$$-\frac{1}{4}\int_0^{\eta_0}\int_{\xi_0}^{\infty}V(\xi_0,\ \eta_0;\ \xi,\ \eta)\,q\left(\frac{\xi-\eta}{2}\right)\tilde{K}(\xi,\ \eta)\,d\xi\,d\eta \tag{1.8}$$

*has one and only one solution* $\tilde{K}(\xi_0,\ \eta_0)$. *When* $\eta_0\geqslant 0$, $\tilde{K}(\xi_0,\ \eta_0)$ *satisfies the inequatity*

$$|\tilde{K}(\xi_0,\ \eta_0)|\leqslant\frac{1}{2}\sigma\left(\frac{\xi_0}{2}\right)e^{\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right)}. \tag{1.9}$$

*Futhermore, if* $q(x)\equiv 0$ *when* $x\geqslant a$, *then* $\tilde{K}(\xi_0,\ \eta_0)\equiv 0$ *when* $\xi_0\geqslant 2a$. (1.10)

*Proof* By the method of successive approximation, Let

$$\tilde{K}(\xi_0,\ \eta_0)=\frac{1}{4}\int_{\xi_0}^{\infty}V(\xi_0,\ \eta_0;\ \xi,\ 0)\,q\left(\frac{\xi}{2}\right)d\xi, \tag{1.11}$$

$$\widetilde{K}_n(\xi_0,\ \eta_0)=\frac{1}{4}\int_0^{\eta_0}\int_{\xi_0}^{\infty}V(\xi_0,\ \eta_0;\ \xi,\ \eta)q\left(\frac{\xi-\eta}{2}\right)\widetilde{K}_{n-1}(\xi,\ \eta)d\xi\,d\eta, \tag{1.12}$$

$$\widetilde{K}(\xi_0,\ \eta_0)=\sum_{n=0}^{\infty}\widetilde{K}_n(\xi_0,\ \eta_0). \tag{1.13}$$

From the first inequality of (1.6), it yields

$$|\widetilde{K}(\xi_0,\ \eta_0)|\leqslant\frac{1}{4}\int_{\xi_0}^{\infty}\left|q\left(\frac{\xi}{2}\right)\right|d\xi=\frac{1}{2}\sigma\left(\frac{\xi_0}{2}\right),$$

$$|\widetilde{K}_1(\xi_0,\ \eta_0)|\leqslant\frac{1}{4}\int_0^{\eta_0}\int_{\xi_0}^{\infty}\left|q\left(\frac{\xi-\eta}{2}\right)\right|\frac{1}{2}\sigma\left(\frac{\xi}{2}\right)d\xi\,d\eta.$$

Let

$$J=\frac{1}{4}\int_0^{\eta_0}\int_{\xi_0}^{\infty}\left|q\left(\frac{\xi-\eta}{2}\right)\right|d\xi\,d\eta, \tag{1.14}$$

$$|J|\leqslant\int_{\xi_0}^{\infty}d\xi\int_{\frac{\xi-\eta_0}{2}}^{\frac{\xi}{2}}\frac{1}{2}|q(s)|\,ds\leqslant\int_{\xi_0}^{\infty}d\xi\int_{\frac{\xi-\eta_0}{2}}^{\infty}\frac{1}{2}|q(s)|\,ds$$

$$=\int_{\xi_0}^{\infty}\frac{1}{2}\sigma\left(\frac{\xi-\eta_0}{2}\right)d\xi=\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right).$$

Thus $$|\widetilde{K}_1(\xi_0,\ \eta_0)|\leqslant\frac{1}{2}\sigma\left(\frac{\xi_0}{2}\right)\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right).$$

