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UNBOUNDED SELF-ADJOINT OPERATOR IN Π_k SPACE

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In [2], triangle models have been constructed for self-adjoint operators in Π_k space, and from these models spectral families have been given in explicit form. Thus the structure of these operators is completely clear. Based on the models, as we shall see below, it is possible to discuss some basic properties of the self-adjoint operators. In this paper we use the results of [2] and further discuss the n-th powers, n-th roots and more general operational calculus of the self-adjoint operators in Π_k space. We also consider the properties of the spectral families and obtain a spectral representation of the resolvent.

With the triangle models, the main results of [2] can be described as follows: If A is a self-adjoint operator in Π_k space, then

$$\Pi_k = N \oplus \{Z + Z^*\} \oplus P, \tag{1}$$

$$A = \{S, A_N, A_P, F, G, Q\}, \tag{2}$$

where $N(P, Z)$ is a negative (positive, neutral) subspace, Z^* is the dual subspace of Z, A_N (A_P) is a self-adjoint operator in Hilbert space $N(P)$, $S(F, G, Q)$ is an operator from $Z(N, P, Z^*)$ to Z. It can be seen readily that under the standard decomposition (1), for $\lambda \in \rho(A)$, $(\lambda - A)^{-1}$ may be expressed in terms of a triangle model

$$\begin{aligned}(\lambda-A)^{-1} = \{&(\lambda-S)^{-1}, (\lambda-A_N)^{-1}, (\lambda-A_P)^{-1}, (\lambda-S)^{-1}F(\lambda-A_N)^{-1}, \\ &(\lambda-S)^{-1}G(\lambda-A_P)^{-1}, (\lambda-S)^{-1}[-Q-F(\lambda-A_N)^{-1}F^* \\ &+G(\lambda-A_P)^{-1}G^*](\lambda-S^*)^{-1}\}.\end{aligned} \tag{3}$$

Suppose that for $\lambda \in \sigma_P(A)$, the radical subspace corresponding to λ is not positive, then we say λ is a critical point of A and denote the set of these points by $C(A)$. If $C(A) = \{\lambda_1, \cdots, \lambda_n\} \subset R$, $C(A) \cap [\mu, \nu) = \{\lambda_1\}$, then

$$\begin{aligned}E_{\mu\nu} = \{&P_1, P_{\mu\nu}^{A_N}, P_{\mu\nu}^{A_P}, P(F, A_N), P(G, A_P), \\ &-P_Q(F, A_N) + P_Q(G, A_P) + Q_1\},\end{aligned} \tag{4}$$

where E_t is the spectral family of A, P_i is some parallel projections defined in Z whose range is the radical subspace of S corresponding to λ_i, $P_\lambda^{A_N}$ and $P_\lambda^{A_P}$ are the spectral

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families of A_N and A_P. And with the notations

$$R_B(\lambda_l, \lambda_m, j, k) = (\lambda_m - B)^{-k}\Big[(\lambda_l - B)^{-j} - \sum_{i=0}^{k-1} C_{i+j-1}^{i} \frac{(B-\lambda_m)^i}{(\lambda_l-\lambda_m)^{i+j}}\Big] P_{\mu\nu}^B,$$

$$H(\lambda_l, E, B) = \sum_j P_l(\lambda_l - S)^{j-1} E \mathscr{P}_{\mu\nu}^B (\lambda_l - B)^{-j},$$

$$\mathscr{P}_{\mu\nu}^B = \begin{cases} I - P_{\mu\nu}^B, & l=1, \\ P_{\mu\nu}^B, & l \neq 1, \end{cases}$$

we write

$$P(E, B) = \sum_l (1-2\delta_{1l}) H(\lambda_l, E, B),$$

$$P_Q(E, B) = \sum_{\substack{l,m\neq 1\\ l=m=1}} (1-2\delta_{1l}) H(\lambda_l, E, B)[H(\lambda_m, E, B)]^*$$
$$+\sum_{l\neq 1}\sum_{j,k} \{(\lambda_1 - S)^j P_1 E R_B(\lambda_l, \lambda_1, k, j) E^*(\lambda_l - S^*)^k P_l^*$$
$$+ (\lambda_l - S)^j P_l E R_B(\lambda_l, \lambda_1, j, k) E^*(\lambda_1 - S^*)^k P_1^*\},$$

$$Q_1 = \sum_{l\neq 1}\sum_{j,k} \frac{C_{k+j-2}^{k-1}}{(\lambda_l - \lambda_1)^{j+k-1}} [(-1)^j (\lambda_1 - S)^{j-1} P_1 Q (\lambda_l - S^*)^{k-1} P_l^*$$
$$+ (-1)^k (\lambda_l - S)^{j-1} P_l Q (\lambda_1 - S^*)^{k-1} P_1].$$

1. The n-th powers of self-adjoint operators in Π_k space

First of all, we shall use the triangle models to study the n-th powers of the unbounded self-adjoint operators in Π_k space.

It should be noted that for an unbounded operator in Hilbert space, $\mathscr{D}(A^2) = \{0\}$ is possible even if A is densely defined. If A is an operator which is defined in Hilbert space H and can be regarded as a self-adjoint operator in a Π_k space, i.e.

$$A = JA^*J, \tag{5}$$

where J satisfies $J^2 = I$, $J^* = J$ and the dimension of the eigenspace corresponding to -1 is finite, then A^2 is self-adjoint in Π_k iff. $(A^2)^* = (A^*)^2$. But in general cases, we only know $(A^*)^2 \subset (A^2)^*$.

It should also be observed that if A is an unbounded self-adjoint operator in Π_k space which can be expressed by (2) under the standard decomposition (1), A^2 may not be defined for some vectors in Z^*. For example, let $\Pi_k = \{Z + Z^*\} \oplus P$, dim $Z = 1$, $P = L^2[0, 1]$, z and z^* be dual basis. We define an operator $A = \{S, A_P, G, Q\} = \{0, A_P, G, 0\}$ where A_P is a multiplicative operator in $L^2[0, 1]$: $(A_P f)(t) = \frac{1}{t} f(t)$, $f \in \mathscr{D}(A_P) = \left\{f(t) \,\Big|\, \int \left|\frac{1}{t} f(t)\right|^2 dt < \infty\right\}$, and G is an operator such that for any $f \in \mathscr{D}(A_P)$, $Gf = (f, f_0)z$ where $f_0 \in L^2[0, 1]$ satisfying $\int \left|\frac{1}{t} f_0(t)\right|^2 dt = \infty$. From (1), it is clear that A is a self-adjoint operator in Π_k space(1). So we have $(f, f_0) = \langle Gf, z^* \rangle =$

(1) From the fact that A_P is self-adjoint in P, we can directly prove A to be self-adjoint without using the results of [2].

(f, G^*z^*), which implies $G^*z^*=f_0$, where $\langle \cdot, \cdot \rangle$ is an inner product defined in Z (cf. [1]). This fact shows that $Az^*=G^*z^*=f_0$. Since $\int \left| \frac{1}{t} f_0(t) \right|^2 dt=\infty$, so $f_0 \bar{\in} \mathscr{D}(A_P)$. It follows that $z^* \bar{\in} \mathscr{D}(A^2)$.

Theorem 1. *If A is a self-adjoint operator in Π_k space, then for each natural number n, A^n is a self-adjoint operator.*

Proof It follows from [2], that under the standard decomposition (1) A has the form (2). We divide the proof into two steps:

(i) We first prove that for any natural number n, $\mathscr{D}(A^n)$ is dense in Π_k. Since dim $(N\oplus Z)=k<\infty$ and $A(N\oplus Z)\subset N\oplus Z$, thus $N\oplus Z\subset \mathscr{D}(A^2)$. Moreover, since $\mathscr{D}(G)\supset \mathscr{D}(A_P)$, therefore $\mathscr{D}(A_P^2)\subset \mathscr{D}(A^2)$. By [2], for $z^*\in Z^*$ and $x\in \mathscr{D}(A_P)$ we have

$$A(x+z^*)=A_Px+G^*z^*+S^*z^*+Gx-F^*z^*+Qz^*. \tag{6}$$

The last four terms of the right side of (6) belong to $\mathscr{D}(A)$. Suppose that $A_P=\int \lambda\, dE_\lambda$ is the spectral representation of A_P, write $P_0=(E_\infty-E_1)+(E_{-1}-E_{-\infty})$. Since $G^*z^*\in P$ (cf. [2]), it is clear that $(I-P_0)G^*z^*\in \mathscr{D}(A_P^2)$. If we set $x_{z^*}=-P_0A_P^{-1}G^*z^*$, then it is readily seen that $x_{z^*}\in \mathscr{D}(A_P)$, and hence $A_Px_{z^*}+G^*z^*=(I-P_0)G^*z^*$, i.e. $x_{z^*}+z^*\in \mathscr{D}(A^2)$. For each vector z_i^* of the basis $\{z_1^*, \cdots, z_l^*\}$ of Z^*, we choose a vector x_{z^*} as before. Let $Z'^*=\text{span}\ \{z_i^*+x_{z_i^*}\}\subset \mathscr{D}(A^2)$. Obviously, $N\oplus(\{Z+Z'^*\}\dotplus \mathscr{D}(A_P^2))$ is dense in Π_k, $N\oplus(\{Z+Z'^*\}\dotplus \mathscr{D}(A_P^2))\subset \mathscr{D}(A^2)$. Hence A^2 is densely defined.

We repeat the preceding process for A^3, A^4, $\cdots$. It may be proved that for any natural number n, A^n is densely defined.

(ii) Next we shall show that A^2 is self-adjoint in Π_k. Since $A=A^\dagger$, for any natural number n we have $A^n\subset (A^n)^\dagger$, i.e. $A^n$is a symmetric operator. Since A^2 is a symmetric operator, A is self-adjoint, it follows that the number of their eigenvalues in upper (lower) half plane is finite. Thus there exists a point $\rho e^{i\theta}$, $\rho>0$, $\theta\neq 0, \pi$, such that $\pm\sqrt{\rho}\, e^{\pm\frac{i\theta}{2}}$ are not eigenvalues of A and $\rho e^{\pm i\theta}$ are not eigenvalues of A^2. For $x\in \mathscr{D}(A^2)$ we have

$$(A^2-\rho e^{\pm i\theta})x=(A+\sqrt{\rho}\, e^{\pm\frac{i\theta}{2}})(A-\sqrt{\rho}\, e^{\pm\frac{i\theta}{2}})x$$

and hence $\mathscr{R}(A^2-\rho e^{\pm i\theta})=(A+\sqrt{\rho}\, e^{\pm\frac{i\theta}{2}})(A-\sqrt{\rho}\, e^{\pm\frac{i\theta}{2}})\mathscr{D}(A^2)$. To prove that A^2 is self-adjoint it will suffice to show that $U=(A^2-\rho e^{i\theta})(A^2-\rho e^{-i\theta})^{-1}$ is unitary. First we prove $\overline{\mathscr{R}(A^2-\rho e^{\pm i\theta})}=\Pi_k$.

Since $\pm\sqrt{\rho}\, e^{\frac{i\theta}{2}}$ are not eigenvalues of S and A_N, Z and $N\oplus Z$ are invariant subspaces of A, it follows at once that $(A^2-\rho e^{i\theta})Z=Z$, $(A^2-\rho e^{i\theta})(N\oplus Z)=N\oplus Z$, i.e. $N\oplus Z\subset \mathscr{R}(A^2-\rho e^{i\theta})$. For any $x\in \mathscr{D}(A_P^2)$, $z\in Z$, we have

$$(A^2-\rho e^{i\theta})(x+z)=(A_P^2-\rho e^{i\theta})x+GA_Px+SGx+(S^2-\rho e^{i\theta})z. \tag{7}$$

For a fixed x, put $z=-(S^2-\rho e^{i\theta})^{-1}(GA_Px+SGx)$. We note $\overline{\mathscr{R}(A_P^2-\rho e^{i\theta})}=P$. By (7), we have $\overline{\mathscr{R}(A^2-\rho e^{i\theta})}\supset P$. Using conclusions of (i), for any $z^*\in Z^*$ we have

$$(A^2-\rho e^{i\theta})(z^*+x_{z^*})=(S^{*2}-\rho e^{i\theta})z^*+p+n+z,$$

where $p\in P$, $n\in N$, $z\in Z$. Since $\overline{\mathscr{R}(A^2-\rho e^{i\theta})}\supset P\oplus N\oplus Z$, it follows that we can choose sequences of vectors $\{p_m\}$, $\{n_m\}$, $\{z_m\}$ such that

$$(A^2-\rho e^{i\theta})(x_{z^*}+z^*+p_m+n_m+z_m)\to(S^{*2}-\rho e^{i\theta})z^*,$$

as $n\to\infty$. By assumptions $\pm\rho e^{\frac{i\theta}{2}}\overline{\in}\sigma(S^*)$, we have $(S^{*2}-\rho e^{i\theta})Z^*\subset Z^*$, this means $Z^*\subset\overline{\mathscr{R}(A^2-\rho e^{i\theta})}$, and therefore $\overline{\mathscr{R}(A^2-\rho e^{i\theta})}=\Pi_k$.

Replacing θ by $-\theta$, we have $\overline{\mathscr{R}(A^2-\rho e^{-i\theta})}=\Pi_k$.

