Chin. Ann. Math.
30B(2), 2009, 129–138
DOI: 10.1007/s11401-008-0253-0

Chinese Annals of Mathematics, Series B

Classification of Quasifinite Modules with Nonzero Central Charges for EALAs of Type A with Coordinates in Quantum Torus**

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Abstract The author first constructs a Lie algebra $\mathfrak{L} := \mathfrak{L}(q, w_d)$ from rank 3 quantum torus, which is isomorphic to the core of EALAs of type A_{d-1} with coordinates in quantum torus C_{q^d}, and then gives the necessary and sufficient conditions for the highest weight modules to be quasifinite. Finally the irreducible $\mathbb{Z}$-graded quasifinite $\mathfrak{L}$-modules with nonzero central charges are classified.

Keywords Core of EALAs, Graded modules, Quasifinite module, Highest weight module, Quantum torus
2000 MR Subject Classification 17B10, 17B60, 17B65, 17B68

1 Introduction

Extended affine Lie algebras (EALAs) are generalizations of affine Kac-Moody Lie algebras introduced in [1]. They were constructed and studied in many papers, for more details we refer the reader to, for example, [2–7]. The structure theory of the EALAs of type A_{d-1} is tied up with Lie algebra $\mathrm{gl}_d(\mathbb{C}) \otimes C_q$, where C_q is the quantum torus. Quantum torus defined in [8] are noncommutative analogue of Laurent polynomial algebras. The universal central extension of the derivation Lie algebra of rank 2 quantum torus is known as the q-analog Virasoro-like algebra (see [9]), which is studied in many papers (see [15–18]). Representations for Lie algebras coordinated by certain quantum tori have been studied by many people (see [10, 14]).

In this paper, we first construct a Lie algebra $\mathfrak{L}$ from rank 3 quantum torus, which is isomorphic to the core of EALAs of type A_{d-1} with coordinates in quantum torus C_{q^d}, and then give the necessary and sufficient conditions for the highest weight modules to be quasifinite. Finally, the irreducible $\mathbb{Z}$-graded quasifinite modules of $\mathfrak{L}$ with nonzero central charges are classified. The results generalize those in [23] from $d = 2$ to arbitrary $d \geq 2$.

This paper is organized as follows. In Section 2, we first recall some concepts and results about quantum torus in [2, 20, 23], then introduce the $\mathbb{Z}^2$-extragraded Lie algebras $\widetilde{L}$ and give the necessary and sufficient conditions for the highest weight modules to be quasifinite. And the necessary and sufficient conditions for the $\mathbb{Z}$-graded highest weight $\mathfrak{L}$-modules to be quasifinite

Manuscript received June 29, 2008. Published online February 18, 2009.
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**Project supported by the Post Doctorate Research Grant from the Ministry of Science and Technology of China (No. 20060390526) and the National Natural Science Foundation of China (No. 10601057).

are obtained in Theorem 2.3. In Section 3, we give the classification of irreducible $\mathbb{Z}$-graded quasifinite $\mathfrak{L}$-modules with nonzero central charges in Theorem 3.2.

2 Highest Weight Quasifinite Modules over $\mathfrak{L}(q, w_d)$

Through this paper, we use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ to denote the sets of complex numbers, integers, nonnegative integers and positive integers respectively. For any additive group G, we denote $G^* := G \setminus \{0\}$. For any Lie algebra L, denote its center by $Z(L)$, and $L' := [L, L]$. For any set S, we define the Kronecker delta

$$\delta_{x,S} = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For any $m, n \in \mathbb{N}$, denote the maximal common factor of m, n by $\langle m, n\rangle$.

2.1 Basics

Let $\mathcal{Q} = (q_{i,j})_{i,j=1}^n$ be an $n \times n$ matrix over $\mathbb{C}$ satisfying

$$q_{i,i} = 1, \quad q_{i,j} = q_{j,i}^{-1}, \tag{2.2}$$

where n is a positive integer. The $\mathcal{Q}$-quantum torus $C_\mathcal{Q}$ is the unital associative algebra over $\mathbb{C}$ generated by $t_1^{\pm 1}, \cdots, t_n^{\pm 1}$ and subject to the defining relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i t_j = q_{i,j} t_j t_i. \tag{2.3}$$

For any $\mathbf{m} \in \mathbb{Z}^n$, we always write

$$\mathbf{m} = (m_1, \cdots, m_n), \quad t^a = t_1^{m_1} \cdots t_n^{m_n}. \tag{2.4}$$

For any $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^n$, we define the functions $\sigma_\mathcal{Q}(\mathbf{m}, \mathbf{n})$ and $f_\mathcal{Q}(\mathbf{m}, \mathbf{n})$ by

$$t^{\mathbf{m}} t^{\mathbf{n}} = \sigma_\mathcal{Q}(\mathbf{m}, \mathbf{n}) t^{\mathbf{m}+\mathbf{n}}, \quad t^{\mathbf{m}} t^{\mathbf{n}} = f_\mathcal{Q}(\mathbf{m}, \mathbf{n}) t^{\mathbf{n}} t^{\mathbf{m}}. \tag{2.5}$$