If $$|\widetilde{K}_{n-1}(\xi_0,\ \eta_0)|\leqslant\frac{1}{2}\sigma\left(\frac{\xi_0}{2}\right)\cdot\frac{1}{(n-1)!}\left[\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right)\right]^{n-1},$$

then

$$|\widetilde{K}_n(\xi_0,\ \eta_0)|\leqslant\frac{1}{4}\int_0^{\eta_0}\int_{\xi_0}^{\infty}\left|q\left(\frac{\xi-\eta}{2}\right)\right|\frac{1}{2}\sigma\left(\frac{\xi}{2}\right)\frac{1}{(n-1)!}\left[\sigma_2\left(\frac{\xi-\eta}{2}\right)\right]^{n-1}d\xi\,d\eta$$

$$\leqslant\frac{1}{2(n-1)!}\sigma\left(\frac{\xi_0}{2}\right)\int_{\xi_0}^{\infty}d\xi\int_{\frac{\xi-\eta_0}{2}}^{\infty}\frac{1}{2}|q(s)|\,[\sigma_2(s)]^{n-1}ds$$

$$\leqslant\frac{1}{2(n-1)!}\sigma\left(\frac{\xi_0}{2}\right)\int_{\xi_0}^{\infty}\left[\sigma_2\left(\frac{\xi-\eta_0}{2}\right)\right]^{n-1}d\xi\int_{\frac{\xi-\eta_0}{2}}^{\infty}\frac{1}{2}|q(s)|\,ds$$

$$=\frac{1}{2(n-1)!}\sigma\left(\frac{\xi_0}{2}\right)\int_{\xi_0}^{\infty}\frac{1}{2}\left[\sigma_2\left(\frac{\xi-\eta_0}{2}\right)^{n-1}d\xi\,\sigma\left(\frac{\xi-\eta_0}{2}\right)\right]d\xi$$

$$=\frac{1}{2(n-1)!}\sigma\left(\frac{\xi_0}{2}\right)\int_{\frac{\xi_0-\eta_0}{2}}^{\infty}[\sigma_2(s)]^{n-1}\sigma(s)ds$$

$$=-\frac{1}{2(n-1)!}\sigma\left(\frac{\xi_0}{2}\right)\int_{\frac{\xi_0-\eta_0}{2}}^{\infty}[\sigma(s)]^{n-1}d\sigma(s)$$

$$=\frac{1}{2n!}\sigma\left(\frac{\xi_0}{2}\right)\left[\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right)\right]^n.$$

It is easily seen that when $\eta_0\geqslant0$, series (1.13) is absolutely and uniformly convergent, and

$$|\widetilde{K}(\xi_0,\ \eta_0)|\leqslant\sum_{n=0}^{\infty}\frac{1}{2}\sigma\left(\frac{\xi_0}{2}\right)\frac{1}{n!}\left[\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right)\right]^n=\frac{1}{2}\sigma\left(\frac{\xi_0}{2}\right)e^{\sigma_2\left(\frac{\xi_0-\eta_0}{2}\right)}$$

We have proved that the function $\widetilde{K}(\xi_0,\ \eta_0)$ satisfies inequality (1.9) and is a solution of (1.8). (1.9) implies obviously the uniqueness of the solution for equation (1.8) and the conclusion (1.10).

Differentiating equation (1.8) directly and using the inequalities of (1.6), we have.

**Lemma 1.3.** 1) *Function* $\tilde{K}(\xi_0, \eta_0)\in C^2$ *on the domain* $\eta_0\geqslant 0$, $\eta_0\leqslant\xi_0<\infty$,

2) *When* $\xi_0-\eta_0=const$, *and* $\xi_0+\eta_0\to+\infty$, *we have*

$$\frac{\partial\tilde{K}}{\partial\xi_0}=O(\eta_0^2),\quad \frac{\partial\tilde{K}}{\partial\eta_0}=O(\xi_0\eta_0),\quad \frac{\partial^2\tilde{K}}{\partial\xi_0\partial\eta_0}=O(\eta_0),$$

$$\frac{\partial^2\tilde{K}}{\partial\xi_0^2}=O(\eta_0^4),\quad \frac{\partial^2\tilde{K}}{\partial\eta_0^2}=O(\xi_0^2+\eta_0^4). \tag{1.15}$$

**Theorem 1.1.** *If hypothesis (I) is fulfilled, then the solution* $\tilde{K}(\xi_0, \eta_0)$ *of equation* (1.8) *is a solution of the following differential equation*

$$1)\quad \frac{\partial^2\tilde{K}}{\partial\eta_0\partial\xi_0}-\frac{1}{4}\eta_0\tilde{K}+\frac{1}{4}q\left(\frac{\xi_0-\eta_0}{2}\right)\tilde{K}=0,\quad \text{when } \eta_0\geqslant 0 \tag{1.16}$$