Using the preceding method, we can prove that A^n is self-adjoint by induction. Q.E.D.

Theorem 1 can also be provéd using the spectral representation of self-adjoint operators in Π_k. But the proof given above is more straightforward.

Theorem 2. *If A is a self-adjoint operator in Π_k space, then for all natural numbers $\{n\}$, there exists a common standard decomposition of $\{A^n\}$, $\Pi_k=N\oplus\{Z+Z^*\}\oplus P$ such that A^n $(n\geqslant 2)$ can be obtained by operational calculus from form (2) of A, moreover*

$$A^n=\Big\{S^n,\ A_N^n,\ A_P^n,\ \sum_{i=0}^{n-1}S^iFA_N^{n-1-i},\ \sum_{i=0}^{n-1}S^iGA_P^{n-1-i},$$
$$\sum_{i=0}^{n-1}S^iQS^{*n-1-i}-\sum_{i+j+k=n-2}S^i(FA_N^jF^*+GA_P^jG^*)S^{*k}\Big\}. \qquad (8)$$

Proof Put $U_\pm=(A\pm\sqrt{\rho}\,e^{\frac{i\theta}{2}})(A\pm\sqrt{\rho}\,e^{\frac{i\theta}{2}})^{-1}$, $U=(A^2-\rho e^{i\theta})(A^2-\rho e^{-i\theta})^{-1}$. It is evident that $U_\pm$ and U are unitary operators in Π_k. Since

$$U_+U_-=(A+\sqrt{\rho}\,e^{\frac{i\theta}{2}})(A+\sqrt{\rho}\,e^{-\frac{i\theta}{2}})^{-1}(A-\sqrt{\rho}\,e^{\frac{i\theta}{2}})(A-\sqrt{\rho}\,e^{-\frac{i\theta}{2}})^{-1}$$
$$=(A+\sqrt{\rho}\,e^{\frac{i\theta}{2}})(A-\sqrt{\rho}\,e^{\frac{i\theta}{2}})(A+\sqrt{\rho}\,e^{-\frac{i\theta}{2}})^{-1}(A-\sqrt{\rho}\,e^{-\frac{i\theta}{2}})^{-1}=U$$

and similarly $U_-U_+=U$, it follows that U_+, U_- and U are commutative, hence the Cayley transformations of A and A^2 are commutative.

Similarly, by induction we can prove that Cayley transformations of all A^n are commutative. So there exists a common standard decomposition of all A^n $(n=1, 2, \cdots)$ (to do this, it will suffice to combine the results of [4] with [2]), $\Pi_k=N\oplus\{Z+Z^*\}\oplus P$. Suppose that under this decomposition, $A=\{S, A_N, A_P, F, G, Q\}$. In particular, we note $Z^*\subset\mathscr{D}(A^n)$, $n=1, 2, \cdots$. Thus, by induction we obtain (8). Q.E.D.

Corollary 1. *If A is a self-adjoint operator in Π_k, then for any n,*

$$\sigma(A^n)=\{\lambda^n|\lambda\in\sigma(A)\}. \qquad (9)$$

Proof According to the conclusions of [2], we have

$$\sigma(A)=\sigma(S)\cup\sigma(S^*)\cup\sigma(A_N)\cup\sigma(A_P),$$
$$\sigma(A^n)=\sigma(S^n)\cup\sigma(S^{*n})\cup\sigma(A_N^n)\cup\sigma(A_P^n).$$

So (9) holds. Q.E.D.

Corollary 2. *If A is an unbounded self-adjoint operator in Π_k, then for any natural number n, A^n must be unbounded.*

Proof As A is unbounded, $\sigma(A_P)$ is an unbounded set and hence $\sigma(A_P^n)$ is unbounded. From [2], A^n must be an unbounded operator. Q.E.D.

Corollary 3. *If A is a quasi-nilpotent self-adjoint operator in Π_k, then A must be nilpotent.*

By (8), the proof is immediate.

In [7], corollary 3 was also proved by a different method.

2. The n-th roots of self-adjoint operators in Π_k space

Suppose that A and A_1 are operators in Π_k, A_1 is said to be a n-th root of A if $A_1^n = A$ holds. Evidently, for any fixed self-adjoint operator A, it is possible that its n-th root does not exist. If it exists, it need not be unique.

Theorem 3. *If A is a self-adjoint operator in Π_k, $\sigma(A)\subset[0, +\infty)$, $0\bar{\in}\sigma_P(A)$, then there exists a unique self-adjoint operator A_1 such that*

$$A_1^n = A, \quad \sigma(A_1)\subset[0, +\infty).$$

Proof Denote the spectral family of A by E_t. Choose $M>0$ such that $[0, M)\supset C(A)$. Decompose Π_k into $\Pi_k = E[0, M)\Pi_k \oplus E[M, \infty)\Pi_k$. We first note that $E[0, M)\Pi_k$ is a Π_k space, the restriction of A to it is a bounded self-adjoint operator, $E[M, \infty)\Pi_k$ is an ordinary Hilbert space and the restriction of A to it is self-adjoint. So we can discuss the n-th roots of A in these two subspaces respectively. Since, for any positive operator in Hilbert space, there exists the unique n-th positive root, hence it will suffice to discuss the reduced part of A on $E[0, M)\Pi_k$. This observation enables us to assume that A is a bounded self-adjoint operator in Π_k.

Suppose we have the standard decomposition (1), $A=\{S, A_N, A_P, F, G, Q\}$. We shall prove, under the same decomposition, that there exists an operator $A_1=\{S, A_{1N}, A_{1P}, F_1, G_1, Q_1\}$ satisfying $A_1^n = A$ and $\sigma(A_1)\subset[0, \infty)$. By (8), $A_1^n = A$ is equivalent to the system of equations

$$\begin{cases} S_1^n = S, \ A_{1N}^n = A_N, \ A_{1P}^n = A_P; \\ \sum_{i=0}^{n-1} S_1^i F_1 A_{1N}^{n-1-i} = F, \ \sum_{i=0}^{n-1} S_1^i G_1 A_{1P}^{n-1-i} = G; \\ \sum_{i=0}^{n-1} S_1^i Q_1 S_1^{*n-1-i} = Q - \sum_{i+j+k=n-2} S_1^i (F_1 A_{1N}^j F_1^* + G_1 A_{1P}^j G_1^*) S_1^{*k}. \end{cases} \tag{11}$$

Since S is a linear operator from finite-dimensional space Z to Z, $\sigma(S)\subset\sigma_P(A)\subset(0, \infty)$, it follows that we can choose one of its n-th roots which only has positive eigenvalues. Denote this root by S_1. For positive operators A_N and A_P in Hilbert spaces

N and P, there exist the n-th positive roots respectively. Denote them by A_{1N}, A_{1P}. Now, applying S_1, A_{1N}, A_{1P} given before, we shall prove that there exist F_1, G_1, Q_1 satisfying the last three equations of (11).

First, we choose vectors $\{z_i\}_{i=1}^n$, one of the bases of Z, such that S_1 has the Jordan canonical form. For F: $N\to Z$, we can find $y_1, \cdots, y_k\in N$ such that $F_n=\sum_{i=1}^k(n, y_i)z_i$, $\forall n\in N$. Assume $F_1n=\sum_{i=1}^k(n, x_i)z_i$. From the fourth equation of (11), we have

$$\sum_{i=1}^k\sum_{j=0}^{n-1}(A_{1N}^{n-j-1}n, x_i)S_1^jz_i=\sum_{i=1}^k(n, y_i)z_i.$$

For convenience, we introduce an operator S_1^*: $S_1^*x_i=\alpha_{ii}x_i+\alpha_{i+1,i}x_{i+1}$, where (α_{ij}) is the adjoint matrix of the Jordan canonical form of S_1. From direct calculation we have

$$\sum_{i=1}^k\left(n, \sum_{j=0}^{n-1}A_{1N}^{n-1-j}S_1^{*j}x_i\right)z_i=\sum_{i=1}^k(n, y_i)z_i,$$

This means

$$\sum_{j=0}^{n-1}A_{1N}^{n-1-j}S_1^{*j}x_i=y_i \quad (i=1, \cdots, k).$$

According to its Jordan canonical form, this system of equations can be divided into several sub-systems which contain less equations and are independent of each other. Take one system of them, assume that it corresponds to the eigenvalue η of S_1, denote the element of the highest order by x_i. From the existence of $\left(\sum_{j=0}^{n-1}A_{1N}^{n-1-j}\eta^j\right)^{-1}$ we have

$$x_i=\left(\sum_{j=0}^{n-1}A_{1N}^{n-1-j}\eta^j\right)^{-1}y_i.$$

The other $x_{i'}$ in this system of equations can be solved by induction

$$x_{i'}=\left(\sum_{j=0}^{n-1}A_{1N}^{n-1-j}\eta^j\right)^{-1}\left(y_{i'}-\sum_{j=0}^{n-1}A_{1N}^{n-1-j}\sum_{l=0}^{i'+j-i-1}C_j^l\eta^l x_{i'+j-l}\right)$$

here we put $\sum_{l=0}^{i'+j-i-1}=0$ when $i'+j-i-1<0$. This verifies that there exists a solution F_1 satisfying the fourth equation of (11). Similarly there exists G_1 satisfying the fifth equation of (11). Now the right side of the last equation of (11) is given. Represent S_1 in the left side by its Jordan canonical form. Denote the l-th element of the main diagonal of S_1 by η_l, write $Q_1=(q_{lm})$. We have

$$\sum_{i=0}^{n-1}S_1^iQS_1^{*n-1-i}=((i\eta_l^i+(n-1-i)\eta_m^{n-1-i})q_{lm}+[q]_{lm}),$$

where $[q]_{lm}=\sum_{k>l}a_{km}q_{km}+\sum_{k>m}a_{lk}q_{lk}$. If we regard $(q_{11}, \cdots, q_{1n}, q_{21}, \cdots, q_{nn})$ as variables, then the corresponding coefficient matrix is an upper triangular one and the elements of the main diagonal are equal to $i\eta_l^i+(n-1-i)\eta_m^{n-i-1}\neq 0$. This shows Q_1 can be determined uniquely.

It is easy to see that $A_1=\{S_1, A_{1N}, A_{1P}, F_1, G_1, Q_1\}$ satisfies $A_1^n=A$ and $\sigma(A_1)\subset[0, \infty)$.

Using the same argument as $n=2$[3], we can prove that A_1 satisfying the assertion of the present theorem is unique. We omit this process. Q.E.D.

3. Critical points of self-adjoint operators in Π_k space

The essential difference between self-adjoint operators in Π_k and those in Hilbert space is the presence of critical points. So in the study of spectral families in indefinite inner product space, one of the most important task is to discuss the structures of the spectral families in the neighborhood of the critical points. Only when these are clear, is it possible to establish the operational calculus which possesses some good properties.

Choose a basis of Z, $\{z_1, \cdots, z_n\}$, under which S has the Jordan canonical form. Then we can take a system of vectors $\{y_1, \cdots, y_n\}\subset P$ uniquely such that $Gp=\sum_{i=1}^{n}(p, y_i)z_i$, $\forall p\in P$. Suppose that the Jordan blocks of S corresponding to λ_0 are the first k_0 blocks. The ranks of them are $n_1, \cdots, n_{k_0}$ respectively. Put

$$x_i=\sum_{j=1}^{n_i}(\lambda_0-A)^{n_i-j}y_{\sum_{k=1}^{i-1}n_k+j}\quad(1\leqslant i\leqslant k_0).$$

Define set functions μ^i by x_i:

$$\mu^i(\Delta)=\int_{\Delta\setminus\{\lambda_0\}}\frac{d(P_t^{A_P}x_i, x_i)}{(\lambda_0-t)^{2n_i}}\quad(1\leqslant i\leqslant k_0),$$

where Δ is some Borel set in real line. In general, they are σ-finite measures. Under these notations, we have.