Then

$$\sigma_\mathcal{Q}(\mathbf{m}, \mathbf{n}) = \prod_{1 \le i < j \le n} q_{j,i}^{m_j n_i}, \quad f_\mathcal{Q}(\mathbf{m}, \mathbf{n}) = \prod_{i,j=1}^{n} q_{j,i}^{m_j n_i}, \tag{2.6}$$

and $f_\mathcal{Q}(\mathbf{m}, \mathbf{n}) = \sigma_\mathcal{Q}(\mathbf{m}, \mathbf{n}) \sigma_\mathcal{Q}(\mathbf{n}, \mathbf{m})^{-1}$. We define

$$\mathrm{rad} f_\mathcal{Q} = \{\mathbf{m} \in \mathbb{Z}^n \mid f_\mathcal{Q}(\mathbf{m}, \mathbb{Z}^n) = 1\}. \tag{2.7}$$

For the properties of $C_\mathcal{Q}$, $f_\mathcal{Q}$ and $\sigma_\mathcal{Q}$, please refer to [2].

In the case $\mathcal{Q} = \left(\begin{smallmatrix} 1 & q^{-1} \\ q & 1 \end{smallmatrix}\right)$, we will simply denote $C_\mathcal{Q}$, $f_\mathcal{Q}$ and $\sigma_\mathcal{Q}$ by C_q, f_q and σ_q respectively.

Let w_d be a d-th primitive root of unity with $d \geq 2$, q a fixed generic complex number, and

$$\mathcal{Q}_d := \begin{pmatrix} 1 & q^{-1} & 1 \\ q & 1 & w_d^{-1} \\ 1 & w_d & 1 \end{pmatrix}. \tag{2.8}$$

Then $Z(C_{\mathcal{Q}_d}) = \mathbb{C}[t_3^d, t_3^{-d}]$. Let J be the idea of the associative algebra $C_{\mathcal{Q}_d}$ generated by $t_3^d - 1$. Denote $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. Define

$$\overline{C}_{\mathcal{Q}_d} = C_{\mathcal{Q}_d}/J = \mathrm{span}_{\mathbb{C}}\{t_1^i t_2^j t_3^k \mid i, j \in \mathbb{Z}, k \in \mathbb{Z}_d\} \tag{2.9}$$

to be the quotient of $C_{\mathcal{Q}_d}$ and identify t_3^k with its image in $\overline{C}_{\mathcal{Q}_d}$.

The derived Lie subalgebra of $\overline{C}_{\mathcal{Q}_d}$ is

$$\overline{C}'_{\mathcal{Q}_d} = \mathrm{span}_{\mathbb{C}}\{t_1^i t_2^j t_3^k \mid (i, j, k) \in (\mathbb{Z}^2 \times \mathbb{Z}_d)^*\}. \tag{2.10}$$

Let $M_d(C_{q^d})$ be the set of $d \times d$ matrices over C_{q^d}, and E_{ij} be the $d \times d$ matrix whose entry is 1 for the (i, j)-entry and 0 elsewhere.

We have the following proposition from a manuscript of K. M. Zhao.

Proposition 2.1 (a) *$\overline{C}_{\mathcal{Q}_d}$ is a simple associative algebra, and $\overline{C}'_{\mathcal{Q}_d}$ is a simple Lie algebra.*
(b) *$\overline{C}_{\mathcal{Q}_d} \cong M_d(C_{q^d})$ as associative algebras.*

Proof (a) Suppose that H is a nonzero associative ideal of $\overline{C}_{\mathcal{Q}_d}$. We want to show that $H = \overline{C}_{\mathcal{Q}_d}$. Take a nonzero element

$$x = \sum_{i=1}^{r} x_i t_1^{a_i} t_2^{b_i} t_3^{c_i} \in H,$$

where $x_i \in \mathbb{C}^*$ and $(a_i, b_i, c_i) \in \mathbb{Z}^2 \times \mathbb{Z}_d$ are pairwise distinct for $i = 1, \cdots, r$.

We may assume that r is minimal. If $r = 1$, clearly $H = \overline{C}_{\mathcal{Q}_d}$. Now assume that $r > 1$. Without loss of generality, we may also assume that $(a_r, b_r, c_r) = (0, 0, \overline{0})$. Then there exists $(a, b, c) \in \mathbb{Z}^2 \times \mathbb{Z}_d$ such that $0 \neq [t_1^a t_2^b t_3^c, t_1^{a_1} t_2^{b_1} t_3^{c_1}] \in \overline{C}_{\mathcal{Q}_d}$. Now

$$0 \neq [t_1^a t_2^b t_3^c, x] = \sum_{i=1}^{r-1} x_i [t_1^a t_2^b t_3^c, t_1^{a_i} t_2^{b_i} t_3^{c_i}] \in H.$$

This is in contradiction with the choice of x. Thus $\overline{C}_{\mathcal{Q}_d}$ is simple.

Now by using Herstein's theorem in [19] and $\overline{C}'_{\mathcal{Q}_d} \cap Z(C_{\mathcal{Q}_d}) = \{0\}$, we see that $\overline{C}'_{\mathcal{Q}_d}$ is a simple Lie algebra.