*and*

$$\tilde{K}(\xi_0, 0)=\frac{1}{2}\int_{\frac{\xi_0}{2}}^{\infty}q(s)ds. \tag{1.17}$$

*If we let* $\xi_0=x-y$, $\eta_0=y-x$, *and express the function* $\tilde{K}(\xi_0, \eta_0)=\tilde{K}(x-y, y-x)=K(x, y)$ *as a function of* $x, y$, *then the function* $K(x, y)$ *satisfies the following equations*

$$2)\quad \frac{\partial^2 K}{\partial x^2}-\frac{\partial^2 K}{\partial y^2}=[q(x)+x-y]K,\quad \text{when } y\geqslant x. \tag{1.18}$$

*and*

$$K(x, x)=\frac{1}{2}\int_x^{\infty}q(s)ds, \tag{1.19}$$

$$|K(x, x)|\leqslant\frac{1}{2}\sigma\left(\frac{x-y}{2}\right)e^{\sigma_1(x)}. \tag{1.20}$$

*Furthermore, if* $q(x)\equiv 0$ *when* $x\geqslant a$, *then* $K(x, y)=0$ *when* $x+y\geqslant a$. (1.21)

3) *When* $x$ *is fixed, and* $y\to\infty$, *we have*

$$\frac{\partial K}{\partial x}=O(y^2),\quad \frac{\partial K}{\partial y}=O(y^2),\quad \frac{\partial^2 K}{\partial x^2}=O(y^4),\quad \frac{\partial^2 K}{\partial y^2}=O(y^4). \tag{1.22}$$

*Proof* 1) From (1.5) and (1.3) of lemma 2.1, it is seen that

$$\frac{\partial^2 V}{\partial\eta_0\partial\xi_0}-\frac{1}{4}\eta_0 V=0 \tag{1.23}$$

From (1.8), it follows

$$\begin{aligned}\frac{\partial^2\tilde{K}}{\partial\eta_0\partial\xi_0}-\frac{\eta_0}{4}K&=\frac{1}{4}\int_{\xi_0}^{\infty}\left(\frac{\partial^2 V}{\partial\eta_0\partial\xi_0}-\frac{1}{4}\eta_0 V\right)q\left(\frac{\xi}{2}\right)d\xi-\frac{1}{4}q\left(\frac{\xi_0-\eta_0}{2}\right)\tilde{K}(\xi_0, \eta_0)\\&\quad+\int_0^{\eta_0}\int_{\xi_0}^{\infty}\left(\frac{\partial V}{\partial\eta_0\partial\xi_0}-\frac{1}{4}\eta_0 V\right)q\left(\frac{\xi-\eta}{2}\right)\tilde{K}(\xi, \eta)d\xi d\eta\\&=-\frac{1}{4}q\left(\frac{\xi_0-\eta_0}{2}\right)\tilde{K}.\end{aligned}$$

Putting $\eta_0=0$ in (1.8), we have (1.17).

2) From lemma 1.3 (1), we have $K(x, y)\in C^2$ on the domain $y>x$. From (1.16),

(1.17) and (1.9), (1.10), it is seen that $K(x, y)$ as a function of $x$, $y$, satisfies equations (1.18), (1.19) and (1.20), (1.21), respectively.

3) From lemma 1.3(2), it is seen that the function $K(x, y)$ as a function of $x$, $y$, satisfies (1.22).

