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On a Linear Equation Arising in Isometric Embedding of Torus-like Surface

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Abstract The solvability of a linear equation and the regularity of the solution are discussed. The equation is arising in a geometric problem which is concerned with the realization of Alexandroff's positive annul in R^3.

Keywords Alexandroff's positive annul, Weighted norm, Regularity
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1 Introduction

Let T be an annul, $T = \{(x_1, x_2) \mid x_1 \in [-\pi, \pi], x_2 \in [0, 1]\}$, and $(\vec{r}, g)$ is a smooth (analytic) nonnegative annul

$$\vec{r} : T \to R^3, \quad g = \mathrm{d}\vec{r}^2,$$

where $\vec{r}$, g are defined in T and satisfy Alexandroff's assumption:

$$\int_T K \mathrm{d}g = 4\pi \quad \text{and} \quad K = 0,\ \nabla K \neq 0 \quad \text{on } \partial T, \tag{1.1}$$

where K is Gaussian curvature. In what follows, we call the annul $\vec{r}$ Alexandroff's positive annul. And (1.1) is called Alexandroff condition.

In the perturbation of isometric embedding of $\mathrm{d}\vec{r}^2$ in R^3, $\mathrm{d}(\vec{r}+\vec{\tau})^2 = \mathrm{d}\vec{r}^2 + g$, the linearized problem is to find the deformation vector field equation

$$\mathrm{d}\vec{r} \cdot \mathrm{d}\vec{\tau} = q, \tag{1.2}$$

where q is a symmetric tensor of second order.

In the present paper, we will give the necessary and sufficient condition for equation (1.2) to be solvable. Set $\sigma_1 = S^1 \times \{0\}$, $\sigma_2 = S^2 \times \{1\}$, and

$$\oint_{\sigma_j} \frac{1}{\sqrt{|g|}} (\partial_1 q_{2i} - \partial_2 q_{1i} + \Gamma_{2i}^k q_{1k} - \Gamma_{1i}^k q_{2k}) \mathrm{d}x^i = 0, \quad j = 1, 2. \tag{1.3}$$

Denote

$$w(p) = \int_{p_0}^{p} \frac{2}{\sqrt{|g|}} (\partial_1 q_{2i} - \partial_2 q_{1i} + \Gamma_{2i}^k q_{1k} - \Gamma_{1i}^k q_{2k}) \mathrm{d}x^i,$$

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where p, $p_0 \in \sigma_j$. Then

$$\oint_{\sigma_j} \partial_i \vec{r}\Big(g^{ik}(q_{1k}\nu_2 - q_{2k}\nu_1) + \frac{1}{2} w\sqrt{|g|}\, g^{ik}\nu_k\Big) = 0, \tag{1.4}$$

where $\nu = (\nu_1, \nu_2)$ is the unit outwards normal of ∂T.

Theorem 1.1 *Let $\vec{r} \in C^\infty(T, R^3)$ be an Alexandroff's positive annul and $q \in C^\infty(T)$ a symmetric tensor of second order. Then the necessary and sufficient condition for* (1.2) *to admit a solution $\vec{\tau} \in C^\infty(T)$ is that* (1.3)–(1.4) *are valid. Moreover, the solution is unique up to a rigid motion of $\vec{r}$.*

In what follows, we will use the notation in [13].

Remark 1.1 In Theorem 1.1, if the given $\vec{r}$ and the tensor q are analytic, so is the solution $\vec{\tau}$.

2 Geometric and Analysis Preliminaries

Before solving equation (1.2), we need several lemmas.

Lemma 2.1 (see [1]) *If* (1.1) *is fulfilled, then each component of $\vec{r}(\partial T)$ is a planar curve l_i which is determined completely by its metric, and lies on the tangential plane of $\vec{r}(\partial T)P_i$.*

Lemma 2.2 (see [11]) *Let M be a nonnegative compact surface which is of no planar point, $\partial M = \cup\, l_i$, where each l_i is a planar curve, P_i is the plane tangential to M along l_i. Then M is infinitesimal rigid.*

Suppose that $\vec{r} : [-\pi, \pi] \times [0, 1]$ is a smooth isometric embedding of a positive annul which satisfies Alexandroff's assumption. In the geodesic coordinate based on $\partial T : t = 0$, the metric is of form:

$$g = \mathrm{d}t^2 + B^2 \mathrm{d}s^2, \quad (s, t) \in [-\pi, \pi] \times [0, 1],\ B(s, 0) = 1,\ B_t(s, 0) = k_g > 0, \tag{2.1}$$

where B is a smooth periodic function of s. Near $t = 1$, all arguments are similar, so ignored. Under the present circumstance $\Gamma^1_{22} = \Gamma^2_{22} = \Gamma^2_{12} = 0$, where 1 and 2 stand for s and t. Consider the Gauss-Codazzi system

$$\begin{aligned} \partial_2 L - \partial_1 M &= \Gamma^1_{12} L - \Gamma^1_{11} M - \Gamma^2_{11} N, \\ \partial_2 M - \partial_1 N &= -M\Gamma^1_{12}, \\ NL - M^2 &= KB^2 s, \end{aligned} \tag{2.2}$$

where

$$\Gamma^1_{11} = \frac{B_s}{B}, \quad \Gamma^1_{12} = \frac{B_t}{B}, \quad \Gamma^2_{11} = -BB_t.$$