(b) Define an associative algebra embedding $\varphi_1 : \overline{C}_{\mathcal{Q}_d} \to M_d(C_q)$ by

$$\varphi_1(t_1^i t_2^j t_3^k) = t_1^i t_2^j F^j E^k, \tag{2.11}$$

where $E = \sum\limits_{i=1}^{d} w_d^{-i} E_{i,i}$, $F = E_{d,1} + \sum\limits_{i=1}^{d-1} E_{i,i+1} \in M_d(\mathbb{C})$. Then $\varphi_1(\overline{C}_{\mathcal{Q}_d})$ is spanned by

$$E_{i,j}(t_1^m t_2^{dk+j-i}), \quad 1 \leq i, j \leq d, k, m \in \mathbb{Z}. \tag{2.12}$$

Now, we define the associative algebra isomorphism

$$\varphi_2 : \varphi_1(\overline{C}_{\mathcal{Q}_d}) \to M_d(\mathbb{C}[t_1^{\pm 1}, t_2^{\pm d}]) \subset M_d(C_q), \quad E_{i,j}(t_1^m t_2^{dk+j-i}) \mapsto q^{im} E_{i,j}(t_1^m t_2^{dk}).$$

All the verifications are straightforward.

Now we construct our Lie algebra as a central extension of $\overline{C}'_{\mathcal{Q}_d}$, which will be denoted by $\mathfrak{L} := \mathfrak{L}(q, w_d) = \overline{C}'_{\mathcal{Q}_d} + \mathrm{span}_{\mathbb{C}}\{c_1, c_2\}$, with the following Lie bracket

$$[t^{\mathbf{m}}t_3^i, t^{\mathbf{n}}t_3^j] = (w_d^{n_2 i}q^{m_2 n_1} - w_d^{m_2 j}q^{m_1 n_2})t^{\mathbf{m}+\mathbf{n}}t_3^{i+j} + \delta_{\mathbf{m}+\mathbf{n},0}\delta_{i+j,\bar{0}}w_d^{n_2 i}q^{m_2 n_1}(m_1 c_1 + m_2 c_2), \quad (2.13)$$

where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$, $i, j \in \mathbb{Z}_d$.

From Proposition 2.1(b), we may easily deduce that $\mathfrak{L}$ is isomorphic to the core of the EALAs of type A_{d-1} with coordinates in C_{q^d}.

Next, we will recall some concepts about the $\mathbb{Z}$-graded $\mathfrak{L}$-modules in [23].

Fix a $\mathbb{Z}$-basis

$$\mathbf{m}_1 = (m_{11}, m_{12}), \quad \mathbf{m}_2 = (m_{21}, m_{22}) \in \mathbb{Z}^2. \quad (2.14)$$

If we define the degree of the nonzero elements in $\mathrm{span}_{\mathbb{C}}\{t^{i\mathbf{m}_1 + j\mathbf{m}_2}t_3^k \in \mathfrak{L} \mid i, j \in \mathbb{Z}, k \in \mathbb{Z}_d\}$ to be i and the degree of the nonzero elements in $\mathbb{C}c_1 + \mathbb{C}c_2$ to be zero, then $\mathfrak{L}$ can be regarded as a $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{L}_i = \mathrm{span}_{\mathbb{C}}\{t^{i\mathbf{m}_1 + j\mathbf{m}_2}t_3^k \in \mathfrak{L} \mid j \in \mathbb{Z}, k \in \mathbb{Z}_d\} + \delta_{i,0}(\mathbb{C}c_1 + \mathbb{C}c_2). \quad (2.15)$$

Set

$$\mathfrak{L}_+ = \bigoplus_{i>0}\mathfrak{L}_i, \quad \mathfrak{L}_- = \bigoplus_{i<0}\mathfrak{L}_i.$$

Then $\mathfrak{L} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{L}_i$ and $\mathfrak{L}$ has the following triangular decomposition

$$\mathfrak{L} = \mathfrak{L}_- \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_+.$$

Definition 2.1 *For any $\mathfrak{L}$-module V, if $V = \bigoplus_{m\in\mathbb{Z}} V_x$ with $\mathfrak{L}_i V_m = V_{m+i}$, $\forall\, i, m \in \mathbb{Z}$, then V is called a $\mathbb{Z}$-graded $\mathfrak{L}$-module (w.r.t $(\mathbf{m}_1, \mathbf{m}_2)$) and V_m is called a homogeneous subspace of V with degree m. The $\mathfrak{L}$-module V is called*

(i) *a quasi-finite $\mathbb{Z}$-graded module, if $\dim V_m < \infty$, $\forall\, m \in \mathbb{Z}$,*

(ii) *a uniformly bounded module, if there exists some $N \in \mathbb{N}$, such that $\dim V_m < N$, $\forall\, m \in \mathbb{Z}$,*

(iii) *a highest (resp. lowest) weight module, if there exists a nonzero homogeneous vector $v \in V_m$, such that V is generated by v and $\mathfrak{L}_+ v = 0$ (resp. $\mathfrak{L}_- v = 0$),*

(iv) *a generalized highest weight module with highest degree m, if there exist a $\mathbb{Z}$-basis $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{Z}^2$ and a nonzero vector $v \in V_m$, such that V is generated by v and $t^{\mathbf{m}}t_3^i v = 0$, $\forall\, \mathbf{m} \in \mathbb{Z}_+\mathbf{b}_1 + \mathbb{Z}_+\mathbf{b}_2, i \in \mathbb{Z}_d$,*

(v) *an irreducible $\mathbb{Z}$-graded module, if V does not have any nontrivial $\mathbb{Z}$-graded submodule.*

We denote the set of quasi-finite irreducible $\mathbb{Z}$-graded $\mathfrak{L}$-modules by $\mathscr{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$.