Theorem 4. *If A is a self-adjoint operator in Π_k, $C(A)\cap(\mu_1, \nu_1)=\{\lambda_1\}$, then the following four assertions are equivalence:*

(i) $\sup_{\mu,\nu}\{\|E_{\mu\nu}\|\,|\,\lambda_0\in(\mu, \nu)\subset(\mu_1, \nu_1)\}<\infty$, *where* $E_{\mu\nu}=E([\mu, \nu))$;

(ii) $\mu^{(1)}, \cdots, \mu^{(k_0)}$ *are finite measures;*

(iii) $s\text{-}\lim_{\mu,\nu\to\lambda_0}E_{\mu\nu}$ *exists;*

(iv) *the radical subspace Φ_{λ_0} which corresponds to λ_0 is non-degenerate. When these assertions hold, we have $\Phi_{\lambda_0}=E\Pi_k$ where $E=s\text{-}\lim E_{\mu\nu}$.*

Proof We first note that $E_{\mu\nu}=E_{\mu\nu}E_{\mu_1\nu_1}$ holds when $[\mu, \nu)\subset[\mu_1, \nu_1)$. $E_{\mu_1\nu_1}\Pi_k$ is a Π_k-space. As we only consider properties of $E_{\mu\nu}$ in the neighborhood of λ_0, without loss of generality, we may assume there exists only one critical point λ_0 of A, and A is bounded. Thus when $\lambda_0\in(\mu, \nu)$, $P_{\mu\nu}^{A_N}=I_N$, i.e. $P_{\mu\nu}^{A_{\bar{P}}}=0$. Suppose that the highest order of the Jordan blocks of S corresponding to λ_0 is n. By (4), we have

$$E_{\mu\nu}=\{P_0,\ I_N,\ P_{\mu\nu}^{A_P},\ 0,\ -\sum_{j=1}^{n}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P},$$
$$-\sum_{j,k=1}^{n}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-(j+k)}P_{\mu\nu}^{A_P}G^*(\lambda_0-S^*)^{k-1}\}.\qquad(12)$$

Prove (i)$\Leftrightarrow$(ii). Write $P_{\mu\nu}(G, A_P)=\sum_{j=1}^{n}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}$. For convenience, we assume $k_0=1$,

$$P_{\mu\nu}(G,\ A_P)p=\sum_{i,j=1}^{n}(p,\ (\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_i)(\lambda_0-S)^{j-1}z_i$$
$$=\sum_{i,j}(p,\ (\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_i)z_{i-j+1}$$
$$=\sum_{i=1}^{n}\Big(p,\ \sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{i+j-1}\Big)z_i,\quad \forall p\in P. \tag{13}$$

It is easy to see that the last term on the right side of (12) is $-P_{\mu\nu}(G,\ A_P)\times[P_{\mu\nu}(G,\ A_P)]^*$. Therefore the uniform boundedness of $E_{\mu\nu}$ is equivalent to the uniform boundedness of $P_{\mu\nu}(G,\ A_P)$, from (13), this is also equivalent to the boundedness of $\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{i-j+1}$ for any μ, ν, where $i=1,\ \cdots,\ n$. Now we are to prove that this is equivalent to the boundedness of $\sum_{j=1}^{n}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_j$ for any μ, ν. In fact, since $\sum_{j=1}^{n}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_j$ is bounded, then using the identity

$$\sum_{j=1}^{n-1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{j+1}=(\lambda_0-A_P)\sum_{j=1}^{n}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_j-P_{\mu\nu}^{A_P}y_1,$$

it follows that $\sum_{j=1}^{n-1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{j+1}$ is bounded. By induction we have that $\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{i+j-1}(i=1,\ \cdots,\ n)$ are bounded. But $\sum_{j=1}^{n}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_j$ is bounded iff $\int_{[\mu,\nu)^c}d\mu_t$ is bounded where $d\mu_t=\dfrac{d(P_t^{A_P}x,\ x)}{(\lambda_0-t)^{2n}}$, $x=\sum_{j=1}^{n}(\lambda_0-A_P)^{n-j}y_j$. Hence (i)$\Leftrightarrow$(ii).

Next we prove (ii)$\Rightarrow$(iii). In order to prove the existence of s-$\lim E_{\mu\nu}$, from the representation (12) of $E_{\mu\nu}$, it suffices to prove for all $p\in P$, $z^*\in Z^*$, $\lim P_{\mu\nu}(G,\ A_P)p$ and $\lim P_{\mu\nu}(G,\ A_P)[P_{\mu\nu}(G,\ A_P)]z^*$ exist.Let $\{z_i^*\}$ be the dual basis of $\{z_i\}$ in Z^*, then $G^*z_i^*=y_i$. By calculation, we have

$$P_{\mu\nu}(G,\ A_P)p=\sum_{i=1}^{n}\Big(p,\ \sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{i+j-1}\Big)z,$$

$$P_{\mu\nu}(G,\ A_P)[P_{\mu\nu}(G,\ A_P)]^*z^*$$
$$=\sum_{j,k=1}^{n}\sum_{i=1}^{n}((\lambda_0-A_P)^{-(j+k)}P_{\mu\nu}^{A_P}G^*(\lambda_0-S^*)^{k-1}z^*,\ y_i)(\lambda_0-S)^{j-1}z_i$$
$$=\sum_{i=1}^{n}\Big(z^*,\ \sum_{k=1}^{n}\sum_{j=1}^{n-i+1}(\lambda_0-S)^{k-1}G(\lambda_0-A_P)^{-(j+k)}P_{\mu\nu}^{A_P}y_{i+j-1}\Big)z_i,$$

$$\sum_{k=1}^{n}\sum_{j=1}^{n-i+1}(\lambda_0-S)^{k-1}G(\lambda_0-A_P)^{-(j+k)}P_{\mu\nu}^{A_P}y_{i+j-1}$$
$$=\sum_{m=1}^{n}\Big(\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{i+j-1},\ \sum_{k=1}^{n-m+1}(\lambda_0-A_P)^{-k}P_{\mu\nu}^{A_P}y_{m+k-1}\Big)z_m.$$

Write $a_{\mu\nu}^{(i)}=\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P_{\mu\nu}^{A_P}y_{i+j-1}$, $1\leqslant i\leqslant n$. It is enough to prove the existence of $\lim a_{\mu\nu}^{(i)}$. Take a regular decomposition $\Pi_k=H_-\oplus H_+$ such that $H_+\supset P$, $H_-\supset N$. According to this decomposition we introduce a new inner product $[\cdot,\ \cdot]_1$ in Π_k which agrees with $(\cdot,\ \cdot)$ in P. For $[\mu_1,\ \nu_1)\subset(\mu,\ \nu)$, we have

$$\|a^{(i)}_{\mu_1\nu_1}\|_1^2-\|a^{(i)}_{\mu\nu}\|_1^2=\left\|\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P^{A_P}_{\mu_1\nu_1}y_{i+j-1}\right\|_1^2-\left\|\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}P^{A_P}_{\mu\nu}y_{i+j-1}\right\|_1^2$$
$$=\left\|\sum_{j=1}^{n-i+1}(\lambda_0-A_P)^{-j}(P^{A_P}_{\mu_1\nu_1}-P^{A_P}_{\mu\nu})y_{i+j-1}\right\|_1^2$$
$$=\|a^{(i)}_{\mu_1\nu_1}-a^{(i)}_{\mu\nu}\|_1^2\geqslant 0.$$

So $\|a^{(i)}_{\mu\nu}\|_1$ increases as $\nu-\mu$ decrease. By (ii), we have $\sup\limits_{\mu,\nu}\|a^{(i)}_{\mu\nu}\|<\infty$ and hence $\lim\|a^{(i)}_{\mu\nu}\|$ exists. Thus $\lim\|a^{(i)}_{\mu\nu}-a^{(i)}_{\mu_1\nu_1}\|=0$. It follows that $\lim a^{(i)}_{\mu\nu}$ exists, which implies that s-$\lim E_{\mu\nu}$ exists. This verifies (iii).

When (iii) holds, by Banach-Steinhaus Theorem, (i) follows.

Let us prove (iv)$\Rightarrow$(i). Suppose that the radical subspace Φ_{λ_0} corresponding to λ_0 is non-degenerate. Write $H=\Pi_k\ominus\Phi_{\lambda_0}$. With the inner product $(\cdot,\ \cdot)$, H is a Hilbert space. Under the decomposition $\Pi_k=H\oplus\Phi_{\lambda_0}$, we have

$$A=A^{(1)}\oplus A^{(2)},\quad E_{\mu\nu}=E^{(1)}_{\mu\nu}\oplus E^{(2)}_{\mu\nu}.$$

Since $A^{(2)}$ is a self-adjoint operator in Π_k, so there is a standard decomposition $\Phi_{\lambda_0}=N\oplus\{Z+Z^*\}\oplus P_1$. Write $P=P_1\oplus H$, we have $\Pi_k=N\oplus\{Z+Z^*\}\oplus P$. As before, let $\Pi_k=H_-\oplus H_+$ such that $H_-\supset N$ and $H_+\supset P$. Denote the inner product corresponding to this decomposition by $[\cdot,\ \cdot]_1$. Denote the projective operator from P to H by P_H. Since $A|H=A_P|H$, therefore $E^{(1)}_{\mu\nu}=P^{A_P}_{\mu\nu}P_H$. Suppose that $[\mu,\ \nu)\subset(\mu_1,\ \nu_1)$. Because $E_{\alpha\beta}\perp\Phi_{\lambda_0}$ when $\lambda_0\bar{\in}[\alpha,\ \beta)$, we have $E^{(2)}_{\alpha\beta}=0$. For $x\in\Pi_k$,

$$E_{\mu_1\nu_1}x-E_{\mu\nu}x=E_{\mu_1\mu}x+E_{\nu\nu_1}x=\{0,\ 0,\ (P^{A_P}_{\mu_1\mu}+P^{A_P}_{\nu\nu_1})P_H,\ 0,\ 0,\ 0\}x.$$

In virtue of the definition, it is easy to see $[\Phi_{\lambda_0},\ H]=\{0\}$, $E_{\mu\nu}x=E^{(2)}_{\mu\nu}x+P^{A_P}_{\mu\nu}P_Hx$. Hence

$$[E_{\mu\nu}x,\ E_{\mu_1\nu_1}x-E_{\mu\nu}x]_1=[P^{A_P}_{\mu\nu}P_Hx,\ E^{(1)}_{\mu_1\mu}x+E^{(1)}_{\nu\nu_1}x]$$
$$=(P^{A_P}_{\mu\nu}P_Hx,\ E^{(1)}_{\mu_1\mu}x+E^{(1)}_{\nu\nu_1}x)=0.$$

Consequently

$$\|E_{\mu_1\nu_1}x\|_1^2-\|E_{\mu\nu}x\|_1^2=\|E_{\mu_1\nu_1}x-E_{\mu\nu}x\|_1^2\geqslant 0. \tag{14}$$

Thus $\|E_{\mu\nu}x\|_1$ decreases as $\nu-\mu$ decreases, which means $\|E_{\mu\nu}\|_1$ is uniformly bounded. It follows that (i) holds.

Finally, we prove (iii)$\Rightarrow$(iv). When $E=s$-$\lim E_{\mu\nu}$ exists, it is easy to see that E is a projective operator. Thus $E\Pi_k$ is non-degenerate, and $E\Pi_k=\bigcap E_{\mu\nu}\Pi_k$ holds. Since $\Phi_{\lambda_0}\subset E_{\mu\nu}\Pi_k$ and $AE_{\mu\nu}\Pi_k\subset E_{\mu\nu}\Pi_k$, therefore $E\Pi_k$ is a Π_k-space and $AE\Pi_k\subset E\Pi_k$. Thus $\sigma(A|E\Pi_k)\subset\sigma(A|E_{\mu\nu}\Pi_k)\subset[\mu,\ \nu)$. So $\sigma(A|E\Pi_k)=\{\lambda_0\}$ due to the arbitrariness of μ and ν. Write $A_e=A|E\Pi_k$. By what has been shown, it is seen that A_e is a self-adjoint operator of the Π_k-space $E\Pi_k$ and $\sigma(A_e)=\{\lambda_0\}$. By [7], we have $(A_e-\lambda_0)^{2k+1}=0$. Therefore $(A-\lambda_0)^{2k+1}E=0$, which implies

$$\Phi_{\lambda_0}=E\Pi_k.$$

(iv) has been shown. Q.E.D.

Corollary. *The four assertions of the theorem are equivalent to*

(v) $\sup_{\mu,\nu}|(E_{\mu\nu}x, y)|<\infty, \quad \forall x, y\in\Pi_k.$

Remark. In the proof of (iv)⇒(i), according to the decomposition $\Pi_k=H\oplus\Phi_{\lambda_0}$, we have

$$A=A^{(1)}\oplus A^{(2)}=(\lambda_0 I_{\Phi_{\lambda_0}}\oplus A^{(1)})+((A^{(2)}-\lambda_0 I_{\Phi_{\lambda_0}})\oplus 0)$$

so that A is a spectral operator, thus $\|E_{\mu\nu}\|$ is uniformly bounded and $S-\lim E_{\mu\nu}$ exists. But the proof given above is direct. By (14), we can also see that $\lim\|E_{\mu\nu}x\|_1$ exists and that $\lim\|E_{\mu_1\nu_1}x-E_{\mu\nu}x\|=0$. So that the existence of s–$\lim E_{\mu\nu}$ is verified.

In general, $E_{\mu\nu}$ need not be uniformly bounded. So we should discuss the orders of the singularity associated with the critical points of $E_{\mu\nu}$. For $\lambda_0\in C(A)$, the order of the singularity of λ_0 is the smallest integer n satisfying $\overline{\lim_{\varepsilon\to 0}}\|\varepsilon^n E[\lambda_0-\varepsilon, \lambda_0+\varepsilon)\|<\infty$.

Suppose that $\lambda_0\in C(A)$. Denote the highest rank of the Jordan blocks of S corresponding to λ_0 by $r(\lambda_0)$.

Theorem 5. *Suppose that A is a self-adjont operator in Π_k, its spectral family is E_t, $\lambda_0\in C(A)$, $r(\lambda_0)=n$. Under the standard decomposition* (1), *A has the form* (2). *Then*

(i) *At λ_0, the order of the singularity of $E_{\mu\nu}$ does not exceed $2n$;*

(ii) $s\text{-}\lim_{\varepsilon\to 0}\int_{[M_1, \lambda_0-\varepsilon)}(t-\lambda_0)^{2n}dE_t$ *and* $s\text{-}\lim_{\varepsilon\to 0}\int_{[\lambda_0+\varepsilon, M_2)}(t-\lambda_0)^{2n}dE_t$ *exist, where M_1 and M_2 satisfies $[M_1, M_2]\cap C(A)=\{\lambda_0\}$.*

Proof With no loss of generality, it may be assumed that $C(A)=\{\lambda_0\}$, S has only one Jordan block and $A_P=\int t\,dP_t^{A_P}$.