Lemma 2.3 *Suppose that $\vec{r} : [-\pi, \pi] \times [0, 1]$ is a smooth isometric embedding of a positive annul which satisfies Alexandroff's assumption. Then the coefficients of the second fundamental form of $\vec{r}$, L, M and N satisfy*

$$L = M = 0, \quad \partial_t L = \sqrt{K_t B_t}, \quad N = \sqrt{\frac{K_t}{B_t}},$$

at $t = 0$ and $t = 1$.

Proof When $t=0$, $K=L=M=0$. Differentiation of the third expression of (2.2) with respect to t, combined with the first, yields

$$N\partial_t L=\partial_t K,\quad \partial_t L=B_t N,\quad \text{as } t=0.$$

Hence

$$N=\sqrt{\frac{K_t}{B_t}}\quad\text{and}\quad \partial_t L=\sqrt{K_tB_t}\,.$$

Since $L=M=0$ on $t=0$ and $t=1$, we have $\partial_1\vec{n}=0$, i.e., on δ_k, $\vec{n}$ is constant, where $\vec{n}$ is the normal of the surface $\vec{r}$.

Remark 2.1 In fact, at $t=0$ and $t=1$, all the coefficients of the second fundamental form h_{ij}, i, $j=1$, 2 and their derivatives of all orders are determined by the metric.

Lemma 2.4 *On* σ_k, $k=1,2$, *we have*

$$\oint_{\sigma_k}\sqrt{|g|}\,g^{2i}\partial_i\vec{r}=0.$$

Proof Set $\vec{\tau}=\vec{A}\times\vec{r}$, where $\vec{A}$ is an arbitrary constant vector. Then

$$\partial_1\vec{\tau}=\vec{A}\times\partial_1\vec{r}_1$$

and

$$\oint_{\sigma_k}\vec{\tau}_1=0.$$

We have

$$\partial_1\vec{\tau}=(g^{11}\partial_1\vec{\tau}\cdot\partial_1\vec{r}+g^{21}\partial_1\vec{\tau}\cdot\partial_2\vec{r})\partial_1\vec{r}+(g^{12}\partial_1\vec{\tau}\cdot\partial_1\vec{r}+g^{22}\partial_1\vec{\tau}\cdot\partial_2\vec{r})\partial_2\vec{r}+(\vec{n}\cdot\partial_1\vec{\tau})\vec{n},$$

i.e.,

$$\partial_1\vec{\tau}=\vec{A}\cdot\vec{n}\sqrt{g}\,g^{2i}\partial_i\vec{r}+(\vec{n}\cdot\partial_1\vec{\tau})\vec{n}.$$

Noting that $\vec{A}\cdot\vec{n}$ is a constant, we have

$$\vec{A}\cdot\vec{n}\oint_{\sigma_k}\sqrt{|g|}g^{2i}\partial_i\vec{r}+(\vec{n}\cdot\partial_1\vec{\tau})\vec{n}=0.$$

Projection to the plane where δ_k lies yields

$$\vec{A}\cdot\vec{n}\oint_{\sigma_k}\sqrt{|g|}\,g^{2i}\partial_i\vec{r}=0.$$

At the same time, we introduce a class of weighted Sobolev space $\widetilde{H}^1$. Denote by $\widetilde{C}^1(T)$ the space of all functions $u\in C^1(T)$ with u being constant on ∂T. Define a norm

$$\|u\|^2_{\widetilde{H}^1}=\int_T u_t^2+\frac{u_s^2}{\mu(t)}+u^2<\infty, \tag{2.3}$$

where $\mu(t)$ is a smooth function of t, as $0<t<1$, $\mu>0$; and near $t=0$, $\mu=t$; near $t=1$, $\mu=1-t$.