From the definition, one sees that the generalized highest weight modules contain the highest weight modules and the lowest weight modules as their special cases. As the central elements c_1, c_2 of $\mathfrak{L}$ act on irreducible graded modules V as scalars, we shall use the same symbols to denote these scalars.

2.2 Finite dimensional irreducible modules over $\mathfrak{L}_0$

Now we study the finite dimensional irreducible modules over $\mathfrak{L}_0$. Note that by the theory of Verma modules, the irreducible $\mathbb{Z}$-graded highest (or lowest) weight $\mathfrak{L}$-modules are classified by the irreducible modules of $\mathfrak{L}_0$.

Recall $\mathfrak{L}_0 = \mathrm{span}_{\mathbb{C}}\{t^{j\mathbf{m}_2}t_3^k, c_1, c_2 \mid (j,k) \in (\mathbb{Z}\times\mathbb{Z}_d)^*\}$.

Clearly

$$\mathfrak{L}_0' = \mathrm{span}_{\mathbb{C}}\Big\{t^{k\mathbf{m}_2}t_3^i, m_{21}c_1 + m_{22}c_2 \,\Big|\, (k,i) \notin \frac{d}{\langle d, m_{22}\rangle}(\mathbb{Z}\times\mathbb{Z}_d)\Big\}. \tag{2.16}$$

Denote

$$H_0 = \mathrm{span}_{\mathbb{C}}\Big\{t^{k\mathbf{m}_2}t_3^i, c_1, c_2 \,\Big|\, (k,i) \in \frac{d}{\langle d, m_{22}\rangle}(\mathbb{Z}\times\mathbb{Z}_d)\Big\}. \tag{2.17}$$

Clearly H_0 is an ideal of $\mathfrak{L}_0$, and we have

$$\mathfrak{L}_0 = H_0 + \mathfrak{L}_0'. \tag{2.18}$$

Suppose that A is an arbitrary finite dimensional irreducible module over $\mathfrak{L}_0$. Then c_1, c_2 act as scalars on A, and considering the action of the three dimensional Heisenberg Lie subalgebra $\mathrm{span}_{\mathbb{C}}\{t^{\mathbf{m}_2}, t^{-\mathbf{m}_2}, m_{21}c_1 + m_{22}c_2\}$, we have $(m_{21}c_1 + m_{22}c_2)A = 0$. Hence, we can regard A as a module over $\mathfrak{L}_0/(\mathbb{C}(m_{21}c_1+m_{22}c_2))$. Noting that $H_0/(\mathbb{C}(m_{21}c_1+m_{22}c_2)) = Z(\mathfrak{L}_0/(\mathbb{C}(m_{21}c_1+m_{22}c_2))$, we see that elements in H_0 act as scalars on A and A is an irreducible $\mathfrak{L}_0'/(\mathbb{C}(m_{21}c_1+m_{22}c_2))$ module.

Define

$$\phi_A : H_0 \to \mathbb{C} \tag{2.19}$$

by $\phi(x)v = xv$ for all $x \in H_0$ and $v \in A$.

Make A to be an $\mathfrak{L}_0 + \mathfrak{L}_+$-module by defining $\mathfrak{L}_+ A = 0$. Then we have the highest Verma $\mathfrak{L}$-module

$$\widetilde{M}^+(A; \mathbf{m}_1, \mathbf{m}_2) = \mathrm{Ind}_{\mathfrak{L}_0+\mathfrak{L}_+}^{\mathfrak{L}} A. \tag{2.20}$$

$\widetilde{M}^+(A;\mathbf{m}_1,\mathbf{m}_2)$ has a unique maximal $\mathbb{Z}$-graded $\mathfrak{L}$ proper submodule $\widetilde{M}^{+'}(A;\mathbf{m}_1,\mathbf{m}_2)$, and the unique irreducible $\mathbb{Z}$-graded quotient module

$$M^+(A;\mathbf{m}_1,\mathbf{m}_2) := \widetilde{M}^+(A;\mathbf{m}_1,\mathbf{m}_2)/\widetilde{M}^{+'}(A;\mathbf{m}_1,\mathbf{m}_2). \tag{2.21}$$

Similarly, we have the lowest Verma module $\widetilde{M}^-(A;\mathbf{m}_1,\mathbf{m}_2)$ and the irreducible $\mathbb{Z}$-graded quotient module $M^-(A;\mathbf{m}_1,\mathbf{m}_2)$.

Let us recall the properties of finite dimensional irreducible modules in [20].

Theorem 2.1 (see [20]) (a) *Let $F(X) = (X-x_1)\cdots(X-x_r)$, $G(X) = (X-y_1)\cdots(X-y_s) \in \mathbb{C}^* + X\mathbb{C}[X]$, and*

$$I'(F,G) := \mathrm{span}_{\mathbb{C}}\{t^{\mathbf{n}}F(t_1^m), t^{\mathbf{n}}G(t_2^m) \mid \mathbf{n} \in \mathbb{Z}^2 \setminus (m\mathbb{Z}^2)\} \subset C_{w_m}, \tag{2.22}$$

where w_m is an m-th primitive root of unity with $m > 1$. Then the quotient Lie algebra $C_{w_m}'/I'(F,G) \cong \oplus^{rs}\mathrm{sl}_m(\mathbb{C})$ if and only if $(F,F') = (G,G') = 1$.