In view of (12), it is easy to see that the singularity of $E_{\mu\nu}$ is determined by the last two components of the right side of (12). The orders of their singularity at λ_0 do not exceed $2n$. This shows (i).

For a fixed Δ, when $\lambda_0\overline{\in}\Delta$, we have

$$E(\Delta)=\Big\{0,\ 0,\ P_\Delta^{A_P},\ 0,\ \sum_{j=1}^{n}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-j}P_\Delta^{A_P},$$
$$\sum_{j,k=1}^{n}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-(j+k)}P_\Delta^{A_P}G^*(\lambda_0-S^*)^{k-1}\Big\} \tag{18}$$

When $1\leqslant i\leqslant 2n$, $s\text{-}\lim\int_{[M_1, \lambda_0-\varepsilon)\cup[\lambda_0+\varepsilon, M_2)}(t-\lambda_0)^{2n}(\lambda_0-A_P)^{-i}P^{A_P}(dt)$ exists, so that (ii) is verified. Q.E.D.

Based on Th. 5, we can define

$$\int_{M_1}^{M_2}(t-\lambda_0)^{2n}dE_t=\lim_{\varepsilon_1,\varepsilon_2\to 0}\Big[\int_{[M_1, \lambda_0-\varepsilon)}+\int_{[\lambda_0+\varepsilon, M_2)}\Big](t-\lambda_0)^{2n}dE_t. \tag{19}$$

However, for natural numbers $k\geqslant 1$, we may define

$$\int_{M_1}^{M_2}(t-\lambda_0)^{2n+k}dE_t \quad \text{by} \quad s\text{-}\lim_{|\Delta|\to 0}\sum(t_\nu-\lambda_0)^{2n+k}E(\Delta_\nu)$$

directly, because $(t_{\nu_m}^{(m)}-\lambda_0)^{2n+k}E(\Delta_{\nu_m}^{(m)})\to 0$ $(t_{\nu_m}^{(m)}\in\Delta_{\nu_m}^{(m)},\ |\Delta^{(m)}|\to 0)$ where $\lambda_0\in\Delta_{\nu_m}^{(m)}$.

Moreover, if λ_0 is a zero point of continuous function $f(t)$ and there exist a neighborhood U of λ_0, a constant M, for $t\in U$, $|f(t)|\leqslant M|t-\lambda_0|^{2n}$ holds, then we may define $\int_{M_1}^{M_2} f(t)dE_t$ in a similar fashion.

Remark. By (12), it is easy to see that if A_P and G are given suitably then at λ_0 the order of the singularity of $E_{\mu\nu}$ corresponding to $A=\{S, A_N, A_P, F, G, Q\}$ is $2n$.

4. The spectral representation of the resolvent of a self-adjoint operator in Π_k

Using the spectral family (4) of the self-adjoint operator in Π_k, we can express the resolvent in an explicit form. This is

Theorem 6. *Suppose that A is a self-adjoint operator in Π_k $A=\{S, A_N, A_P, F, G, Q\}$, its spectral family is E_t, the set of the critical points $C(A)=\{\lambda_1, \cdots, \lambda_l, \lambda_{l+1}, \bar\lambda_{l+1}, \cdots, \lambda_{l+p}, \bar\lambda_{l+p}\}$, where $\operatorname{Im}\lambda_\nu=0(1\leqslant\nu\leqslant l)$, $r(\lambda_\nu)=n_\nu$. Then*

$$(\lambda-A)^{-1}=\int_{-\infty}^{\infty} K(\lambda, t)dE_t+\sum_{\nu=1}^{l}\sum_{i=1}^{2n_\nu+1}\frac{B_{\nu}}{(\lambda-\lambda_\nu)^i}+\sum_{\nu=l+1}^{l+p}\sum_{i=1}^{n_\nu}\left[\frac{B_{\nu i}}{(\lambda-\lambda_\nu)^i}+\frac{B_{\nu i}^{\dagger}}{(\lambda-\bar\lambda_\nu)^i}\right], \tag{20}$$

where $K(\lambda, t)=\dfrac{1}{\lambda-t}-\sum_{\nu=1}^{l}\delta(t-\lambda_\nu)\sum_{i=1}^{2n_\nu}\dfrac{(t-\lambda_\nu)^{i-1}}{(\lambda-\lambda_\nu)^i}$, $\delta(\lambda)=\begin{cases}1, & |\lambda|<\delta,\\ 0, & |\lambda|\geqslant\delta,\end{cases}$

$0<\delta<\min\limits_{\substack{1\leqslant\mu,\nu\leqslant l\\ \lambda_\mu\neq\lambda_\nu}}|\lambda_\mu-\lambda_\nu|$. *For $1\leqslant\nu\leqslant l$, $B_{\nu i}$ are bounded self-adjoint operators in Π_k and for $l+1\leqslant\nu\leqslant l+p$, $B_{\nu i}=(\lambda_\nu-S)^{i-1}P_{\lambda_\nu}$ where P_{λ_ν}, are some parallel projections whose ranges are the radical subspaces corresponding to λ_ν.*

Proof (1) Suppose that $l+1\leqslant\nu\leqslant l+p$, denote the radical subspaces corresponding to λ_ν, $\bar\lambda_\nu$ respectively by Φ_{λ_ν}, $\Phi_{\bar\lambda_\nu}$. The finite-dimensional spaces $H_\nu=\Phi_{\lambda_\nu}+\Phi_{\bar\lambda_\nu}$ are non-degenerate and $AH_\nu\subset H_\nu$. Denote the projection from Π_k to H_ν by P_{H_ν}. We have

$$(\lambda-A)^{-1}P_{H_\nu}=(\lambda-S)^{-1}P_{\lambda_\nu}+(\lambda-S^*)^{-1}P_{\bar\lambda_\nu}$$
$$=\sum_{i=1}^{n_\nu}\frac{(S-\lambda_\nu)^{i-1}P_{\lambda_\nu}}{(\lambda-\lambda_\nu)^i}+\sum_{i=1}^{n_\nu}\frac{(S^*-\bar\lambda_\nu)^{i-1}P_{\bar\lambda_\nu}}{(\lambda-\bar\lambda_\nu)^i}.$$

Thus, in view of $P_{\lambda_\nu}^*=P_{\bar\lambda_\nu}$, we have found the final sum of the right side of (20).

Now, neglecting the radical subspace corresponding to the complex critical points, it may be assumed that there exist only real critical points.

(2) Take $\Delta=(-M, M)\supset\{\lambda_1, \cdots, \lambda_l\}$. We have the decomposition $\Pi_k=E(\Delta)\Pi_k\oplus E(\Delta^c)\Pi_k$. Since $A|E(\Delta^c)\Pi_k$ is a self-adjoint operator in Hilbert space $E(\Delta^c)\Pi_k$, so we have $(\lambda-A)^{-1}E(\Delta^c)=\int_{\Delta^c}\dfrac{dE_t}{\lambda-t}$, for $\lambda\in\rho(A)\subset\rho(A|E(\Delta^c)\Pi_k)$. Thus, it will be sufficient to prove the theorem in $E(\Delta)\Pi_k$. Since $\sigma(A|E(\Delta)\Pi_k)=\sigma(A)\cap\bar\Delta$, by [2], $A|E(\Delta)\Pi_k$ is bounded. Similarly, by decomposing the space further, it may

be assumed that there exists only one critical point, we denote it by λ_0. For convenience, we suppose S has only one Jordan block whose rank is n.

(3) Write $K(\lambda, t)=\frac{1}{\lambda-t}-\sum_{i=1}^{2n}\frac{(t-\lambda_0)^{i-1}}{(\lambda-\lambda_0)^i}$, define an operator-valued function $F(\lambda)$ in the complex plane as follows:

$$F(\lambda)=\begin{cases}(\lambda-A)^{-1}-\int_{-\infty}^{\infty}K(\lambda, t)dE_t, \quad \lambda\in\rho(A), & (21.1)\\ \int_{\Delta_\lambda}\sum_{i=1}^{2n}\frac{(t-\lambda_0)^{i-1}}{(\lambda-\lambda_0)^i}dE_t+(\lambda-A)^{-1}E(\Delta_\lambda^c) & \\ \quad -\int_{\Delta_\lambda^c}K(\lambda, t)dE_t, \quad \lambda\in\sigma(A)\setminus\{\lambda_0\}, & (21.2)\end{cases}$$

where $\Delta_\lambda=[\mu, \nu)$ satisfies $\lambda\in\Delta_\lambda$, $\lambda_0\bar{\in}\bar{\Delta}_\lambda$, $\Delta_\lambda^c=(-\infty, \infty)\setminus\Delta_\lambda$.

By theorem 5 and its remark, the integrals in (21.1) and (21.2) make sense. The value of $F(\lambda)$ is independent of the choise of Δ_λ. Indeed, suppose that $\Delta_\lambda^1\supset\Delta_\lambda^2$, from (21.2) we have $F_1(\lambda)$ and $F_2(\lambda)$ corresponding to Δ_λ^1 and Δ_λ^2 respectively. By simple calculation we have

$$F_2(\lambda)-F_1(\lambda)=(\lambda-A)^{-1}E(\Delta_\lambda^1\setminus\Delta_\lambda^2)-\int_{\Delta_\lambda^1\setminus\Delta_\lambda^2}\frac{dE_t}{\lambda-t}=0.$$

From the definition, it is clear that $F(\lambda)$ is well defined in complex plane except $\lambda=\lambda_0$. For $\lambda\in\rho(A)$, $F(\lambda)$ is analytic. We note when $\lambda\bar{\in}\Delta=[\mu, \nu)$, $A|E[\mu, \nu)\Pi_k$ is a self-adjoint operator in Hilbert space, whose spectral family is $E[\mu, \nu)E_t$. Thus, for $\lambda\in\sigma(A)\setminus\{\lambda_0\}$, we can choose a neighborhood U of λ and an interval Δ which contains λ, such that for $\lambda\in U$, $F(\lambda)$ has the unified form (21.2) where $\Delta_\lambda=\Delta$. It is then evident that every term of the right side of (21.2) is analytic. Thus $F(\lambda)$ is analytic at λ. By (21.2), we have $\lim_{\lambda\to\infty}\|F(\lambda)\|=0$.

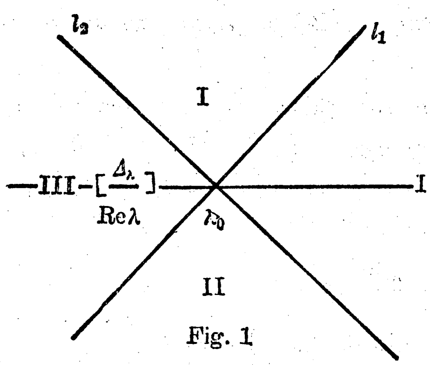

Fig. 1

(4) Now, let us examine the order of the singularity of $F(\lambda)$ at $\lambda=\lambda_0$. Take two straight lines passing through λ_0, whose slopes are $K_{1,2}=\pm1$ respectively. They divide the whole plane into four parts I, II, III, IV (Fig. 1)

(i) $\lambda\in$ I $\cup$ II,

In order to estimate $\left\|\int K(\lambda, t)dE_t\right\|=\left\|\int\frac{(t-\lambda_0)^{2n}dE_t}{(\lambda-\lambda_0)^{2n}(\lambda-t)}\right\|$, by the definition of these integrals and (18), it suffices to estimate

$$\left\|\frac{1}{(\lambda-\lambda_0)^{2n}}\int_{[\lambda_0+0,\infty)}\frac{(t-\lambda_0)^{2n}(A_P-\lambda_0)^{-k}}{\lambda-t}dP_t^{A_P}\right\|,$$

$$\left\|\frac{1}{(\lambda-\lambda_0)^{2n}}\int_{(-\infty,\lambda_0-0)}\frac{(t-\lambda_0)^{2n}(A_P-\lambda_0)^{-k}}{\lambda-t}dP_t^{A_P}\right\|, \quad 1\leqslant k\leqslant 2n,$$

Since we have $\sqrt{2}|\lambda-t|\geqslant|\lambda-\lambda_0|$ when $t\in(-\infty, \infty)$, so that

$$\left\| \frac{1}{(\lambda-\lambda_0)^{2n}} \int_{[\lambda_0+0,\infty)} \frac{(t-\lambda_0)^{2n}(A_P-\lambda_0)^{-k}}{\lambda-t} dP_t^{A_P} p \right\|^2$$

$$\leqslant \frac{M_1^2}{(\lambda-\lambda_0)^{4n}} \left\| (A_P-\lambda_0)^{-k} \int \frac{(t-\lambda_0)^{2n} dP_t^{A_P} p}{\lambda-t} \right\|_1^2$$

$$= \frac{M_1^2}{(\lambda-\lambda_0)^{4n}} \int \frac{|t-\lambda_0|^{2n-k}}{|\lambda-t|^2} d\|P_t^{A_P}p\|_1^2 \leqslant \frac{2M_1^2}{(\lambda-\lambda_0)^{4n+2}} \int |t-\lambda_0|^{2n-k} d\|P_t^{A_P}p\|_1^2$$

$$\leqslant \frac{M_2^2}{(\lambda-\lambda_0)^{4n+2}} \|p\|_1^2 \leqslant \frac{M_2^2}{(\lambda-\lambda_0)^{4n+2}} \|x\|_1^2 \leqslant \frac{M^2}{(\lambda-\lambda_0)^{4n+2}} \|x\|^2,$$

where $\|\cdot\|$ is the original norm in Π_k, $\|\cdot\|_1$ is the norm introduced according to the decomposition $\Pi_k = H_1 \oplus H_2$ where $H_1 \supset N$, $H_2 \supset P$, M, M_1, M_2 are positive constants. For brevity, we shall omit the foot indexes of $\|\cdot\|_1$.