$\widetilde{H}^1$ is the completion of $\widetilde{C}^1(T)$ equipped with the norm defined above $\|\cdot\|_{\widetilde{H}^1}$. By Lemma 2.3, we have

$$\|u\|^2_{\widetilde{H}^1}\approx\int_T h^{ij}u_iu_j+u^2. \tag{2.4}$$

Lemma 2.5 *$\widetilde{H}^1$ is a Banach space, and $\widetilde{H}^1 \hookrightarrow H^1(T)$. By embedding theorem, the embedding $\widetilde{H}^1 \hookrightarrow L^2(T)$ is compact.*

Proof It suffices to prove that it is sequence compact. Assume that there exists a Cauchy sequence u^n in $\widetilde{H}^1$. Then u^n is a Cauchy sequence in H^1 too. Therefore there exists $u \in H^1$ such that $\|u^n - u\|_{H^1} \to 0$. Similarly, there exists $v \in L^2$ such that $\|\frac{u_s^n}{\sqrt{\mu}} - v\|_{L^2} \to 0$. In what follows we will illustrate $\sqrt{\mu}\, v = u_s$. By $\|u^n - u\|_{H^1} \to 0$ we have $\|u_s^n - u_s\|_{L^2} \to 0$. It implies from $\|\frac{u_s^n}{\sqrt{\mu}} - v\|_{L^2} \to 0$ that $\|u_s^n - \sqrt{\mu}\, v\|_{L^2} \to 0$. By the uniqueness of limit, it immediately implies $\sqrt{\mu}\, v = u_s$. It is easily checked that such $u \in \widetilde{H}^1$.

Lemma 2.6 *If $u \in H^1$ and $u = 0$ on $t = 0$, then $\widetilde{u} = \frac{u}{t} \in L^2$.*

Proof For any $u \in \widetilde{C}^1$, $\widetilde{u} = \int_0^1 u_t(x, \zeta t)\mathrm{d}\zeta$,

$$
\begin{aligned}
|\widetilde{u}|^2 &\le \Big(\int_0^1 |u_t(x,\zeta t)|\mathrm{d}\zeta\Big)^2 \\
&= \Big(\int_0^1 \zeta^{-\frac{1}{4}}\zeta^{\frac{1}{4}}|u_t(x,\zeta t)|\mathrm{d}\zeta\Big)^2 \\
&\le \int_0^1 \zeta^{-\frac{1}{2}}\mathrm{d}\zeta \int_0^1 \zeta^{\frac{1}{2}}|u_t(x,\zeta t)|^2\mathrm{d}\zeta \\
&= 2\int_0^1 \zeta^{\frac{1}{2}}|u_t(x,\zeta t)|^2\mathrm{d}\zeta.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|\widetilde{u}\|_0 &= \Big(\iint |\widetilde{u}|^2\Big)^{\frac{1}{2}} \le \sqrt{2}\Big(\int_0^1 \zeta^{\frac{1}{2}}\mathrm{d}\zeta \iint |u_t(x,\zeta t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
&= \sqrt{2}\Big(\int_0^1 \zeta^{-\frac{1}{2}}\mathrm{d}\zeta \int_{-\pi}^{\pi}\int_0^{\zeta} |u_t(x,z)|^2\mathrm{d}z\Big)^{\frac{1}{2}} \\
&\le 2\|u_t\|_0.
\end{aligned}
$$

Since $\widetilde{C}^1$ is dense in $\widetilde{H}^1$, the lemma is proved.

To discuss the regularity of solution later, we need two lemmas as follows. Consider a class of mixed type equations in 2-dimensional Euclidean space

$$Lu = a^{22}u_{tt} + 2a^{12}u_{xt} + a^{11}u_{xx} + b^1u_t + b^2u_x + cu = f. \tag{2.5}$$

Set $\phi = \det(a^{ij})$, $G = S^1 \times (-1, 1)$. Denote $\Sigma_0 = G \cap \{\phi = 0\}$, $G_+ = \overline{G} \cap \{\phi > 0\}$. Assume that

$$D\phi \ne 0, \quad a^{ij}\phi_i\phi_j = 0, \quad \text{on } \Sigma_0. \tag{2.6}$$

[5] defines a kind of characteristic numbers similar to Fichera numbers $D_\alpha\,(\alpha = 1, 2)$

$$
\begin{aligned}
D_1 &= \inf_{\Sigma_0}\Big(b^i\phi_i - \frac{1}{2}\frac{a_j^{ij}\phi_i\phi_j}{|\mathrm{d}\phi|^2}\Big), \\
D_2 &= \inf_{\Sigma_0}\Big(b^i\phi_i - \frac{3}{2}\frac{a_j^{ij}\phi_i\phi_j}{|\mathrm{d}\phi|^2}\Big),
\end{aligned}
$$

where $i = 1, 2$ stand for x and t. Moreover the signs of $D_\alpha\,(\alpha = 1, 2)$ are invariant under the regular transformation of variables. [6, 7] prove the H^1- (L^2-) hypoellipticity.

Lemma 2.7 *If the coefficients and f are C^∞, and $D_1 > 0$, then there is a neighborhood of $(x,t) \in \Sigma_0$, $o(x,t)$ such that any solution u in $H^1(o(x,t))$ is C^∞.*

3 Fundamental Equation

Having the geometric and analysis preliminaries, we will derive the fundamental equation. The process is due to Weyl [10] (or see [9, 11]). Here we use Yau's notation in [12] (also in [13]).