(b) *Let A be an irreducible finite dimensional module over the Lie algebra C_{w_m} with $\dim A > 1$. Then there exist nonzero polynomials $F(X), G(X) \in \mathbb{C}^* + X\mathbb{C}[X]$ with $(F,F') = (G,G') = 1$ such that $I'(F,G) \subset \mathrm{Ann}\, A$, and*

(1) *A is an irreducible module over $C'_{w_m}/I'(F,G)$,*

(2) *elements in $Z(C_{w_m})$ act as scalars on A.*

Proof (a) and (b) are Lemma 2.5 and Theorem 3.2 in [20] respectively.

Corollary 2.1 *Suppose that $\langle d, m_{22}\rangle \neq d$, and A is an irreducible finite dimensional module over $\mathfrak{L}_0$ with $\dim A > 1$. Then there exists a nonconstant polynomial $F(X) \in \mathbb{C}^* + X\mathbb{C}[X]$ with $(F, F') = 1$ such that*

(1) $(m_{21}c_1 + m_{22}c_2)A = 0$,

(2) *A is an irreducible module over the semisimple finite dimensional algebra $\mathfrak{L}'_0/(I'(F) + \mathbb{C}(m_{21}c_1 + m_{22}c_2))$ (which is isomorphic to some direct sums of $\mathrm{sl}_{\frac{d}{\langle d,m_{22}\rangle}}(\mathbb{C})$), where*

$$I'(F) := \mathrm{span}_{\mathbb{C}}\Big\{F((t^{\mathbf{m}_2})^{\frac{d}{\langle d,m_{22}\rangle}})t^{k\mathbf{m}_2}t_3^i \,\Big|\, (k,i) \notin \frac{d}{\langle d, m_{22}\rangle}(\mathbb{Z}\times\mathbb{Z}_d)\Big\}, \tag{2.23}$$

(3) *elements in H_0 act on A as scalars.*

2.3 Highest weight modules over $\mathbb{Z}^2$-extragraded Lie algebra $\widetilde{L}$

Let us recall a $\mathbb{Z}^2$-extragraded Lie algebra in [22] for some special case.

Denote

$$\mathcal{Q}' = (q'_{i,j}) := \begin{pmatrix} 1 & q^{m_{12}m_{21}-m_{11}m_{22}} & w_d^{-m_{12}} \\ q^{-m_{12}m_{21}+m_{11}m_{22}} & 1 & w_d^{-m_{22}} \\ w_d^{m_{12}} & w_d^{m_{22}} & 1 \end{pmatrix},$$

where $\mathbf{m}_1, \mathbf{m}_2$ are defined in (2.14) and q is generic.

We have an associative algebra isomorphism

$$\rho : C_{\mathcal{Q}'} \to C_{\mathcal{Q}_d} \tag{2.24}$$

with $\rho(t_3) = t_3$ and $\rho(t_i) = t^{\mathbf{m}_i}$ for $i = 1, 2$. Further, we have

$$\rho(t_1^i t_2^j t_3^k) = q^{\frac{m_{11}m_{12}i(i-1)+m_{21}m_{22}j(j-1)}{2}+ijm_{12}m_{21}}t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k. \tag{2.25}$$

Define the Lie algebra $\widetilde{L} := \widetilde{L}_{\mathcal{Q}'}$ with $\widetilde{L} = C_{\mathcal{Q}'}$ as vector space and the relations

$$\begin{aligned}[t^{\mathbf{a}}, t^{\mathbf{b}}] &= t^{\mathbf{a}}t^{\mathbf{b}} - t^{\mathbf{b}}t^{\mathbf{a}} + \delta_{a_1+b_1,0}\delta_{\mathbf{a}+\mathbf{b},\mathrm{rad}f_{\mathcal{Q}'}}a_1 t^{\mathbf{a}}t^{\mathbf{b}} \\ &= \sigma_{\mathcal{Q}'}(\mathbf{a},\mathbf{b})(1 - f_{\mathcal{Q}'}(\mathbf{b},\mathbf{a}) + \delta_{a_1+b_1,0}\delta_{\mathbf{a}+\mathbf{b},\mathrm{rad}f_{\mathcal{Q}'}}a_1)t^{\mathbf{a}+\mathbf{b}}, \quad \forall\, \mathbf{a},\mathbf{b}\in\mathbb{Z}^3.\end{aligned} \tag{2.26}$$

Now we need to recall some notations in [21].