Thus, $\left\| \int K(\lambda, t) dE_t \right\| \leqslant \frac{M}{|\lambda-\lambda_0|^{2n+1}}$.

On the other hand, from the representation (3) of $(\lambda-A)^{-1}$, it is easy to see that to estimate the norm of the resolvent of A, it suffices to estimate the resolvents of S, A_N and A_P. From the structure of S, we have

$$\|(\lambda-S)^{-1}\| \leqslant \frac{M}{|\lambda-\lambda_0|^n}.$$

Since $\sqrt{2}\,|\mathrm{Im}\lambda| \geqslant |\lambda-\lambda_0|$, thus

$$\|(A_P-\lambda)^{-1}\| \leqslant \frac{1}{|\mathrm{Im}\,\lambda|} \leqslant \frac{\sqrt{2}}{|\lambda-\lambda_0|}.$$

Similarly, $\|(A_N-\lambda)^{-1}\| \leqslant \frac{\sqrt{2}}{|\lambda-\lambda_0|}$. Hence, for $\lambda \in \mathrm{I} \cup \mathrm{II}$, the growth order of the components of the model (3) does not exceed $2n+1$. It follows that the growth order of $(\lambda-A)^{-1}$ does not exceed $2n+1$. This verifies that the growth order of $F(\lambda)$ does not exceed $2n+1$.

(ii) $\lambda \in \mathrm{III} \cup \mathrm{IV}$.

For $\mathrm{Re}\,\lambda < \lambda_0$ take

$$\Delta_\lambda = \left[\frac{3\mathrm{Re}\,\lambda-\lambda_0}{2}, \frac{\mathrm{Re}\,\lambda+\lambda_0}{2}\right),$$

for $\mathrm{Re}\,\lambda > \lambda_0$ take

$$\Delta_\lambda = \left[\frac{\mathrm{Re}\,\lambda+\lambda_0}{2}, \frac{3\,\mathrm{Re}\,\lambda-\lambda_0}{2}\right).$$

We define F(λ) in the unified form (21.2).

To estimate $\left\| \int_{\Delta_\lambda} \sum_{i=1}^{2n} \frac{(t-\lambda_0)^{i-1}}{(\lambda-\lambda_0)^i} dE_t \right\|$, by (18), it suffices to estimate the norms of

$$\int_{\Delta_\lambda} \frac{(t-\lambda_0)^{i-1}(A_P-\lambda_0)^{-k}}{(\lambda-\lambda_0)^i} dP_t^{A_P}, \quad 1 \leqslant k \leqslant 2n.$$

For $t \in \Delta_\lambda$, $\frac{1}{2\sqrt{2}}|\lambda-\lambda_0| \leqslant |t-\lambda_0| \leqslant \frac{3}{2}|\lambda-\lambda_0|$, thus

$$\left\|\int_{\Delta_\lambda}\frac{(t-\lambda_0)^{i-1}(A_P-\lambda_0)^{-k}}{(\lambda-\lambda_0)^i}\,dP_t^{A_P}p\right\|^2\leqslant\frac{1}{|\lambda-\lambda_0|^{2i}}\int_{\Delta_\lambda}\frac{|t-\lambda_0|^{2i-2}d\|P_t^{A_P}p\|^2}{|t-\lambda_0|^{2k}}$$

$$\leqslant\frac{\left(\frac{3}{2}\right)^{2i-2}(2\sqrt{2})^{2k}}{|\lambda-\lambda_0|^{2k+2}}\int_{\Delta_\lambda}d\|P_t^{A_P}p\|^2\leqslant\frac{M}{|\lambda-\lambda_0|^{2k+2}}\|p\|^2.$$

Hence $\left\|\int_{\Delta_\lambda}\sum_{i=1}^{2n}\frac{(t-\lambda_0)^{i-1}}{(\lambda-\lambda_0)^i}\,dE_t\right\|\leqslant\frac{M}{|\lambda-\lambda_0|^{2n+1}}.$

When $t\in\Delta_\lambda^c$, $|t-\lambda|>\frac{1}{2}|\lambda-\lambda_0|$ holds, by an argument like that given for $\lambda\in\mathrm{I}\cup\mathrm{II}$, we have

$$\left\|\int_{\Delta_\lambda^c}K(\lambda,\ t)dE_t\right\|\leqslant\frac{M}{|\lambda-\lambda_0|^{2n+1}}.$$

We only have to estimate $(\lambda-A)^{-1}E(\Delta_\lambda^c)$. For a fixed neighborhood Δ of λ_0, when $\lambda\in\{\lambda|\ \mathrm{Re}\lambda\in\Delta_\lambda^c\}$, $(\lambda-A)^{-1}E(\Delta)$ is an operator-valued analytic function. When $\lambda\in\{\lambda|\mathrm{Re}\lambda\in\Delta^c\}\backslash\Delta^c$, $(\lambda-A)^{-1}E(\Delta)$ can be given by multiplying the model (3) of $(\lambda-A)^{-1}$ by the model (12) of $E(\Delta)$. Because of continuity, $(\lambda-A)^{-1}E(\Delta)$ has a triangle model in $\{\lambda|\mathrm{Re}\lambda\in\Delta^c\}$ and which is the product of the two models given above. From calculation,

$$\begin{aligned}(\lambda-A)^{-1}E(\Delta)=\Big\{&(\lambda-S)^{-1},\ (\lambda-A_N)^{-1}P_\Delta^{A_N},\ (\lambda-A_P)^{-1}P_\Delta^{A_P},\ (\lambda-S)^{-1}F(\lambda-A_N)^{-1}P_\Delta^{A_N},\\ &\sum_{j=1}^{n}(\lambda-S)^{-1}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-j}P_\Delta^{A_P}+(\lambda-S)^{-1}G(\lambda-A_P)^{-1}P_\Delta^{A_P},\\ &-\sum_{j,k=1}^{n}(\lambda-S)^{-1}(\lambda_0-S)^{j-1}G(\lambda_0-A_P)^{-(j+k)}P_\Delta^{A_P}G^*(\lambda_0-S^*)^{k-1}\\ &-\sum_{j=1}^{n}(\lambda-S)^{-1}G(\lambda-A_P)^{-1}(\lambda_0-A_P)^{-j}P_\Delta^{A_P}G^*(\lambda_0-S^*)^{j-1}\\ &+(\lambda-S)^{-1}[Q-F(\lambda-A_N)^{-1}F^*+G(\lambda-A_P)^{-1}G^*](\lambda-S^*)^{-1}\Big\}.\end{aligned}$$

By induction, it is easy to verify

$$(\lambda-A_P)^{-1}(\lambda_0-A_P)^{-j}=\frac{1}{(\lambda_0-\lambda)^j}(\lambda-A_P)^{-1}-\sum_{i=1}^{j}\frac{1}{(\lambda_0-\lambda)^i}(\lambda_0-A_P)^{i-j-1}.$$

Thus, in the last component of $(\lambda-A)^{-1}E(\Delta)$, the terms which contain $(\lambda-A_P)^{-1}$ is

$$\begin{aligned}&-(\lambda-S)^{-1}G(\lambda-A_P)^{-1}\sum_{j=1}^{n}(\lambda_0-A_P)^{-j}P_\Delta^{A_P}G^*(\lambda_0-S^*)^{j-1}\\ &+(\lambda-S)^{-1}G(\lambda-A_P)^{-1}G^*(\lambda-S^*)^{-1}\\ &=-(\lambda-S)^{-1}G(\lambda-A_P)^{-1}P_\Delta^{A_P}G^*\sum_{j=1}^{n}\frac{1}{(\lambda_0-\lambda)^j}(\lambda_0-S^*)^{j-1}\\ &-(\lambda-S)^{-1}G\sum_{j=1}^{n}\sum_{i=1}^{j}\frac{(\lambda_0-A_P)^{-(j+1)+i}}{(\lambda_0-\lambda)^i}P_\Delta^{A_P}G^*(\lambda_0-S^*)^{j-1}\\ &+(\lambda-S)^{-1}G(\lambda-A_P)^{-1}G^*(\lambda-S^*)^{-1}.\end{aligned}$$

We note in the second sum of the right side, $-n\leqslant-(j+1)+i\leqslant-1$. The sum of the first and the third one is

$$-(\lambda-S)^{-1}G(\lambda-A_P)^{-1}P_{\Delta}^{A_P}G^*(\lambda-S^*)^{-1}+(\lambda-S)^{-1}G(\lambda-A_P)^{-1}G^*(\lambda-S^*)^{-1}$$
$$=(\lambda-S)^{-1}G(\lambda-A_P)^{-1}P_{\Delta}^{A_P}G^*(\lambda-S^*)^{-1}.$$

Put $\Delta=\Delta_\lambda^c$. We should estimate each component of $(\lambda-A)^{-1}E(\Delta_\lambda^c)$. We note first that F, G and S are bounded operators. By straightforward calculation, we have

$$\|(\lambda-S)^{-1}(\lambda_0-S)^j\|\leqslant\frac{M}{|\lambda-\lambda_0|^{n-j}}.$$

Using $|\lambda-t|\geqslant\frac{1}{2}|\lambda-\lambda_0|$ where $t\in\Delta_\lambda^c$, by estimation we have

$$\|(\lambda-A_P)^{-1}P_{\Delta_\lambda^c}^{A_P}p\|^2=\left\|\int_{\Delta_\lambda^c}\frac{dP_t^{A_P}p}{\lambda-t}\right\|^2\leqslant\frac{4}{|\lambda-\lambda_0|^2}\int_{-\infty}^{\infty}d\|P_t^{A_P}p\|^2=\frac{4}{|\lambda-\lambda_0|^2}\|p\|^2.$$

i.e. $\|(\lambda-A_P)^{-1}P_{\Delta_\lambda^c}^{A_P}\|\leqslant\frac{2}{|\lambda-\lambda_0|}$. Similarly, $\|(\lambda-A_N)^{-1}P_{\Delta_\lambda^c}^{A_N}\|\leqslant\frac{2}{|\lambda-\lambda_0|}$ holds. In fact, under the hypothesis $\sigma(A_N)=\{\lambda_0\}$, as A_N is self-adjoint, $\|(\lambda-A_N)^{-1}\|\leqslant\frac{1}{|\lambda-\lambda_0|}$ also holds.

For $t\in\Delta_\lambda$, $|\lambda_0-t|\geqslant\frac{1}{2\sqrt{2}}|\lambda-\lambda_0|$, thus, by estimation,

$$\|(\lambda_0-A_P)^{-1}P_{\Delta_\lambda^c}^{A_P}\|=\|(\lambda_0-A_P)^{-1}P_{\Delta_\lambda}^{A_P}\|=\left\|\int_{\Delta_\lambda}\frac{dP_t^{A_P}}{\lambda_0-t}\right\|\leqslant\frac{2\sqrt{2}}{|\lambda-\lambda_0|}.$$

It follows at once that

$$\|(\lambda-A)^{-1}E(\Delta_\lambda^c)\|\leqslant\frac{M}{|\lambda-\lambda_0|^{2n+1}}.$$

Hence, for $\lambda\in\mathrm{III}\cup\mathrm{IV}$, the growth order of $F(\lambda)$ does not exceed $2n+1$.

(5) Since $F(\lambda)$ is analytic in whole plane except $\lambda=\lambda_0$, $F(\lambda)\to0$ as $\lambda\to\infty$ and λ_0 is a pole of $F(\lambda)$, whose multiplicity does not exceed $2n+1$, thus

$$F(\lambda)=\sum_{\nu=1}^{2n+1}\frac{B_\nu}{(\lambda-\lambda_0)^\nu}.$$

For $\lambda\in\rho(A)$, we have

$$(\lambda-A)^{-1}=\int K(\lambda,t)dE_t+\sum_{\nu=1}^{2n+1}\frac{B_\nu}{(\lambda-\lambda_0)^\nu}.$$

Since $F(\lambda)$ is a self-adjoint operator when λ is a real number, we find

$$\sum_{\nu=1}^{2n+1}\frac{(B_\nu x,y)}{(\lambda-\lambda_0)^\nu}=\sum_{\nu=1}^{2n+1}\frac{(x,B_\nu y)}{(\lambda-\lambda_0)^\nu},\quad\lambda\in R.$$

Hence $B_\nu^\dagger=B_\nu$, i.e. B_ν are self-adjoint.

For general cases, it is now easy to see that (20) holds. Q.E.D.