**Theorem 1.2.** *If $q(x)$ satisfies condition (I), the function $K(x, y)$ is defined as theorem 1.1 and we let*

$$\varphi(x, \lambda)=\varphi_0(x, \lambda)+\int_x^\infty K(x, y)\varphi_0(y, \lambda)dy, \tag{1.24}$$

*then the function $\varphi(x, \lambda)$ is a solution of (0.1) and when $x\to\infty$*

$$\varphi(x, \lambda)/\varphi_0(x, \lambda)\to 1. \tag{1.25}$$

*Proof*

$$\varphi'(x, \lambda)=\varphi_0'(x, \lambda)-K(x, x)\varphi_0(x, \lambda)+\int_x^\infty K_x(x, y)\varphi_0(y, \lambda)dy, \tag{1.26}$$

$$\varphi''(x, \lambda)=\varphi_0''(x, \lambda)-\frac{d}{dx}[K(x, x)\varphi_0(x, \lambda)]-K_x(x, x)\varphi_0(x, \lambda) +\int_x^\infty K_{xx}(x, y)\varphi_0(y, \lambda)dy. \tag{1.27}$$

From (1.20) and (1.22), it is seen that if $x$ is fixed and $y\to\infty$, then

$$K(x, y)=O(1), \quad \frac{\partial K}{\partial x}=O(y^2), \quad \frac{\partial^2 K}{\partial x^2}=O(y^4).$$

From (2.4), it is seen that when $y\to\infty$, $\varphi_0(y, \lambda)$, $\varphi_0'(y, \lambda)$ tend to zero exponentially, so that the last terms of (1.24), (1.26) and (1.27) are integrable. From $-\varphi_0''(y, \lambda)-y\varphi_0(y, \lambda)=\lambda\varphi_0(y, \lambda)$ and (1.24) we have

$$-\lambda\varphi(x, \lambda)=-\lambda\varphi_0(x, \lambda)-\int_x^\infty K(x, y)y\varphi_0(y, \lambda)dy +\int_x^\infty K(x, y)\varphi_0''(y, \lambda)dy, \tag{1.28}$$

where

$$\int_x^\infty K(x, y)\varphi_0''(y, \lambda)dy=K(x, y)\varphi_0'(y, \lambda)\Big|_x^\infty-\int_x^\infty K_y(x, y)\varphi_0'(y, \lambda)dy$$
$$=K(x, y)\varphi_0'(y, \lambda)\Big|_x^\infty-K_y(x, y)\varphi_0(y, \lambda)\Big|_x^\infty-\int_x^\infty K_{yy}(x, y)\varphi_0(y, \lambda)dy. \tag{1.29}$$

In the same way, it follows that the last terms of (1.28) and (1.29) are integrable and when $y\to\infty$

$$K(x, y)\varphi_0'(y, \lambda)\to 0, \quad K_y(x, y)\varphi_0(y, \lambda)\to 0.$$

From (1.24) and (1.27)—(1.29), we have

$$-\varphi''(x, \lambda)+[x+q(x)]\varphi(x, \lambda)-\lambda\varphi(x, \lambda)$$
$$=-\varphi_0''+[x+q]\varphi_0-\lambda\varphi_0-2\frac{d}{dx}K(x, x)\varphi_0(x, \lambda)$$
$$-\int_x^\infty\Big[\frac{\partial^2 K}{\partial x^2}-\frac{\partial^2 K}{\partial y^2}-(x+q(x)-y)K\Big]\varphi_0(y, \lambda)dy.$$

From $-\varphi_0''+x\varphi_0=\lambda\varphi_0$ and (1.18), it is seen that $\varphi(x, \lambda)$ satisfies equation (0.1).

When $x>0$ and is sufficiently large, $\varphi_0(x, \lambda)$ is a monotone decreasing positive function, so that

$$\left|\int_x^\infty K(x, y)\varphi_0(y, \lambda)dy\right| \leqslant \varphi_0(x, \lambda)\int_x^\infty |K(x, y)|dy = \varphi_0(x, \lambda)\sigma_2(x)e^{\sigma_1(x)}.$$

When $x\to\infty$, $\sigma_2(x)\to 0$, $\sigma(x)\to 0$, thus when $x\to\infty$,

$$\varphi(x, \lambda)/\varphi_0(x, \lambda) = 1 - \left[\int_x^\infty K(x, y)\varphi_0(y, \lambda)dy\right]\Big/\varphi_0(x, \lambda)\to 1,$$

theorem 1.2 is proved.

**Remarks** If $q(x)\equiv 0$ when $x\geqslant a$, then $K(x, y)\equiv 0$ when $x\geqslant a$, $y>x$. From (1.24), we have $\varphi(x, \lambda)=\varphi_0(x, \lambda)$ when $x\geqslant a$.