Definition 2.2 (a) *The algebra of exp-polynomial functions in r' variables $m_1, m_2, \cdots, m_{r'}$ is the algebra of functions $f(m_1,\cdots,m_{r'}) : \mathbb{Z}^{r'} \to \mathbb{C}$ generated as an algebra by functions m_j and a^{m_j} for various constants $a \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $j = 1,\cdots,r'$.*

(b) *Let $G = \bigoplus_{(i,\mathbf{a})\in\mathbb{Z}^{n+1}} G_{i,\mathbf{a}}$ be any $\mathbb{Z}^{n+1}$-graded Lie algebra, $\mathcal{K} = \{K_i \mid i \in \mathbb{Z}\}$ be a family of finite sets and*

$$\mathcal{B} = \{g_i^{(k_i)}(\mathbf{a}) \mid k_i \in K_i, (i,\mathbf{a}) \in \mathbb{Z}^{n+1}\} \tag{2.27}$$

be any homogenous spanning set of G with $g_i^{(k_i)}(\mathbf{a}) \in G_{i,\mathbf{a}}$. Then G is said to be a $\mathbb{Z}^n$-extragraded Lie algebra with respect to $\mathcal{K}$ and $\mathcal{B}$, if there exists a family of exp-polynomial *functions* $\{f_{i,j,i+j}^{k_i,k_j,k_{i+j}}(\mathbf{a},\mathbf{b})\}$ *in the* $2n$ *variables* a_l, b_l, $l=1,2,\cdots,n$, *where* $k_i \in K_i$, $\forall\, i\in\mathbb{Z}$, *such that*

$$[g_i^{(k_i)}(\mathbf{a}), g_j^{(k_j)}(\mathbf{b})] = \sum_{k_{i+j}\in K_{i+j}} f_{i,j,i+j}^{k_i,k_j,k_{i+j}}(\mathbf{a},\mathbf{b}) g_{i+j}^{(k_{i+j})}(\mathbf{a}+\mathbf{b}). \tag{2.28}$$

(c) *Let* G *be a* $\mathbb{Z}^n$*-extragraded Lie algebra w.r.t* $\mathcal{B}$ *and* $\mathcal{K}$ *as defined in* (b), *and* $G_0 := \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} G_{0,\mathbf{a}}$. *A finite dimensional nonzero module* A *over* G_0 *is called an exp-polynomial* G_0*-module, if there exists a basis* $\{v_i\}_{i\in J}$, *and there exists a family of* n*-variable exp-polynomial functions* $h_{s,j}^k(\mathbf{a})$ *for* $k\in K_0$, $j,s\in J$, *such that*

$$g_0^k(\mathbf{a})v_j = \sum_{j\in J} h_{s,j}^k(\mathbf{a}) v_s.$$

Lemma 2.1 (see [22]) *Suppose that* $\psi:\mathbb{Z}^n\to\mathbb{C}$ *is a function,* $h_i(t) = \sum_{j=0}^{m_i} x_{i,j}t^j = \prod_{j=1}^{l_i}(t-y_{i,j})^{s_{i,j}}$, $i=1,\cdots,n$, *are polynomials in* $\mathbb{C}[t]$, *where* $s_{i,j}, m_i\in\mathbb{N}$, *and* $x_{i,j}, y_{i,j}\in\mathbb{C}$ *with* $x_{i,0}x_{i,m_i}\neq 0$. *For* $k=1,2,\cdots,n$, *let*

$$\begin{aligned}\mathcal{F}_k &= \{f_{k,0}(r), f_{k,1}(r),\cdots, f_{k,m_k-1}(r)\}\\ &:= \{y_{k,1}^r, ry_{k,1}^r,\cdots, r^{s_{k,1}-1}y_{k,1}^r; y_{k,2}^r,\cdots, r^{s_{k,2}-1}y_{k,2}^r;\cdots; y_{k,l_k}^r, ry_{k,l_k}^r,\cdots, r^{s_{k,l_k}-1}y_{k,l_k}^r\}\end{aligned}$$

be a set of functions in $r\in\mathbb{Z}$. *Then*

$$\sum_{j=0}^{m_i} x_{i,j}\psi(\mathbf{a}+j\overline{\varepsilon_i}) = 0, \quad \forall\,\mathbf{a}\in\mathbb{Z}^n,\ i=1,2,\cdots,n, \tag{2.29}$$

if and only if there exist $\prod_{i=1}^n m_i$ *complex numbers* $z_{(b_1,\cdots,b_n)}$, $0\le b_i\le m_i-1$, $i=1,\cdots,n$, *such that*

$$\psi(\mathbf{a}) = \sum_{b_1=0}^{m_1-1}\cdots\sum_{b_n=0}^{m_n-1} z_{(b_1,\cdots,b_n)}\prod_{i=1}^n f_{i,b_i}(a_i), \quad \forall\,\mathbf{a}\in\mathbb{Z}^n. \tag{2.30}$$

Lemma 2.2 (see [22]) $\widetilde{L}_{\mathcal{Q}'}$ *is a* $\mathbb{Z}^2$*-extragraded Lie algebra with respect to* $\mathcal{K}$ *and* $\mathcal{B}$, *where*

$$\mathcal{K} = \{K_i \mid i\in\mathbb{Z}\}, \quad \mathcal{B} = \{g_i^{(k_i)}(\mathbf{a}) \mid k_i\in K_i, (i,\mathbf{a})\in\mathbb{Z}^3\}, \tag{2.31}$$

$$K_0 = \{1,2\} \text{ and } K_i = \{1\}, \quad \forall\, i\neq 0, \tag{2.32}$$

$$g_0^{(1)}(\mathbf{a}) = \delta_{\mathbf{a},\frac{d}{\langle d,m_{22}\rangle}\mathbb{Z}^2}(1-q^{(m_{11}m_{22}-m_{12}m_{21})a_1}w_d^{m_{12}a_2} + \delta_{a_1,0}\delta_{a_2,d\mathbb{Z}})t^{(0,\mathbf{a})}, \tag{2.33}$$

$$g_0^{(2)}(\mathbf{a}) = (1-\delta_{\mathbf{a},\frac{d}{\langle d,m_{22}\rangle}\mathbb{Z}^2})t^{(0,\mathbf{a})}, \tag{2.34}$$

$$g_i^{(1)}(\mathbf{a}) = t^{(i,\mathbf{a})}, \quad \forall\, i\neq 0. \tag{2.35}$$