Choose the local coordinate (x_1, x_2) on T, and assume

$$q = q_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where $q_{ij} = q_{ji}$, and

$$\mathrm{d}\vec{r} = \partial_i\vec{r}\mathrm{d}x^i, \quad \mathrm{d}\vec{\tau} = \partial_i\vec{\tau}\mathrm{d}x^i.$$

We always denote $|g| = \det(g_{ij})$.

Then equation (1.2) can be rewritten as

$$\partial_1\vec{r}\cdot\partial_1\vec{\tau} = q_{11}, \tag{3.1}$$

$$\partial_1\vec{r}\cdot\partial_2\vec{\tau} + \partial_2\vec{r}\cdot\partial_1\vec{\tau} = 2q_{12}, \tag{3.2}$$

$$\partial_2\vec{r}\cdot\partial_2\vec{\tau} = q_{22}. \tag{3.3}$$

To solve (3.1)–(3.3), we shall introduce a new variable. Let $\vec{n}$ be the normal of the surface $\vec{r}$

$$u_i = \vec{n}\cdot\partial_i\vec{\tau}, \quad i = 1, 2 \tag{3.4}$$

and

$$w = \frac{1}{\sqrt{|g|}}(\partial_2\vec{r}\cdot\partial_1\vec{\tau} - \partial_1\vec{r}\cdot\partial_2\vec{\tau}). \tag{3.5}$$

Note that $u_i\mathrm{d}x_i = \vec{n}\cdot\mathrm{d}\vec{\tau}$ is a globally defined 1-form on T. We can check easily that w is a globally defined function on T. Moreover, w satisfies

$$\sqrt{|g|}\,w + 2q_{12} = 2\partial_2\vec{r}\cdot\partial_1\vec{\tau}, \tag{3.6}$$

$$\sqrt{|g|}\,w - 2q_{12} = -2\partial_1\vec{r}\cdot\partial_2\vec{\tau}. \tag{3.7}$$

Therefore $\{u_1, u_2, w\}$ determines $\{\partial_1\vec{\tau}, \partial_2\vec{\tau}\}$. In fact, by (3.1)–(3.3) and (3.6)–(3.7), we have

$$\begin{aligned}\partial_1\vec{\tau} &= g^{ij}q_{1j}\partial_i\vec{r} + \frac{1}{2}w\sqrt{|g|}\,g^{2i}\partial_i\vec{r} + u_1\vec{n},\\ \partial_2\vec{\tau} &= g^{ij}q_{2j}\partial_i\vec{r} - \frac{1}{2}w\sqrt{|g|}\,g^{1i}\partial_i\vec{r} + u_2\vec{n}.\end{aligned} \tag{3.8}$$

The functions u_1, u_2 and w will be new dependent variables and we proceed to find the differential equations which they satisfy. These could be derived as the integrable condition for these equations to be integrable.

Differentiating (3.7) with respect to x_1, we have

$$\sqrt{|g|}\,\partial_1 w + \partial_1\sqrt{|g|}\,w = 2\partial_1 q_{12} - 2\partial_{11}\vec{r}\cdot\partial_2\vec{\tau} - 2\partial_1\vec{r}\cdot\partial_{12}\vec{\tau}. \tag{3.9}$$

Differentiating (3.1) with respect to x_2, we get

$$\partial_{12}\vec{r}\cdot\partial_1\vec{\tau} + \partial_1\vec{r}\cdot\partial_{12}\vec{\tau} = \partial_2 q_{11}. \tag{3.10}$$

Combining (3.10) with (3.9), we obtain

$$\partial_1 w = \frac{1}{\sqrt{|g|}}(-\partial_1\sqrt{|g|}\,w + 2(\partial_1 q_{12} - \partial_2 q_{11}) + 2(\partial_{12}\vec{r}\cdot\partial_1\vec{\tau} - \partial_{11}\vec{r}\cdot\partial_2\vec{\tau})).$$

Similarly, we obtain

$$\partial_2 w = \frac{1}{\sqrt{|g|}}(-\partial_2\sqrt{|g|}\,w + 2(\partial_1 q_{22} - \partial_2 q_{12}) + 2(\partial_{22}\vec{r}\cdot\partial_1\vec{\tau} - \partial_{12}\vec{r}\cdot\partial_2\vec{\tau})).$$