## 2. Completeness.

When $q(x)=0$, equation (0.1) becomes

$$-\varphi''(x, \lambda)+x\varphi(x, \lambda)=\lambda\varphi(x, \lambda). \tag{2.1}$$

From Ref. [3], equation (2.1) is in the limit point case at two singular points. There is a solution

$$\varphi_0(x, \lambda) = -\sqrt{\pi}\,\mathrm{Ai}(z) = -\frac{1}{\pi}\sqrt{\frac{2}{3}}\,K_{1/3}(\xi) = -\frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{1}{3}x^3 - xz\right)dx, \tag{2.2}$$

where

$$\xi = \frac{2}{3}z^{3/2}, \quad z = x-\lambda. \tag{2.3}$$

When $\mathrm{Im}\,\lambda>0$,

$$\varphi_0(x, \lambda)\sim -\frac{1}{2}x^{-\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}},$$

$$\varphi_0'(x, \lambda)\sim \frac{1}{2}x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}, \quad x\to+\infty, \tag{2.4}$$

which is of class $L^2(0, \infty)$. Let

$$Q_0(x, \lambda) = \sqrt{\pi}\,\mathrm{Bi}(z)\sim x^{-\frac{1}{4}}e^{\frac{2}{3}x^{\frac{3}{2}}}, \quad x\to+\infty, \tag{2.5}$$

then $Q_0(x, \lambda)$ is a linearly independent solution of $\varphi_0(x, \lambda)$ and

$$W[Q_0(x, \lambda), \varphi_0(x, \lambda)] = \varphi_0'Q_0 - Q_0'\varphi_0 = 1. \tag{2.6}$$

Let

$$\psi_0(x, \lambda) = Q_0(x, \lambda) - M_1(\lambda)\varphi_0(x, \lambda) = Q_0 - i\varphi_0, \tag{2.7}$$

then

$$\psi_0(x, \lambda)\sim(-x)^{-\frac{1}{4}}e^{i(-x)^{\frac{3}{2}}}, \quad x\to-\infty, \tag{2.8}$$

when $\mathrm{Im}\,\lambda>0$, $\psi_0(x, \lambda)$ is of class $L(-\infty, 0)$.

According to the notation used in chap. 3 of Ref. [3], here

$$M_2(\lambda) = +\infty, \quad M_1(\lambda) = -i,$$

$$\xi(\lambda)=0, \quad \eta(\lambda)=0, \quad \zeta(\lambda)=\int_0^\lambda du = \lambda, \tag{2.9}$$

From § 4 of Ref. [4], we have

**Theorem 2.1.** *If* $f(x)\in L^2(-\infty, \infty)$, *and let*

$$F_0(\lambda)=\text{l.i.m}\int_{-\infty}^{\infty} f(x)\varphi_0(x, \lambda)dx, \tag{2.10}$$

*then*

$$f(y)=\text{l.i.m}\frac{1}{\pi}\int_{-\infty}^{\infty} F_0(\lambda)\varphi_0(y, \lambda)d\lambda, \tag{2.11}$$

*i.e.,*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_0(x, \lambda)\varphi_0(y, \lambda)d\lambda=\delta(x-y). \tag{2.12}$$

**Lemma 2.1.** *When* $q(x)$ *satisfies the condition*

$$\int_{-\infty}^{\infty}\frac{|q(x)|}{1+|x|^{\frac{1}{2}}}dx<\infty, \tag{2.13}$$

*then equation* (0.1) *is in the limit point case at two singular points.*

*Proof* When $\operatorname{Im}\lambda>0$, $x>x_0>0$, and $x_0$ is sufficiently large, we define

$$\Phi(x, \lambda)=Q_0(x, \lambda)-\int_{x_0}^{x}[\varphi_0(x, \lambda)Q_0(y, \lambda)-Q_0(x, \lambda)\varphi_0(y, \lambda)]q(y)\Phi(y, \lambda)dy, \tag{2.14}$$

it is easy to verify that $\Phi(x, \lambda)$ is a solution of equation (0.1).