$\widetilde{L}$ has a natural $\mathbb{Z}$-gradation with $\widetilde{L}_i = \mathrm{span}_{\mathbb{C}}\{t_1^i t_2^j t_3^k \mid (j,k)\in\mathbb{Z}^2\}$. Similarly, we have the notations of $\mathbb{Z}$-graded modules and quasi-finite modules over $\widetilde{L}$. And for any irreducible module A over $\widetilde{L}_0$, make A to be an $\widetilde{L}_0+\widetilde{L}_+$-module by defining $\widetilde{L}_+A=0$. Then we have the Verma module

$$\widetilde{M}^+_{\widetilde{L}_{\mathcal{Q}'}}(A) = \mathrm{Ind}_{\widetilde{L}_0+\widetilde{L}_+}^{\widetilde{L}} A \tag{2.36}$$

and the unique irreducible $\mathbb{Z}$-graded quotient module $M^+_{\widetilde{L}_{\mathcal{Q}'}}(A)$. Similarly, we have $M^-_{\widetilde{L}_{\mathcal{Q}'}}(A)$.

Theorem 2.2 *Let A be any finite dimensional irreducible $\widetilde{L}_0$ module. Then $M^{\pm}_{\widetilde{L}_{\mathcal{Q}'}}(A)$ is quasifinite, if and only if there exists some 2-variable exp-polynomial function $\psi : \mathbb{Z}^2 \to \mathbb{C}$, such that*

$$(t_2^{\frac{d}{\langle d,m_{22}\rangle}i} t_3^{\frac{d}{\langle d,m_{22}\rangle}j})v = \frac{\psi((i,j))}{1 - q^{\frac{(m_{11}m_{22}-m_{12}m_{21})di}{\langle d,m_{22}\rangle}} w_d^{\frac{m_{12}dj}{\langle d,m_{22}\rangle}} + \delta_{i,0}\delta_{\frac{j}{\langle d,m_{22}\rangle},\mathbb{Z}}} v \tag{2.37}$$

for all $(i,j) \in \mathbb{Z}^2$ and $v \in A$.

Proof If $\dim A = 1$, then the theorem follows from [22, Theorem 2.11]. So we may assume that $\dim A > 1$. Suppose that $M^{\pm}_{\widetilde{L}_{\mathcal{Q}'}}(A)$ is quasifinite. Fix $0 \neq v \in A$. Let $\widetilde{H} := \mathbb{C}[t_1^{\pm 1}, t_2^{\pm\frac{d}{\langle d,m_{22}\rangle}}, t_3^{\pm\frac{d}{\langle d,m_{22}\rangle}}] \subset \widetilde{L}_{\mathcal{Q}'}$. Then $U(\widetilde{H})v$ is a quasifinite $\mathbb{Z}$-graded $\widetilde{H}$ module. And from [22, Theorem 2.11], we get (2.37).

On the other hand, suppose that (2.37) holds. Note that

$$\widetilde{L}_0 \cong C_{w_d^{m_{22}}}, \quad \widetilde{L}_0 = \widetilde{L}_0' \oplus Z(\widetilde{L}_0). \tag{2.38}$$

From Theorem 2.1(a), we have a nonconstant polynomial $F_1(X), G_1(X) \in \mathbb{C}^* + X\mathbb{C}[X]$, such that

$$\Big\{t_2^i t_3^j F_1(t_2^{\frac{d}{\langle d,m_{22}\rangle}}), t_2^i t_3^j G_1(t_3^{\frac{d}{\langle d,m_{22}\rangle}}) \,\Big|\, (i,j) \notin \frac{d}{\langle d,m_{22}\rangle}\mathbb{Z}^2\Big\} \subset \mathrm{Ann}A. \tag{2.39}$$

Now, by Lemma 2.1, it is direct to check that A is an exp-polynomial module (see Definition 2.2(c)). Hence the theorem follows from [21, Theorem 1.7].

2.4 Irreducible quasifinite highest weight modules over $\mathfrak{L}(q, w_d)$

Define a Lie algebra surjective homomorphism $\varrho : \widetilde{L}_{\mathcal{Q}'} \to \mathfrak{L}/\mathbb{C}(m_{21}c_1 + m_{22}c_2)$ by

$$\varrho(t_1^i t_2^j t_3^k) = \begin{cases} q^{\frac{m_{11}m_{12}i(i-1)+m_{21}m_{22}j(j-1)+2ijm_{12}m_{21}}{2}} t^{i\mathbf{m}_1 + j\mathbf{m}_2} t_3^k, & (i,j,k) \notin (0,0,d\mathbb{Z}), \\ m_{11}c_1 + m_{12}c_2, & (i,j,k) \in (0,0,d\mathbb{Z}). \end{cases} \tag{2.40}$$