5. The operational calculus for self-adjoint operators in Π_k space

With the spectral representation of the resolvent, we can further discuss the operational calculus. Suppose that A is a bounded operator in Π_k, $f(\lambda)$ is analytic in a neighborhood Ω of $\sigma(A)$. Γ is an admissible contour in Ω surrounding $\sigma(A)$. According to the ordinary definition, $f(A)=\frac{1}{2\pi i}\oint_\Gamma f(\lambda)(\lambda-A)^{-1}d\lambda$. With these notation, we have

Theorem 7. *Suppose that A is a self-adjoint operator in Π_k, $C(A)=\{\lambda_1, \cdots, \lambda_l, \lambda_{l+1}, \bar{\lambda}_{l+1}, \cdots, \lambda_{l+p}, \bar{\lambda}_{l+p}\}$, $r(\lambda_\nu)=n_\nu$, then*

$$f(A)=\int_{-\infty}^{\infty}\Big[f(t)-\sum_{\nu=1}^{l}\delta(t-\lambda_\nu)\sum_{i=0}^{2n_\nu-1}\frac{f^{(i)}(\lambda_\nu)}{i!}(t-\lambda_\nu)^i\Big]dE_t+\sum_{\nu=1}^{l}\sum_{i=0}^{2n_\nu}\frac{f^{(i)}(\lambda_\nu)}{i!}B_{\nu i}$$
$$+\sum_{\nu=l+1}^{l+p}\sum_{i=0}^{n_\nu-1}\Big[\frac{f^{(i)}(\lambda_\nu)}{i!}B_{\nu i}+\frac{f^{(i)}(\bar{\lambda}_\nu)}{i!}B_{\nu i}^{\dagger}\Big]. \tag{22}$$

Proof For brevity, we assume $C(A)=\{\lambda_0\}$. Using Fubini's theorem, for $x, y\in\Pi_k$, we have

$$(f(A)x, y)=\frac{1}{2\pi i}\oint_\Gamma f(\lambda)\int_{-\infty}^{\infty}\Big[\frac{1}{\lambda-t}-\sum_{i=1}^{2n}\frac{(t-\lambda_0)^{i-1}}{(\lambda-\lambda_0)^i}\Big]d(E_tx, y)d\lambda$$
$$+\frac{1}{2\pi i}\sum_{i=1}^{2n+1}\oint\frac{f(\lambda)d\lambda}{(\lambda-\lambda_0)^i}(B_ix, y)$$
$$=\int_{-\infty}^{\infty}\frac{1}{2\pi i}\oint_\Gamma\Big[\frac{f(\lambda)}{\lambda-t}-\sum_{i=1}^{2n}\frac{f(\lambda)}{(\lambda-\lambda_0)^i}(t-\lambda_0)^{i-1}\Big]d\lambda d(E_tx, y)$$
$$+\sum_{i=1}^{2n+1}\frac{1}{2\pi i}\oint_\Gamma\frac{f(\lambda)d\lambda}{(\lambda-\lambda_0)^i}(B_ix, y).$$
$$(\text{denote } B_i \text{ by } B_{i-1})=\int_{-\infty}^{\infty}\Big[f(t)-\sum_{i=0}^{2n-1}\frac{f^{(i)}(\lambda_0)}{i!}(t-\lambda_0)^i\Big]d(E_tx, y)$$
$$+\sum_{i=0}^{2n}\frac{f^{(i)}(\lambda_0)}{i!}(B_ix, y).$$

It follows that

$$f(A)=\int_{-\infty}^{\infty}\Big[f(t)-\sum_{i=0}^{2n-1}\frac{f^{(i)}(\lambda_0)}{i!}(t-\lambda_0)^i\Big]dE_t+\sum_{i=0}^{2n}\frac{f^{(i)}(\lambda_0)}{i!}B_i \tag{23}$$

Q.E.D.

Corollary 1. *Under the same conditions of theorem 7, in addition, $C(A)=\{\lambda_0\}$, we have*

$$(\lambda-A)^{-1}=\int\Big[\frac{1}{t-\lambda}-\sum_{i=0}^{2n}\frac{(t-\lambda_0)^i}{(\lambda-\lambda_0)^{i+1}}\Big]dE_t+\sum_{i=0}^{2n}\frac{(A-\lambda_0)^i}{(\lambda-\lambda_0)^{i+1}}, \tag{24}$$

$$f(A)=\int\Big[f(t)-\sum_{i=0}^{2n}\frac{f^{(i)}(\lambda_0)}{i!}(t-\lambda_0)^i\Big]dE_t+\sum_{i=0}^{2n}\frac{f^{(i)}(\lambda_0)}{i!}(A-\lambda_0)^i. \tag{25}$$

Proof Take $f(t)=(t-\lambda_0)^k$, substitute it into (23), we have

$$B_k=\begin{cases}(A-\lambda_0)^k, & 0\leqslant k\leqslant 2n-1,\\ (A-\lambda_0)^{2n}-\int_{-\infty}^{\infty}(t-\lambda_0)^{2n}dE_t, & k=2n.\end{cases}$$

Substitute them into the representations of $(\lambda-A)^{-1}$ and $f(A)$, it follows that (24) and (25) hold. Q.E.D.

Remark. Using the triangle model, we can find B_{2n} concretely. Indeed, take $\Delta=[\mu, \nu)$ such that $\lambda_0\in(\mu, \nu)$, then

$$B_{2n}=(A-\lambda_0)^{2n}-\int_{-\infty}^{\infty}(t-\lambda_0)^{2n}dE_t=(A-\lambda_0)^{2n}E(\Delta)-\int_\Delta(t-\lambda_0)^{2n}dE_t.$$

From (8) and (12), we have

$$(A-\lambda_0)^{2n}E(\Delta)=\Big\{0,\ 0,\ (A_P-\lambda_0)^{2n}P_\Delta^{A_P},\ 0,\ \sum_{i=1}^{n}(S-\lambda_0)^{i-1}G(A_P-\lambda_0)^{2n-i}P_\Delta^{A_P},$$
$$-(S-\lambda_0)^{n-1}FF^*(S^*-\lambda_0)^{n-1}$$
$$-\sum_{i,j=1}^{n}(S-\lambda_0)^{i-1}G(A_P-\lambda_0)^{2n-(i+j)}P_\Delta^{A_P}G^*(S^*-\lambda_0)^{j-1}\Big\}.$$

Let $|\Delta|\to 0$, by the definition, we have $\lim\limits_{|\Delta|\to 0}\int_\Delta (t-\lambda_0)^{2n}dE_t=0$. So

$$B_{2n}=\lim_{|\Delta|\to 0}(A-\lambda_0)^{2n}E(\Delta)$$
$$=\{0,\ 0,\ 0,\ 0,\ 0,\ -(S-\lambda_0)^{n-1}[FF^*$$
$$+G(P_{\lambda_0}^{A_P}-P_{\lambda_0-0}^{A_P})G^*](S^*-\lambda_0)^{n-1}\}.$$

As above, under the basis $\{z_i\}$ of Z, S has the Jordan canonical form. For convenience, we assume it has only one Jordan block. Suppose that F and G are

$$Fn=\Sigma(n,\ x_i)z_i,\quad Gp=\Sigma(p,\ y_i)z_i,\quad \forall n\in N,\ p\in P,$$

where $x_i\in N$, $y_i\in P$. Then, for $x=n+z+z^*+p\in\Pi_k$, we have

$$B_{2n}x=-(S-\lambda_0)^{n-1}[FF^*+G(P_{\lambda_0}^{A_P}-P_{\lambda_0-0}^{A_P})G^*](S^*-\lambda_0)^{n-1}z^*$$
$$=-[(x_n,\ x_n)+((P_{\lambda_0}^{A_P}-P_{\lambda_0-0}^{A_P})y_n,\ y_n)](z^*,\ z_n)z_n.$$

Corollary 2. *Under the same conditions of theorem 7, in addition, if $C(A)$ is real, $k_\nu\geqslant 1$, $1\leqslant\nu\leqslant l$, then*

$$\prod_{\nu=1}^{l}(A-\lambda_\nu)^{2n_\nu+k_\nu}=\int_{-\infty}^{\infty}\prod_{\nu=1}^{l}(t-\lambda_\nu)^{2n_\nu+k_\nu}dE_t. \tag{26}$$

Proof Taking $f(t)=\prod\limits_{\nu=1}^{l}(t-\lambda_\nu)^{2n_\nu+k_\nu}$ and substituting it into (23), the present corollary follows at once. Q.E.D.

The operational calculus mentioned above was restricted to analytic functions. Next, we shall extend this notion to a still more general case. For convenience, we assume there exists only one critical point, 0.

First, we introduce two special algebras, Ω_n and ω_n. Their definitions are:

$\Omega_n=\{(f,\ \{a_i\}_{i=0}^{2n})\,|\,f$ is any Borel measurable function, $\{a_i\}$ is any system of constants$\}$. For $F=(f,\ \{a_i\})\in\Omega_n$, $G=(g,\ \{b_i\})\in\Omega_n$, the operations of F and G are defined by

$$\alpha F+\beta G=(\alpha f+\beta g,\ \{\alpha a_i+\beta b_i\}),$$
$$F\cdot G=\Big(f\cdot g,\ \Big\{\sum_{j=0}^{i}a_jb_{i-j}\Big\}\Big).$$

Moreover, we define the conjugate of F by

$$\bar{F}=(\bar{f},\ \{\bar{a}_i\}).$$

It is obvious that Ω_n is a commutative algebra with identical element $(1,\ \{1,\ 0,\ \cdots,\ 0\})$ and null element $(0,\ \{0,\ \cdots,\ 0\})$. We also define ω_n as a subalgebra of Ω_n:

$$\omega_n=\Big\{F=(f,\ \{a_i\})\in\Omega_n \mid \text{ for a neighborhood of 0 corresponding to } F, \text{ soch}$$
$$\text{that } \Big|f(t)-\sum_{i=0}^{2n}a_it^i\Big|\leqslant M_F\cdot t^{2n+1} \text{ holds, where } M_F \text{ depends on } F\Big\}.$$

If f is Borel measurable and in a neighborhood of 0 it has continuous derivative of the $2n+1$-th order, then $(f,\ \{a_i\}_{i=0}^{2n})\in\omega_n$, where $a_i=\dfrac{f^{(i)}(0)}{i!}(0\leqslant i\leqslant 2n+1)$.

Definition. *Suppose that A is a self-adjoint operator, $C(A)=\{0\}$, $r(0)=n$, the spectral family of A is E_t. For $F=(f,\ \{a_i\})\in\omega_n$, we define*

$$F(A)=\int_{-\infty}^{\infty}\Big[f(t)-\sum_{i=0}^{2n}a_it^i\Big]dE_t+\sum_{i=0}^{2n}a_iA^i,$$

The domain of this operator is

$$\mathscr{D}(F(A))=\mathscr{D}(A^{2n})\cap\Big\{x\in\Pi_k\Big|\int\Big|f(t)-\sum_{i=0}^{2n}a_it^i\Big|^2d\|E_tx\|^2<\infty\Big\}.$$

By theorem 5 and its remark, this definition makes sense. Moreover, if f is analytic, set $F=\Big(f,\ \Big\{\dfrac{f^{(i)}(0)}{i!}\Big\}\Big)$, then we have $F(A)=f(A)$, the right side is defined in the ordinary sense.

Take $M>0$, put $\Delta=[-M,\ M)$. We have $A=A|E(\Delta)\Pi_k\oplus A|E(\Delta^c)\Pi_k$. It is easy to see that we may discuss the operational calculus in $E(\Delta)\Pi_k$ and $E(\Delta^c)\Pi_k$ respectively. However, for $A|E(\Delta^c)\Pi_k$, it has been done by the self-adjoint operator theory in Hilbert space. So, we assume A is bounded.