We put

$$\Phi_1(x, \lambda)=x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\Phi(x, \lambda).$$

From (2.4) and (2.5), when $x_0<y<x$, we have

$$\left|x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}[\varphi_0(x, \lambda)Q_0(y, \lambda)-Q_0(x, \lambda)\varphi(y, \lambda)]y^{-\frac{1}{4}}e^{\frac{2}{3}x^{\frac{3}{2}}}\right|\leqslant|y|^{-\frac{1}{2}}.$$

From (2.14), we have

$$|\Phi_1(x, \lambda)|\leqslant M+\int_{x_0}^{x}|y|^{-\frac{1}{2}}|q(y)||\Phi(y, \lambda)|dy, \quad M>0.$$

Applying Bellman inequality, we get

$$|\Phi_1(x, \lambda)|\leqslant Me^{\int_{x_0}^{x}|y^{-\frac{1}{2}}q(y)|dy}.$$

It has been shown that the function $\Phi_1(x, \lambda)$ is bounded for $[x_0, \infty)$ under the condition (2.13), thus $\Phi(x, \lambda)$ does not belong to $L^2(x_0, \infty)$.

We note the asymptotic

$$\varphi_0(x, \lambda)\sim-(-x)^{-\frac{1}{4}}\sin\left[\frac{2}{3}(-x)^{\frac{3}{2}}+\frac{\pi}{4}\right],$$

$$Q_0(x, \lambda)\sim(-x)^{-\frac{1}{4}}\cos\left[\frac{2}{3}(-x)^{\frac{3}{2}}+\frac{\pi}{4}\right], \quad x\to-\infty, \tag{2.15}$$

and when $\operatorname{Im}\lambda>0$, $x<-x_0<0$, we define

$$\Psi(x, \lambda)=Q_0(x, \lambda)-\int_{x}^{-x_0}[\varphi_0(x, \lambda)Q_0(y, \lambda)-Q_0(x, \lambda)\varphi_0(y, \lambda)]q(y)\Psi(y, \lambda)dy.$$

By the same method, we can prove that $\Psi(x, \lambda)$ is a solution of equation (0.1) and $\Psi(x, \lambda)$ does not belong to $L^2(-\infty, -x_0)$. From Ref. [3], we have proved lemma 2.2.

We note that when $q(x)$ satisfies condition (I), it satisfies condition (2.13) also.

In § 1, we have proved the relation

$$\varphi(x, \lambda)=\varphi_0(x, \lambda)+\int_x^\infty K(x, y)\varphi_0(y, \lambda)dy,$$

where $\varphi(x, \lambda)$ is a solution of equation (0.1) and is class $L^2(0, \infty)$ when $\mathrm{Im}\lambda>0$. If $Q(x, \lambda)$ which satisfies the relation $W[\varphi, Q]=1$, is another solution of (0.1) and we let

$$\psi(x, \lambda)=Q(x, \lambda)-M_1(\lambda)\varphi(x, \lambda), \tag{2.16}$$

which is of class $L^2(-\infty, 0)$, when $\mathrm{Im}\,\lambda>0$, then $M_2(\lambda)=\infty$ and

$$\xi(\lambda)=0, \quad \eta(\lambda)=0, \quad \zeta(\lambda)=\lim_{\delta\to\infty}\int_0^\lambda -\mathrm{Im}\,M_1(u-i\delta)du. \tag{2.17}$$

From § 4 of Ref. [4], we have

**Theorem 2.2.** *If $f(x)$ $L\in(-\infty, \infty)$, we let*

$$F(\lambda)=\mathrm{l.i.m}\int_{-\infty}^\infty f(x)\varphi(x, \lambda)dx \tag{2.18}$$

*then*

$$f(y)=\mathrm{l.i.m}\frac{1}{\pi}\int_{-\infty}^\infty F(\lambda)\varphi(y, \lambda)d\zeta(\lambda), \tag{2.19}$$

*i.e.,*

$$\frac{1}{\pi}\int_{-\infty}^\infty \varphi(x, \lambda)\varphi(y, \lambda)d\zeta(\lambda)=\delta(x-y). \tag{2.20}$$