Note that for any $i = 1, 2$,

$$\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}} = \Gamma^1_{1i} + \Gamma^2_{2i},$$

and (h^{ij}) is the inverse of (h_{ij}). Then a straightforward calculation, using the Gauss equation, gives

$$\begin{aligned}\partial_1 w &= -2K\sqrt{|g|}\,h^{2i}u_i + c_1,\\ \partial_2 w &= 2K\sqrt{|g|}\,h^{1i}u_i + c_2,\end{aligned} \tag{3.11}$$

where

$$c_i = \frac{2}{\sqrt{|g|}}(\partial_1 q_{2i} - \partial_2 q_{1i} + \Gamma^k_{2i}q_{1k} - \Gamma^k_{1i}q_{2k}). \tag{3.12}$$

Note that $c_i \mathrm{d}x^i$ is a globally defined 1-form in T. From (3.11), we can solve for u_1 and u_2 in terms of w and obtain

$$\begin{aligned}u_1 &= \frac{\sqrt{|g|}}{2}h^{2i}\partial_i w - \frac{\sqrt{|g|}}{2}h^{2i}c_i,\\ u_2 &= -\frac{\sqrt{|g|}}{2}h^{1i}\partial_i w + \frac{\sqrt{|g|}}{2}h^{1i}c_i.\end{aligned} \tag{3.13}$$

Hence, in order to solve for $\vec{\tau}$, we need only to solve for w. Now we derive an equation for w. Differentiating (3.4) appropriately and taking the difference, we get

$$\partial_1 u_2 - \partial_2 u_1 = \partial_1\vec{n}\cdot\partial_2\vec{\tau} - \partial_2\vec{n}\cdot\partial_1\vec{\tau}.$$

Using Weingarten equation, by a straightforward calculation, we obtain

$$\frac{1}{\sqrt{|g|}}(\partial_1 u_2 - \partial_2 u_1) = Hw + E, \tag{3.14}$$

where H is the mean curvature of the convex surface and E is given by

$$E = \frac{1}{\sqrt{|g|}}(h^i_2 q_{1i} - h^i_1 q_{2i}), \tag{3.15}$$

where

$$h^j_i = g^{jk}h_{ki}, \quad i, j = 1, 2.$$

The expression in the left-hand side of (3.14) and also E in (3.15) are globally defined in T.

We should note that (3.11) and (3.15) form the system for u_1, u_2 and w. (3.8) is an integrable system, and the integrable condition can be obtained by inserting (3.13) into (3.14):

$$-\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,h^{ij}\partial_j w) - 2Hw = -\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,h^{ij}c_j) + 2E. \tag{3.16}$$

Set

$$Lw = -\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,h^{ij}\partial_j w) - 2Hw.$$

The operator L is invariant under the change of coordinates and is elliptic in T and degenerate on ∂T. It is also formally self-adjoint.

4 The Solvability and Regularity

We will find the necessary condition of solvability.

The proof for necessity of Theorem 1.1 Under the coordinate of (2.1), by Lemma 2.3 and the first equation of (3.11), we have $\partial_1 w = c_1$ on σ_k. By periodicity, we have

$$\oint_{\sigma_k} c_1 = 0, \tag{4.1}$$

where σ_k is defined by $\sigma_1 = S^1 \times \{0\}$, $\sigma_2 = S^2 \times \{1\}$. Solving $\partial_1 w = c_1$ on σ_k, we see that w is unique up to a constant. By the first equation of (3.8), we have

$$\oint_{\sigma_k} \Big(g^{ij}q_{1j}\partial_i\vec{r} + \frac{1}{2}w\sqrt{|g|}\,g^{2i}\partial_i\vec{r} + u_1\vec{n}\Big) = 0.$$

Projection to the plane where δ_k lies yields

$$\oint_{\sigma_k} g^{ij}q_{1j}\partial_i\vec{r} + \frac{1}{2}w\sqrt{|g|}\,g^{2i}\partial_i\vec{r} = 0. \tag{4.2}$$

By Lemma 2.4,

$$\oint_{\sigma_k} \sqrt{|g|}\,g^{2i}\partial_i\vec{r} = 0.$$

(4.1)–(4.2) is the form of (1.3)–(1.4) under the coordinate (2.1).

By Lemma 2.2, we can obtain the uniqueness part of the theorem above.

The proof for sufficiency of Theorem 1.1 In what follows, we will prove the sufficiency part of the theorem. The proof is split into three steps:

(1) Introduce a boundary value problem for equation (3.16), and find the conjugate problem;

(2) Prove that the problem is of Fredholm type;

(3) Compute the kernel of conjugate problem, and prove that the right-hand side of (3.16) is perpendicular to the kernel which guarantees the existence of the solution, and the solution generates $\vec{\tau}$.

Here and thereafter, the integration on T is with respect to the metric g. Without loss of generality, suppose that $c_j = 0$ on σ_k, otherwise replace w with $w-\phi$, where ϕ is such a smooth function on σ_k that $\partial_j\phi = c_j$, $k, j = 1, 2$.