Theorem 8. *Suppose that A is a bounded self-adjoint operator in Π_k, $C(A)=\{0\}$, $r(0)=n$, F, $G\in\omega_n$, then*

$$\bar{F}(A)=[F(A)]^\dagger,$$
$$(\alpha F+\beta G)(A)=\alpha[F(A)]+\beta[G(A)], \tag{28}$$
$$(FG)(A)=F(A)G(A).$$

Proof The first two formulas of (28) are obvious. We shall only prove $(FG)(A)=F(A)G(A)$, which is equivalent to $(F^2)(A)=[F(A)]^2$. Suppose that $F=(f,\ \{a_i\})$. Using the same argument given above, with no loss of generality, we may assume for all $t\in R$

$$\Big|f(t)-\sum_{i=0}^{2n}a_it^i\Big|<M_Ft^{2n+1} \tag{29}$$

holds. For any fixed $\mu>0$, write $\Delta=[-\mu,\ \mu)$. By the definition, it is easy to see that $F(A)$ and $(F^2)(A)$ are reduced by $E(\Delta)$. We have

$$[F(A)]^2=[F(A)]^2E(\Delta)+[F(A)]^2E(\Delta^c),$$
$$(F^2)(A)=(F^2)(A)E(\Delta)+(F^2)(A)E(\Delta^c).$$

Write $AE(\Delta)=A_\Delta$, $AE(\Delta^c)=A_{\Delta^c}$, we have

$$F(A)E(\Delta^c)=\int_{\Delta^c}\Big[f(t)-\sum_{i=0}^{2n}a_it^i\Big]dE_t+\sum_{i=0}^{2n}a_iA^i_{\Delta^c}=\int_{\Delta^c}f(t)\,dE_t,$$

Similarly $(F^2)(A)E(\Delta^c)=\int_{\Delta^c} f^2(t)dE_t$. But $\{E_tE(\Delta^c)\}$ is a spectral family in ordinary Hilbert space, so we have $\left[\int_{\Delta^c} f(t)dE_t\right]^2=\int_{\Delta^c} f^2(t)dE_t$, i.e.

$$[F(A)]^2E(\Delta^c)=(F^2)(A)E(\Delta^c).$$

Write

$$\xi_1(t)=f(t)-\sum_{i=0}^{2n} a_it^i, \quad \xi_2(t)=(f^2)(t)-\sum_{i=0}^{2n} b_it^i,$$

where

$$b_i=\sum_{j=0}^{i} a_ja_{i-j} \quad (0\leqslant i\leqslant 2n).$$

We have

$$[F(A)]^2E(\Delta)=\left[\int_\Delta \xi_1(t)dE_t+\sum_{i=0}^{2n} a_iA_\Delta^i\right]^2$$

$$=\int_\Delta \xi_1(t)dE_t\left[\int_\Delta \xi_1(t)dE_t+2\sum_{i=0}^{2n} a_iA_\Delta^i\right]+\sum_{i,j=0}^{2n} a_ia_jA_\Delta^{i+j},$$

$$(F^2)(A)E(\Delta)=\int_\Delta \xi_2(t)dE_t+\sum_{k=0}^{2n} b_kA_\Delta^k.$$

By (29), it is easy for us to show that there exists a constant M_1 such that for $t\in\Delta$ $|\xi_1(t)|\leqslant M_1t^{2n+1}$ and $|\xi_2(t)|\leqslant M_1t^{2n+1}$hold. Thus, for $1\leqslant i\leqslant 2n$, we have

$$\left\|\int_{[\varepsilon,\mu)} \xi_1(t)A_P^{-i}\,dP_t^{A_P}p\right\|^2=\left\|\int_{[\varepsilon,\mu)} \xi_1(t)t^{-i}\,dP_t^{A_P}p\right\|^2\leqslant\int_{[\varepsilon,\mu)} (M_1t^{2n+1-i})^2\,d\|P_t^{A_P}p\|^2$$

$$\leqslant M_1^2\mu^2\int_{[\varepsilon,\mu)} t^{4n-2i}d\|P_t^{A_P}p\|^2\leqslant M_2^2\mu^2\|p\|^2. \tag{30}$$

Similarly, we have

$$\left\|\int_{[-\mu,\varepsilon)} \xi_1(t)A_P^{-i}\,dP_t^{A_P}p\right\|\leqslant M_3\mu\|p\|. \tag{31}$$

Now, we can estimate $\left\|\int_\Delta \xi_1(t)dE_t\right\|$. From $\int_\Delta \xi_1(t)dE_t=\lim_{\varepsilon\to 0}\left[\int_{[-\mu,-\varepsilon)}+\int_{[\varepsilon,\mu)}\right]\xi_1(t)dE_t$, using an argument similar to the proof for theorem 5, we note it will suffice to estimate $\left\|\int_{[\varepsilon,\mu)} \xi_1(t)A_P^{-i}dP_t^{A_P}\right\|$ and $\left\|\int_{[-\mu,-\varepsilon)} \xi_1(t)A_P^{-i}\,dP_t^{A_P}\right\|$. By (30) and (31), we have

$$\left\|\int_\Delta \xi_1(t)dE_t\right\|\leqslant M_4\mu.$$

Similarly, we have $\left\|\int_\Delta \xi_2(t)dE_t\right\|\leqslant M_5\mu$. Since A_Δ is bounded, hence

$$\left\|\int_\Delta \xi_1(t)dE_t\left[\int_\Delta \xi_1(t)dE_t+2\sum_{i=0}^{2n} a_iA_\Delta^i\right]\right\|\leqslant M_6\mu.$$

On the other hand

$$\sum_{i,j=0}^{2n} a_ia_jA_\Delta^{i+j}=\sum_{i+j=0}^{2n} a_ia_jA_\Delta^{i+j}+\sum_{\substack{0\leqslant i,j\leqslant 2n\\ i+j\geqslant 2n+1}} a_ia_jA_\Delta^{i+j}=\sum_{k=0}^{2n} b_kA_\Delta^k+\sum_{\substack{0\leqslant i,j\leqslant 2n\\ i+j\geqslant 2n+1}} a_ia_jA_\Delta^{i+j}.$$

By the corollary of theorem 7, for $k\geqslant 1$, $A_\Delta^{2n+k}=\int_\Delta t^{2n+k}dE_t$ holds. The same argument can be applied to show

$$\|A_\Delta^{2n+k}\|=\left\|\int_\Delta t^{2n+k}\,dE_t\right\|\leqslant M_7\mu.$$

Thus from what have been proved it follows that

$$\|[F(A)]^2-(F^2)(A)\|\leqslant C\mu,$$

where C is a constant independent of μ. Hence $[F(A)]^2=(F^2)(A)$ due to the arbitrariness of μ. Q.E.D.

Theorem 9. *Suppose that A is a self-adjoint operator in Π_k, $C(A)=\{0\}$, $r(0)=n$, $F_1=(f_1, \{a_i\})\in\omega_n$, $F_2=(f_2, \{a_i\})\in\omega_n$, $f_1(t)$ and $f_2(t)$ are continuous in $(-\infty, \infty)$, in addition, $f_1(t)\equiv f_2(t)$ where $t\in\sigma(A)$, then $F_1(A)=F_2(A)$.*

Proof By the hypothesis and the definition of $F_1(A)$ and $F_2(A)$, we have

$$(F_1-F_2)(A)=\int_{-\infty}^{\infty}(f_1-f_2)(t)dE_t=\left(\int_{-\infty}^{-0}+\int_{+0}^{+\infty}\right)(f_1-f_2)(t)dE_t.$$

For any fixed $\varepsilon>0$, $\sigma(A|E(-\infty, -\varepsilon))=\sigma(A)\cap(-\infty, -\varepsilon]$, $\sigma(A|E[\varepsilon, \infty))=\sigma(A)\cap[\varepsilon, \infty)$. However, in $\sigma(A|E(-\infty, -\varepsilon))\cup\sigma(A|E[\varepsilon, \infty))$, $f_1(t)\equiv f_2(t)$ holds, so that $\left(\int_{-\infty}^{-\varepsilon}+\int_{+\varepsilon}^{\infty}\right)(f_1-f_2)(t)dE_t=0$ and hence $(F_1-F_2)(A)=0$. By theorem 8, $F_1(A)=F_2(A)$ holds. Q.E.D.

We now try to establish the spectral mapping theorem. To do this, we first prove

Lemma. *Suppose that A is a bonuded self-adjoint operator in Π_k, $C(A)=\{0\}$, $r(0)=n$, $F=(f, \{a_i\})\in\omega_n$, where f is a continuous function. Then $F(A)$ is bounded.*

Proof Write $\xi(t)=f(t)-\sum_{i=0}^{2n}a_it^i$. With no loss of generality, we may assume (29) holds in $[-r_A-1, r_A+1]$ Where r_A is the spectral radius of A. To prove

$$F(A)=\int_{-r_A-1}^{r_A+1}\xi(t)dE_t+\sum_{i=0}^{2n}a_iA^i$$

is bounded, it suffices to show $\int_{-r_A-1}^{-\varepsilon}\xi(t)dE_t$ and $\int_{\varepsilon}^{r_A+1}\xi(t)dE_t$ are uniformly bounded for $\varepsilon>0$. By (18), it is easy to see that this is equivalent to the uniform boundedness of $\int_{-r_A-1}^{-\varepsilon}\xi(t)A_P^{-i}dP_t^{A_P}$ and $\int_{\varepsilon}^{r_A+1}\xi(t)A_P^{-i}dP_t^{A_P}$, where $1\leqslant i\leqslant n$. Now, the present lemma follows by the estimation like that given in theorem 8. Q.E.D.

Theorem 10. *Suppose that A is a bounded self-adjoint operator in Π_k, $C(A)=\{0\}$, $r(0)=n$, $F=(f, \{a_i\})\in\omega_n$ where f is a continuous function. Then*

$$\sigma(F(A))=\{f(t)\,|\,t\in\sigma(A)\}.$$

Proof First, we prove $\{f(t)\,|\,t\in\sigma(A)\}\subset\sigma(F(A))$. For $\lambda\in\sigma(A)\backslash\{0\}$, take $\delta>0$ such that $0\overline{\in}[\lambda-\delta, \lambda+\delta]$. Since Hilbert space $E[\lambda-\delta, \lambda+\delta)\Pi_k$ reduces A and $F(A)$, hence

$$f(\lambda)\in\{f(t)\,|\,t\in\sigma(A|E[\lambda-\delta, \lambda+\delta)\Pi_k)\subset\sigma(F(A)|E[\lambda-\delta, \lambda+\delta)\Pi_k)\subset\sigma(F(A)).$$

On the other hand, the critical point $0\in\sigma_P(A)$, thus there exists $x_0\in\Pi_k$ such that $Ax_0=0$. Since $\sigma_A(x_0)=\{0\}$, it follows that for any $\delta>0$, $x_0\in E[\lambda_0-\delta, \lambda_0+\delta)\Pi_k$ holds, hence for any $\varepsilon>0$, $E_tx_0=\begin{cases}0, & t\in(-\infty, -\varepsilon),\\ x_0, & t\in[\varepsilon, +\infty).\end{cases}$ We have

$$F(A)x_0=\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\left[f(t)-\sum_{i=0}^{2n}a_it^i\right]dE_tx_0+\sum_{i=0}^{2n}a_iA^ix_0=f(0)x_0.$$

This means $f(0)\in\sigma_P(F(A))$. So we have $\{f(t)\,|\,t\in\sigma(A)\}\subset\sigma(F(A))$.

Conversely, suppose that $r\overline{\in}\{f(t)\,|\,t\in\sigma(A)\}$ and hence $f(0)\neq r$. In view of the continuity of $f(t)$, there exists a closed interval $\varDelta$ which contains 0 as an interior point and satisfies $f(t)\neq r$ for $t\in\varDelta$. Suppose that

$$(-r_A-1,\ r_A+1)\setminus(\sigma(A)\cup\varDelta)=\bigcup_{\nu=1}^{\infty}(a_\nu,\ b_\nu),$$

where r_A is the spectral radius of A. We define

$$g(t)=\begin{cases}\dfrac{1}{r-f(t)}, & t\in\sigma(A)\cup\varDelta,\\ \dfrac{1}{r-f(a_\nu)}\dfrac{b_\nu-t}{b_\nu-a_\nu}-\dfrac{1}{r-f(b_\nu)}\dfrac{t-a_\nu}{b_\nu-a_\nu}, & t\in(a_\nu,\ b_\nu),\ \nu=1,\ 2,\ \cdots.\end{cases}$$

It is easy to see that $g(t)$ is continuous in $(-r_A-1,\ r_A+1)$. Develop $\dfrac{1}{r-\sum_{i=0}^{2n}a_it^i}$ in a neighbourhood of 0, denote the first $2n+1$ coefficients by $\{b_i\}_{i=0}^{2n}$. We observe that $G=(g,\ \{b_i\})\in\omega_n$. Consider $F\cdot G=(f\cdot g,\ \{c_i\})$. We have $g(t)(r-f(t))=1$ for $t\in\sigma(A)\cup\varDelta$ and $\{c_i\}=\{1,\ 0,\ \cdots,\ 0\}$. By theorem 8 and 9, we have $G(A)(r-F(A))=(r-F(A))G(A)=I$. So $G(A)=(r-F(A))^{-1}$. By Lemma, $G(A)$ is bounded This verifies $r\in\rho(F(A))$.

Summarizing what have beed proved, we have $\sigma(F(A))=\{f(t)\,|\,t\in\sigma(A)\}$. Q.E.D.

Theorem 11. *Under the hypotheses of Theorem 10, $F(A)$ has a triangle model*

$$F(A)=\{F(S),\ a_0I_N,\ f(A_P),\ [F(A)]_{\mathscr{F}},\ [F(A)]_{\mathscr{G}},\ [F(A)]_{\mathscr{Q}}\}\tag{32}$$

where

$$F(S)=\sum_{i=0}^{n-1}a_iS^i,\qquad [F(A)]_{\mathscr{F}}=\sum_{i=1}^{n}a_iS^{i-1}F,$$

$$[F(A)]_{\mathscr{G}}=\sum_{j=1}^{n-1}S^{j-1}G\int_{-\infty}^{\infty}\Big[f(t)-\sum_{i=0}^{j-1}a_it^i\Big]t^{-j}dP_t^{A_P},$$

$$[F(A)]_{\mathscr{Q}}=\sum_{i=0}^{2n}a_i\sum_{j=1}^{i}S^{j-1}QS^{*i-j}-\sum_{i=2}^{2n}a_i\sum_{j=0}^{i-2}S^jFF^*S^{*i-2-j}$$
$$+\sum_{j,k=1}^{n}S^{j-1}G\int_{-\infty}^{\infty}\Big[f(t)-\sum_{i=0}^{j+k-1}a_it^i\Big]t^{-(j+k)}dP_t^{A_P}G^*S^{*k}.$$

Proof By the definition of $F(A)$, (27), the model (18) of $E(\varDelta)$ for $0\overline{\in}\varDelta$, and the formula (8) of $(A-\lambda_0)^n$, it is easy to verify (32). Q.E.D.