**Theorem 2.3.** *If $q(x)$ satisfies condition (I), then the function $K(x, y)$ satisfies the following integral equation*

$$f(x, y)+K(x, y)+\int_x^\infty K(x, t)f(t, y)dt=0, \tag{2.21}$$

*where*

$$f(x, y)=\int_{-\infty}^\infty \varphi_0(x, \lambda)\varphi_0(y, \lambda)d\rho(\lambda), \tag{2.22}$$

$$\rho(\lambda)=\frac{1}{\pi}(\zeta(\lambda)-\lambda). \tag{2.23}$$

*Proof* (1.24) may be viewed as a transformation of Volltera's type and the inverse transformation may be expressed as

$$\varphi_0(y, \lambda)=\varphi(y, \lambda)+\int_y^\infty K_1(y, t)\varphi(t, \lambda)dt. \tag{2.24}$$

From (2.20), we have

$$\frac{1}{\pi}\int_{-\infty}^\infty \varphi(x, \lambda)\varphi_0(y, \lambda)d\zeta(\lambda)=\delta(x-y)+\int_x^\infty K_1(y, t)\,\delta(y-t)dt=\delta(x-y), \quad y>x. \tag{2.25}$$

From (1.24) and (2.12), one has also

$$\frac{1}{\pi}\int_{-\infty}^\infty \varphi(x, \lambda)\varphi_0(y, \lambda)d\lambda=\delta(x-y)+\int_x^\infty K(x, y)\delta(y-t)dt$$
$$=\delta(x-y)-K(x, y), \quad y>x. \tag{2.26}$$

Substract (2.26) from (2.25), we have

$$\int_{-\infty}^{\infty} \varphi(x, \lambda)\varphi_0(y, \lambda)d\rho(\lambda)=K(x, y). \tag{2.27}$$

From (1.24), we have

$$\int_{-\infty}^{\infty} \varphi(x, \lambda)\varphi_0(y, \lambda)d\rho(\lambda)=\int_{-\infty}^{\infty} \varphi_0(x, \lambda)\varphi_0(y, \lambda)d\rho(\lambda)$$
$$+\int_{x}^{\infty} K(x, y)\int_{-\infty}^{\infty} \varphi_0(y, \lambda)\varphi_0(t, \lambda)d\rho(\lambda).$$

From (2.22) and (2.23), follows (2.21).

Now if we give a spectral function $\zeta(\lambda)$, by sustituting it into (2.23) and (2.22) and solving $K(x, y)$ from equation (2.21), then we can determine the function $q(x)$ from (1.19)

$$q(x)=-2\frac{d}{dx}K(x, x).$$

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# 在全实轴上的一个特殊的二阶微分方程反问题

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## 摘 要

为了解轴对称的 KDV 方程要考虑以下问题

$$-\varphi''(x, \lambda)+Q(x)\varphi(x, \lambda)=\lambda\varphi(x, \lambda) \quad (-\infty<x<\infty), \tag{0.1}$$

$$Q(x)=x+q(x). \tag{0.2}$$

Блох[2] 曾考虑以上二端奇型反问题，他指出函数 $Q(x)$ 可由 $2\times 2$ 的谱矩阵来确定。本文指出当 $Q(x)=x+q(x)$，而 $q(x)$ 满足以下条件时

$$q(x)\in C^1(-\infty, \infty), \quad \int_{-\infty}^{\infty}|s^i q(s)|ds<\infty, \quad i=0, 1, \tag{I}$$

则函数 $q(x)$ 可由一个谱函数来确定，在 §1 我们引进黎曼函数证明了函数 $\varphi_0(x, \lambda)$ 和 $\varphi(x, \lambda)$ 间变换的存在性，其中 $\varphi_0(x, \lambda)=-\sqrt{\pi}\,\mathrm{Ai}(x-\lambda)$ 是方程(0.1)当 $Q(x)=x$ 时的解，$\varphi(x, \lambda)$ 是方程(0.1)当 $Q(x)=x+q(x)$时的解。在 §2 中，根据 Titchmarsh-Kodaira 理论给出对一个谱函数的完备性。最后推导出类似于 Gel'fand-Levitan 方程。