Theorem 2.3 *Let A be an irreducible finite dimensional $\mathfrak{L}_0$ module. Then the highest weight $\mathfrak{L}$-module $M^{\pm}(A; \mathbf{m}_1; \mathbf{m}_2)$ is quasifinite, if and only if there exist 1-variable exp-polynomial functions $\psi_0, \psi_2, \cdots, \psi_{\langle d,m_{22}\rangle - 1} : \mathbb{Z} \to \mathbb{C}$, such that*

$$\phi_A(m_{11}c_1 + m_{12}c_2) = \psi_0(0), \quad \phi_A(m_{21}c_1 + m_{22}c_2) = 0, \tag{2.41}$$

$$\phi_A(t^{\frac{id\mathbf{m}_2}{\langle d,m_{22}\rangle}} t_3^{\frac{kd}{\langle d,m_{22}\rangle}}) = \frac{\psi_k(i)}{(1 - q^{\frac{(m_{11}m_{22}-m_{12}m_{21})di}{\langle d,m_{22}\rangle}} w_d^{\frac{m_{12}dk}{\langle d,m_{22}\rangle}}) q^{\frac{m_{21}m_{22}id(id-\langle d,m_{22}\rangle)}{2\langle d,m_{22}\rangle^2}}} \tag{2.42}$$

for all $(i,k) \notin (0, \langle d,m_{22}\rangle\mathbb{Z}_d)$, $k = 0, \cdots, \langle d,m_{22}\rangle - 1$, where ϕ_A is defined in (2.19).

Proof Note that we may regard $\mathfrak{L}$-module $M^{\pm}(A; \mathbf{m}_1; \mathbf{m}_2)$ as $M^{\pm}_{\widetilde{L}_{\mathcal{Q}'}}(A)$ via the surjective homomorphism ϱ defined in (2.41). The theorem follows from Theorem 2.2 and Lemma 2.1.

3 Classification of Irreducible Quasifinite $\mathbb{Z}$-graded Modules with Nonzero Central Charges for $\mathfrak{L}(q, w_d)$

In this section, we will give the classification of irreducible quasifinite $\mathbb{Z}$-graded modules with nonzero charges for $\mathfrak{L}(q, w_d)$.

We will omit the details of the proof in this section, since they are almost the same as in [23].

We need to point out that "t_3" in this paper is corresponding to "t_0" in [23].

Proposition 3.1 *If $V \in \mathscr{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$, then V is a generalized highest weight module or a uniformly bounded module.*

Proof If V is not a generalized highest weight module, then one may deduce that $\dim V_m \leq d(\dim V_0 + \dim V_1)$ in the same way as in the proof of [23, Proposition 2.7].

Lemma 3.1 *If V is a nontrivial irreducible generalized highest weight $\mathbb{Z}$-graded $\mathfrak{L}$-module corresponding to a $\mathbb{Z}$-basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{Z}^2$, then*

(a) *For any $0 \neq v \in V$, there is some $p \in N$ such that $t^{m_1\mathbf{b}_1+m_2\mathbf{b}_2}t_3^i \cdot v = 0$ for all $m_1, m_2 \geq p$ and $i \in \mathbb{Z}_d$.*

(b) *For any $0 \neq v \in V$ and $m_1, m_2 > 0$, $i \in \mathbb{Z}_d$, we have $t^{-m_1\mathbf{b}_1-m_2\mathbf{b}_2}t_3^i \cdot v \neq 0$.*

Proof The proof is the same as [23, Lemma 4.1].

Lemma 3.2 *If $V \in \mathscr{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$ is a generalized highest weight $\mathfrak{L}(q, w_d)$-module, then V must be a highest or lowest weight module.*

Proof The proof is the same as [23, Lemma 4.2].

From the above lemma and the results in Section 2, we have the following theorem.

Theorem 3.1 *If V is a quasifinite irreducible $\mathbb{Z}$-graded $\mathfrak{L}$-module w.r.t $(\mathbf{m}_1, \mathbf{m}_2)$, then V is either $M^+(A; \mathbf{m}_1, \mathbf{m}_2)$, $M^-(A; \mathbf{m}_1, \mathbf{m}_2)$ with ϕ_A satisfying* (2.41) *and* (2.42)*, or a uniformly bounded module.*

Then we have the same result as [23, Theorem 4.4].

Theorem 3.2 *If $V \in \mathcal{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$ is an irreducible $\mathfrak{L}(q, w_d)$-module with nontrivial central charges, then there exists some finite dimensional irreducible $\mathfrak{L}_0$ module A with ϕ_A satisfying* (2.41) *and* (2.42)*, such that $V \cong M^+(A; \mathbf{m}_1, \mathbf{m}_2)$ or $V \cong M^-(A; \mathbf{m}_1, \mathbf{m}_2)$.*

One can also construct a class of highest weight $\mathbb{Z}^2$-graded $\mathfrak{L}(q, w_d)$-modules $V_{\mathbb{Z}^2} = V \otimes \mathbb{C}[x^{\pm 1}]$ from the $\mathbb{Z}$-graded module V w.r.t $(\mathbf{m}_1, \mathbf{m}_2)$ as follows:

$$t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k \cdot (v \otimes x^r) = (t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k \cdot v) \otimes x^{r+j}.$$

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