Step 1 Boundary value problem and adjoint problem

The boundary value problem

$$\begin{aligned} &Lw = F, \quad w \in \widetilde{H}^1, \\ &\oint_{\sigma_k} \sqrt{|g|}\,h^{ij}\partial_j w\nu_i = 0, \quad k = 1, 2, \end{aligned} \tag{4.3}$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward norm of ∂T. Here

$$F = -\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\, h^{ij}(c_j - \partial_j\phi)) + 2E - 2H\phi. \tag{4.4}$$

We claim that problem (4.3) is self-adjoint, i.e., its adjoint problem is

$$\begin{aligned} &Lu = G, \quad u \in \widetilde{H}^1, \\ &\oint_{\sigma_k} \sqrt{|g|}\, h^{ij}\partial_j u\nu_i = 0, \quad k = 1, 2. \end{aligned} \tag{4.5}$$

In fact, suppose $w, u \in C^2(T)$ and w, u are constants on ∂T, then integral by parts yields

$$\begin{aligned} \int_T Lw \cdot u - Lu \cdot w &= \int_T \frac{1}{\sqrt{|g|}}(\partial_i(\sqrt{|g|}\, h^{2j}\partial_j w \cdot u) - \partial_i(\sqrt{|g|}\, h^{2j}\partial_j u \cdot w) \\ &= \int_{\partial T}(\sqrt{|g|}\, h^{2j}\partial_j w \cdot u - \sqrt{|g|}\, h^{2j}\partial_j u \cdot w) \\ &= 0. \end{aligned}$$

Hence the adjoint problem of (4.3) is (4.5).

Step 2 Fredholm properties

In $\widetilde{H}^1$, we consider the bilinear form

$$Q(u, w) = \int_T h^{ij}\partial_j w\partial_i u + (\lambda - 2H)w \cdot u, \tag{4.6}$$

where $\lambda = 2\max\limits_{\overline{T}} H + 1$. Then

$$Q(u, u) \geq C\|u\|^2_{\widetilde{H}^1}. \tag{4.7}$$

The weak solution form of equation (4.3) is

$$Q(u, w) = \int_T fu, \quad \forall\, u \in \widetilde{H}^1, \tag{4.8}$$

where $f = F + \lambda w$. By Riesz representation theorem, for any $f \in L^2$, there exists a $w \in \widetilde{H}^1$, i.e., there exists a bounded operator $(L + \lambda)^{-1} : L^2 \mapsto \widetilde{H}^1$, and because $\widetilde{H}^1 \hookrightarrow L^2$ is compact embedding, $(L + \lambda)^{-1}$:$L^2 \mapsto L^2$ is a compact operator. Then $(L + \lambda)w - \lambda w = F$, i.e. $(I - \lambda(L + \lambda)^{-1})w = (L + \lambda)^{-1}F$. Hence L is of Fredholm type.

Step 3 The solvability

To prove that equation (4.3) is solvable, it suffices to verify that F is perpendicular to the kernel of its adjoint problem. Since the problem is self-adjoint, we need only to compute the kernel of (4.3). Recall that $q_{ij} = 0$, and

$$\oint_{\sigma_k} \sqrt{|g|}\, h^{ij}\partial_j w\nu_i = \oint_{\sigma_k} u_2\nu_1 + u_1\nu_2 = 0. \tag{4.9}$$

Then $\oint_{\sigma_k} \partial_i\vec{\tau}\mathrm{d}x^i = 0$, and the generated $\partial_1\vec{\tau}$ and $\partial_2\vec{\tau}$ by (3.8) could generate $\vec{\tau}$ by $\mathrm{d}\vec{\tau} = \partial_i\vec{\tau}\mathrm{d}x^i$. By Lemmas 2.1 and 2.2, $\vec{r}$ is infinitesimal rigid. Such a $\vec{\tau}$ must come from the rigid motion of $\vec{r}$:

$$\vec{\tau} = \vec{A} \times \vec{r} + \vec{B},$$

where $\vec{A}$ and $\vec{B}$ are constant vectors. By (3.5), we have

$$w = 2\vec{A} \cdot \frac{1}{\sqrt{|g|}}(\partial_1 \vec{r} \times \partial_2 \vec{r}) = 2\vec{A} \cdot \vec{n}.$$

Therefore, (4.3) possesses a solution w, if

$$\int_T \Big(-\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\, h^{ij}(c_j - \partial_j \phi)) + 2E - 2H\phi \Big) \vec{A} \cdot \vec{n} = 0$$

for any constant $\vec{A}$, or simply

$$\int_T \Big(-\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\, h^{ij}(c_j - \partial_j \phi)) + 2E - 2H\phi \Big) \cdot \vec{n} = 0. \tag{4.10}$$

Note that the expression in the parenthesis in (4.10) is an invariant scalar function on T. Hence we need to verify

$$\int_T h^{ij}(c_j - \partial_j \phi)\partial_i \vec{n} + (2E - 2H\phi)\vec{n} = 0. \tag{4.11}$$