Corollary. *Suppose that $F=(f,\ \{a_i\})\in\omega_n$, f is a real function, $Im\ a_i=0$ $(0\leqslant i\leqslant 2n)$ then*

$$C(F(A))=\{f(t)\,|\,t\in C(A)\}.$$

Proof Since f is a real function, $\mathrm{Im}\,a_i=0$ $(0\leqslant i\leqslant 2n)$, by Theorem 8, it follows that $F(A)$ is a self-adjoint operator in Π_k. In view of the triangle model of $F(A)$, we have

$$C(F(A))=\sigma(F(S))=\{a_0\}.$$

Since $F\in\omega_n$, thus $a_0=f(0)$. This verifies our assertion. Q.E.D.

In particular, suppose that A is a self-adjoint operator whose radical subspace corresponding to critical points are non-degenerate, $C(A)=\{0\}$, $r(0)=n$. For $F=(f, \{a_i\})\in\Omega_n$, we define

$$F(A)=\int_{(-\infty,\infty)\setminus\{0\}} f(t)\,dE_t+\sum_{i=0}^{2n} a_iA^iE(\{0\}).$$

It is easy to show that (a) for $F\in\omega_n$, this definition agrees with the preceding one; (b) Theorem 8 and 9 still hold.

Theorem 12. *Suppose that A is a self-adjoint operator in a separable Π_k space, $C(A)=\{0\}$, $r(0)=n$, the radical subspace corresponding to 0 is non-degenerate, T is a closed operator which is densely defined. Then $T\in\{A\}''$ if and only if there exists a $F\in\Omega_n$ such that $T=F(A)$, where $\{A\}''=\{T\mid BT\subset TB$, where B is any bounded operator satisfying $BA\subset AB\}$.*

Proof The sufficiency is obvious. We shall prove the necessity.

Write $\Delta=\{0\}$, $\Delta^c=(-\infty,\infty)\setminus\{0\}$. $E(\Delta^c)\Pi_k$ is a Hilbert space with inner product $(\cdot,\cdot)$, and $A|E(\Delta^c)\Pi_k$ is a self-adjoint operator in this space. Since $T\in\{A\}''$, we have $TE(\Delta^c)\in\{AE(\Delta^c)\}''$, hence there exists a Borel measurable function f such that

$$TE(\Delta^c)=\int_{\Delta^c} f(t)dE_t.$$

On the other hand, $A|E(\Delta^c)\Pi_k$ is a quasi-nilpotent operator in $E(\Delta)\Pi_k$. From $r(0)=n$, we have, in fact $(A|E(\Delta)\Pi_k)^{2n+1}=0$. However, $T|E(\Delta)\Pi_k\in\{A|E(\Delta)\Pi_k\}''$. By [6], it follows that there exists a system of constants $\{a\}_{i=0}^{2n}$ such that

$$TE(\Delta)=\sum_{i=0}^{2n} a_iA^iE(\Delta).$$

Thus, for $F=(f, \{a_i\})\in\Omega_n$, we have

$$T=\int_{\Delta^c} f(t)\,dE_t+\Big(\sum_{i=0}^{2n} a_iA^i\Big)E(\Delta)=F(A). \qquad \text{Q.E.D.}$$

At the end of this paper, by the way, we point out that the integral representations of the conditional positive-definite functions can be given immediately by the results proved above. But we omit these complicated statements.

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Π_k 空间上的无界自共轭算子

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摘　要

如果 A 是 Π_k 空间上的自共轭算子，由文[1]可知存在空间的一个标准分解

$$\Pi_k = N \oplus \{Z + Z^*\} \oplus P,$$

在此分解下，A 有三角模型 $A = \{S, A_N, A_P, F, G, Q\}$. 利用三角模型，我们直接证明了

定理 1 设 A 是 Π_k 上的自共轭算子，n 是任何自然数，那末 A^n 也是自共轭算子.

定理 2 设 A 是 Π_k 上的自共轭算子，那末对所有的 $A^n(n=1, 2, \cdots)$，存在一个公共的标准分解，在此分解下

$$A^n = \Big\{S^n, A_N^n, A_P^n, \sum_{i=0}^{n-1} S^i F A_N^{n-1-i}, \sum_{i=0}^{n-1} S^i G A_P^{n-1-i},$$
$$\sum_{i=0}^{n-1} S^i Q S^{*n-1-i} - \sum_{i+j+k=n-2} S^i (F A_N^j F^* + G A_P^j G^*) S^{*k}\Big\}.$$

定理 3 设 A 是 Π_k 空间上的自共轭算子，$\sigma(A) \subset [0, \infty)$，$0 \overline{\in} \sigma_P(A)$，那末存在唯一的自共轭算子 A_1，满足 $A_1^n = A$，$\sigma(A_1) \subset [0, \infty)$.

其次，我们研究了谱系在临界点附近的性状. 记临界点全体为 $C(A)$. 对 $\lambda_0 \in C(A)$，记 S 与 λ_0 相应的最高阶根向量的阶数为 $r(\lambda_0)$.

定理 4 设 A 是 Π_k 空间上的无界自共轭算子，$C(A) \cap (\mu_1, \nu_1) = \{\lambda_0\}$，那末以下四个命题等价：

(i) $\sup\limits_{\mu,\nu} \{\|E_{\mu\nu}\| \,|\, \lambda_0 \in (\mu, \nu) \subset (\mu_1, \nu_1)\} < \infty$；(ii) $\mu^{(1)}, \cdots, \mu^{(k_0)}$ 是全有限测度；(iii) $s\text{-}\lim E_{\mu\nu}$ 存在；(iv) A 与 λ_0 相应的根子空间 Φ_{λ_0} 非退化；这里 $\mu^{(1)}, \cdots, \mu^{(k_0)}$ 是由 A_P 和 G 导出的测度.

定理 5 设 A 是 Π_k 上自共轭算子，$\lambda_0 \in C(A)$，$r(\lambda_0) = n$，那末

(i) $E_{\mu\nu}$ 在 λ_0 处的奇性次数不超过 $2n$，

(ii) $s\text{-}\lim\limits_{\varepsilon\to 0} \int_{[M_1, \lambda_0-\varepsilon)} (t-\lambda_0)^{2n} dE_t$，$s\text{-}\lim\limits_{\varepsilon\to 0} \int_{[\lambda_0+\varepsilon, M_2)} (t-\lambda_0)^{2n} dE_t$ 存在. 这里 M_1, M_2 满足 $[M_1, M_2] \cap C(A) = \{\lambda_0\}$

定理 6 设 A 是 Π_k 上自共轭算子，临界点集 $C(A) = \{\lambda_1, \cdots, \lambda_l, \lambda_{l+1}, \bar{\lambda}_{l+1}, \cdots, \lambda_{l+p}, \bar{\lambda}_{l+p}\}$，这里 $\operatorname{Im}\lambda_\nu = 0 (1 \leqslant \nu \leqslant l)$，$r(\lambda_\nu) = n_\nu$，那末有

$$(\lambda - A)^{-1} = \int_{-\infty}^{\infty} K(\lambda, t) dE_t + \sum_{\nu=1}^{l} \sum_{i=1}^{2n_\nu+1} \frac{B_{\nu i}}{(\lambda-\lambda_\nu)^i} + \sum_{\nu=l+1}^{l+p} \sum_{i=1}^{n_\nu} \Big[\frac{B_{\nu i}}{(\lambda-\lambda_\nu)^i} + \frac{B_{\nu i}^\dagger}{(\lambda-\bar{\lambda}_\nu)^i}\Big]$$

这里 $$K(\lambda, t) = \frac{1}{\lambda - t} - \sum_{\nu=1}^{l} \delta(t-\lambda_\nu) \sum_{i=1}^{2n_\nu} \frac{(t-\lambda_\nu)^{i-1}}{(\lambda-\lambda_\nu)^i}, \quad \delta(\lambda) = \begin{cases} 1, & |\lambda| < \delta \\ 0, & |\lambda| \geqslant \delta \end{cases}$$

$0<\delta<\min\limits_{\substack{1\leqslant\mu,\nu\leqslant l\\ \lambda_\mu\neq\lambda_\nu}}|\lambda_\mu-\lambda_\nu|$. 对 $1\leqslant\nu\leqslant l$, $B_{\nu i}$ 是 Π_k 上的有界自共轭算子，而当 $l+1\leqslant\nu\leqslant l+p$ 时，$B_{\nu i}=(\lambda_\nu-S)^{i-1}P_{\lambda_\nu}$, P_{λ_ν} 是以与 λ_ν 相应的根子空间为值域的某些平行投影.

定理 7 在定理 6 的条件下，有

$$f(A)=\int_{-\infty}^{\infty}\Big[f(t)-\sum_{\nu=1}^{l}\delta(t-\lambda_\nu)\sum_{i=0}^{2n_\nu-1}\frac{f^{(i)}(\lambda_\nu)}{i!}(t-\lambda_\nu)^i\Big]dE_t$$
$$+\sum_{\nu=1}^{l}\sum_{i=0}^{2n_\nu}\frac{f^{(i)}(\lambda_0)}{i!}B_\nu+\sum_{\nu=l+1}^{l+p}\sum_{i=0}^{n_\nu-1}\Big[\frac{f^{(i)}(\lambda_\nu)}{i!}B_{\nu i}+\frac{f^{(i)}(\bar\lambda_\nu)}{i!}B_{\nu i}^{\dagger}\Big]$$

这里 $f(\lambda)$ 在 $\sigma(A)$ 的一个邻域内解析.

为了建立更一般的算子演算，我们引入两个特殊的代数：

$\Omega_n=\{(f,\ \{a_i\}_{i=0}^{2n})\,|\,f$ 为 Borel 可测函数，$\{a_i\}$ 为一组常数$\}$. 对 $F=(f,\ \{a_i\})\in\Omega_n$, $G=(g,\ \{b_i\})\in\Omega_n$, 定义

$$\alpha F+\beta G=(\alpha f+\beta G,\ \{\alpha a_i+\beta b_i\}),$$
$$F\cdot G=\Big(f\cdot g,\ \Big\{\sum_{j=0}^{i}a_jb_{i-j}\Big\}\Big),\quad \bar F=(\bar f,\ \{\bar a_i\}).$$

显然 Ω_n 是一个交换代数，它的子代数 ω_n 定义为

$$\omega_n=\Big\{F=(f,\ \{a_i\})\in\Omega_n\,\Big|\ \text{在 0 点的一个与 } F \text{ 有关的邻域中，}$$
$$\text{成立}\Big|f(t)-\sum_{i=0}^{2n}a\,t^i\Big|\leqslant M_F|t|^{2n+1},\ M_F\text{ 与 } F \text{ 有关}\Big\}$$

定义 设 A 是 Π_k 上自共轭算子，$C(A)=\{0\}$, $r(0)=n$, 对 $F=(f,\ \{a_i\})\in\omega_n$, 定义

$$F(A)=\int_{-\infty}^{\infty}\Big[f(t)-\sum_{i=0}^{2n}a_it^i\Big]dE_t+\sum_{i=0}^{2n}a_iA^i$$
$$\mathscr{D}(F(A))=\mathscr{D}(A^{2n})\cap\Big\{x\in\Pi_k\Big|\int_{-\infty}^{\infty}\Big|f(t)-\sum_{i=0}^{2n}a_it^i\Big|^2d\|E_tx\|^2<\infty\Big\}.$$

如果 f 解析，$F=\Big(f,\ \Big\{\dfrac{f^{(i)}(0)}{i!}\Big\}\Big)$, 那末可得 $F(A)=f(A)$.

定理 8 设 A 是有界自共轭算子，$C(A)=\{0\}$, $r(0)=n$, $F,\ G\in\omega_n$, 那末

$$\bar F(A)=[F(A)]^{\dagger},\quad (\alpha F+\beta G)(A)=\alpha F(A)+\beta G(A),$$
$$(FG)(A)=F(A)G(A).$$

定理 9 设 A 是 Π_k 上自共轭算子，$C(A)=\{0\}$, $r(0)=n$, $F_1=(f_1,\ \{a_i\})\in\omega_n$, $F_2=(f_2,\ \{a_i\})\in\omega_n$, f_1,f_2 在 $(-\infty,\ \infty)$ 连续，在 $\sigma(A)$ 上恒等，那末 $F_1(A)=F_2(A)$.

定理 10 设 A 是 Π_k 上有界自共轭算子，$C(A)=\{0\}$, $r(0)=n$, $F=(f,\ \{a_i\})\in\Omega_n$, f 是连续函数，那末 $\sigma(F(A))=\{f(t)\,|\,t\in\sigma(A)\}$.

在定理 11 中，我们建立了 $F(A)$ 的三角模型，并由此证明当 $F=\bar F$ 时，$C(F(A))=\{f(t)\,|\,t\in C(A)\}$.

定理 12 设 A 是可析 Π_k 空间上的自共轭算子，$C(A)=\{0\}$, $r(0)=n$, 与 0 相应的根子空间非退化，T 是稠定闭算子，那末 $T\in\{A\}''$ 的充要条件是存在 $F\in\Omega_n$, 使 $T=F(A)$. 这里 $\{A\}''=\{T\,|$ 对任何满足 $BA\subset AB$ 的有界算子 B, 均有 $BT\subset TB\}$.