By the Weingarten equation, we have

$$h^{ij} c_j \partial_i \vec{n} = -h^{ij} c_j h_i^k \partial_k \vec{r} = -h^{ij} c_j h_{il} g^{lk} \partial_k \vec{r} = -c_j g^{jk} \partial_k \vec{r}.$$

Hence, (4.11) becomes

$$\int_T -g^{ij}(c_j - \partial_j \phi)\partial_i \vec{r} + (2E - 2H\phi)\vec{n} = 0. \tag{4.12}$$

Inserting (3.12) into (4.12), we have

$$\begin{aligned} &\int_T -\frac{2}{\sqrt{|g|}} g^{ij}(\partial_1 q_{2j} - \partial_2 q_{1j})\partial_i \vec{r} - \frac{2}{\sqrt{|g|}} g^{ij}(\Gamma_{2j}^k q_{1k} - \Gamma_{1j}^k q_{2k})\partial_i \vec{r} + 2E\vec{n} \\ = &\int_T -g^{ij}\partial_j \phi \partial_i \vec{r} + 2H\phi \vec{n}. \end{aligned} \tag{4.13}$$

We concentrate on the first integral of the left-hand side of (4.13). An integration by parts shows

$$\begin{aligned} &-\int_T \frac{2}{\sqrt{|g|}} g^{ij}(\partial_1 q_{2j} - \partial_2 q_{1j})\partial_i \vec{r} \\ = &\int_T \frac{2}{\sqrt{|g|}}(\partial_1(g^{ij}\partial_i \vec{r})q_{2j} - \partial_2(g^{ij}\partial_i \vec{r})q_{1j}) + \int_{\partial T} 2g^{ij}\partial_i \vec{r}(q_{1j}\nu_2 - q_{2j}\nu_1) \\ = &\,\mathrm{I} + \int_{\partial T} 2g^{ij}\partial_i \vec{r}(q_{1j}\nu_2 - q_{2j}\nu_1), \end{aligned}$$

where

$$\mathrm{I} = \int_T \frac{2}{\sqrt{|g|}}((g^{ij}\partial_{1i} r)q_{2j} - g^{ij}\partial_{2i} r q_{1j}) + \int_T \frac{2}{\sqrt{|g|}}(\partial_1 g^{ij}\partial_i \vec{r} q_{2j} - \partial_2 g^{ij}\partial_i \vec{r} q_{1j}) \tag{4.14}$$

and

$$\int_T -g^{ij}\partial_j \phi \partial_i \vec{r} + 2H\phi \vec{n} = -\int_{\partial T} \sqrt{|g|}\, \phi g^{ij}\partial_i \vec{r}\nu_j.$$

With the Gauss equation and the identity

$$\partial_k g^{ij} = -g^{il}\Gamma^j_{kl} - g^{jl}\Gamma^i_{kl},$$

a straightforward calculation shows that the integral in (4.14) is just the opposite of the sum of the last integral in the right-hand side of (4.13). Hence (4.13) is equivalent to

$$\int_{\partial T} \frac{\sqrt{|g|}}{2}\phi g^{ij}\partial_i\vec{r}\nu_j + g^{ij}\partial_i\vec{r}(q_{1j}\nu_2 - q_{2j}\nu_1) = 0.$$

Noting the choice of ϕ, we see that the equation above is nothing else but (3.5).

Therefore, we have proven (4.10) and the existence of the solution. Such a solution w generates u_1 and u_2 by (3.11). By (3.8), they generate $\vec{\tau}_1$ and $\vec{\tau}_2$. In what follows, we prove

$$\oint_{\sigma_k} \partial_1\vec{\tau}\mathrm{d}x^1 + \partial_2\vec{\tau}\mathrm{d}x^2 = 0$$

in the coordinate of (2.3). In fact,

$$u_1 = \frac{\sqrt{|g|}}{2}h^{2i}\partial_i(w+\phi) - \frac{\sqrt{|g|}}{2}h^{2i}c_i$$

satisfies

$$\oint_{\sigma_k} u_1 = \oint_{\sigma_k} \sqrt{|g|}\,h^{ij}\partial_j\phi\nu_i = 0. \tag{4.15}$$

Therefore,

$$\begin{aligned}\oint_{\sigma_k} \partial_1\vec{\tau}\mathrm{d}x^1 + \partial_2\vec{\tau}\mathrm{d}x^2 &= \oint_{\sigma_k} \partial_1\vec{\tau}\\ &= \oint_{\sigma_k} \Big(g^{ij}q_{1j}\partial_i\vec{r} + \frac{1}{2}\phi\sqrt{|g|}g^{2i}\partial_i\vec{r} + u_1\vec{n}\Big)\\ &= \vec{n}\oint_{\sigma_k} u_1\\ &= 0,\end{aligned}$$

where we use (4.2). Thus, by $\mathrm{d}\vec{\tau} = \partial_i\vec{\tau}\mathrm{d}x^i$, $\vec{\tau} \in H^1$ is generated.

In what follows, we will use the results in [7, 8] to deal with the regularity of solution.

In the previous discussion, we use the geodesic coordinate based on any component σ_k of ∂T. Then the first fundamental form is

$$I = B^2\mathrm{d}s^2 + \mathrm{d}t^2.$$

(3.16) can be rewritten as

$$\begin{aligned}F = {} & \frac{1}{B}\Big(\partial_1\Big(B\frac{N}{KB^2}\partial_1 w\Big) - \partial_1\Big(B\frac{M}{KB^2}\partial_2 w\Big)\\ & -\partial_2\Big(B\frac{M}{KB^2}\partial_1 w\Big) + \partial_2\Big(B\frac{N}{KB^2}\partial_2 w\Big)\Big) + 2Hw.\end{aligned} \tag{4.16}$$

As we knew, the solution to (4.16) suffices $w \in \widetilde{H}^1$.

To improve the regularity, we introduce an auxiliary function $\widetilde{w}$, $w = K\widetilde{w} + \phi$, where $w = \phi$ on σ_k, $\phi \in C^\infty(T)$ or $\phi \in C^\omega$. We will illustrate the L^2-boundedness of $\widetilde{w}$ (see [7]). Because $K\widetilde{w} = w - \phi$, $\nabla K \neq 0$ and $\widetilde{w} \sim \frac{w-\phi}{t}$, by Lemma 2.6, we have $\widetilde{w} \in L^2$.

Next we consider the equation satisfied by $\widetilde{w}$:

$$Lw = L(K\widetilde{w} + \phi) = F, \tag{4.17}$$

i.e.,

$$\begin{aligned}\widetilde{F} = &\frac{N}{KB^2}(K\partial_{11}\widetilde{w} + 2\partial_1 K\partial_1\widetilde{w} + \widetilde{w}\partial_{11}K) + \frac{1}{B}\partial_1\Big(\frac{N}{KB}\Big)(K\partial_1\widetilde{w} + \widetilde{w}\partial_1 K)\\ &- 2\frac{M}{KB^2}(K\partial_{12}\widetilde{w} + \partial_1 K\partial_2\widetilde{w} + \partial_2 K\partial_1\widetilde{w} + \widetilde{w}\partial_{12}K)\\ &- \frac{1}{B}\Big(\partial_1\Big(\frac{M}{KB}\Big)(K\partial_2\widetilde{w} + \widetilde{w}\partial_2 K) + \partial_2\Big(\frac{N}{KB}\Big)(K\partial_1\widetilde{w} + \widetilde{w}\partial_1 K)\Big)\\ &+ \frac{L}{KB^2}(K\partial_{22}\widetilde{w} + 2\partial_2 K\partial_2\widetilde{w} + \widetilde{w}\partial_{22}K) + \frac{1}{B}\partial_2\Big(\frac{L}{KB}\Big)(K\partial_2\widetilde{w} + \widetilde{w}\partial_2 K) + 2KH\widetilde{w},\end{aligned}$$

where

$$\widetilde{F} = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,h^{ij}(c_j - \partial_j\phi)) - 2E + 2H\phi$$

is the function of (q_{ij}), ϕ and $\vec{r}$. Note that $\partial_1\phi = c_1$ on σ_k, and $\widetilde{F}$ is smooth or analytic if (q_{ij}), ϕ and $\vec{r}$ are smooth or analytic. Now we compute D_3 in (4.17), where D_3 is the characteristic number defined in [5]:

$$\begin{aligned}D_3 &= \inf_{\sigma_i}\Big(b^2 - \frac{3}{2}\partial_2 a_{22}\Big)\\ &= \inf_{\sigma_i}\Big(-2\frac{M\partial_1 K}{KB^2} - \frac{K}{B}\partial_1\Big(\frac{M}{KB}\Big) + 2\frac{L}{KB^2}\partial_2 K + \frac{K}{B}\partial_2\Big(\frac{L}{KB}\Big) - \frac{3}{2}\partial_2\Big(\frac{L}{B^2}\Big)\Big)\\ &= \inf_{\sigma_i}\frac{1}{2}\frac{\partial_2 L}{B^2}\\ &> 0,\end{aligned}$$

where we use Lemma 2.3.

By Lemmas 2.6 and 2.7, if (q_ij) and $\vec{r}$ are smooth, then $\widetilde{w} \in C^\infty$ and therefore $w \in C^\infty$; if q and $\vec{r}$ are analytic, then $w \in C^\omega$. Thus $\vec{\tau} \in C^\infty(C^\omega)$. We complete the proof of Theorem 1.1.

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