Chin. Ann. Math.
30B(2), 2009, 173–178
DOI: 10.1007/s11401-008-0092-z

Chinese Annals of Mathematics, Series B

Kähler Finsler Metrics Are Actually Strongly Kähler***

Bin CHEN*   Yibing SHEN**

Abstract In this paper, the Kähler conditions of the Chern-Finsler connection in complex Finsler geometry are studied, and it is proved that Kähler Finsler metrics are actually strongly Kähler.

Keywords Complex Finsler metric, Chern-Finsler connection, Kähler Finsler metric
2000 MR Subject Classification 53B40, 32Q99

1 Introduction

Complex Finsler manifolds are complex manifolds with complex Finsler metrics, which are more general than Hermitian metrics. In recent years, the complex Finsler geometry has attracted a renewed interest as examples of such metrics appear in a natural way in the geometric theory of several complex variables and mathematical physics (see [1, 3, 4, 6]).

As is well known, in the usual Hermitian geometry, the vanishing of the torsion of the Hermitian connection is equivalent to the metric being Kähler. In the complex Finsler geometry, there is a Chern-Finsler connection associated to a strongly pseudoconvex Finsler metric F on a complex manifold M. The torsions and curvatures of the Chern-Finsler connection enjoy a number of nice properties (see [4]). Since the torsion of the Chern-Finsler connection has a horizontal part and a mixed part, the situation for a strongly pseudoconvex Finsler metric to be Kähler is a bit subtler. In many works, there are three kinds of metric notions called respectively the strongly Kähler, Kähler and weakly Kähler according to the vanishing of some parts of the torsion of the Chern-Finsler connection (see [1]).

The three Kähler conditions have their deep impact on complex Finsler geometry. In [1], a Kobayashi rigidity result is proved for certain weakly Kähler Finsler metrics. Some great results for Kähler Finsler metrics are then obtained in [2, 3]. Under the strongly Kähler hypothesis, a Hodge decomposition theorem is verified in [6], and a version of Kodaira vanishing theorem is recently shown in [5]. Therefore, it is important to understand the relations among the three Kähler conditions. The main purpose of this note is to prove the following

Theorem 1.1 *The Kähler Finsler metrics are actually strongly Kähler.*

Hence, there are only two kinds of Kähler Finsler metrics with respect to the Chern-Finsler connection in the complex Finsler geometry.

Manuscript received March 10, 2008. Published online February 18, 2009.
*Center of Mathematical Science, Zhejiang University, Hangzhou 310027, China.
E-mail: chenbinn@zju.edu.cn
**Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
E-mail: yibingshen@zju.edu.cn
***Project supported by the National Natural Science Foundation of China (No. 10571154).

On the other hand, the curvature of the Chern-Finsler connection has four parts, and we define a complex analogue of the Landsberg curvature in real Finsler geometry, called the weak $h\overline{v}$-curvature. We shall prove the following

Theorem 1.2 *If the weak $h\overline{v}$-curvature of the Chern-Finsler connection of a complex Finsler metric F vanishes identically, then F is weakly Kähler if and only if it is Kähler.*

Hence, if F has vanishing $h\overline{v}$-curvature, then all the three Kähler conditions are equivalent.

2 Chern-Finsler Connection

Let M be a complex manifold of complex dimension n. Denote the holomorphic tangent bundle of M by $\pi : T^{1,0}M \to M$. For a local complex coordinate system $z = (z^1, \cdots, z^n)$ on M, a holomorphic tangent vector v of M is written as

$$v = v^i \partial_i, \quad \partial_i := \frac{\partial}{\partial z^i},$$

and we may take $(z, v) = (z^1, \cdots, z^n, v^1, \cdots, v^n)$ as a local coordinate system for $T^{1,0}M$. Throughout this paper, we shall use the following convention of the index range unless otherwise stated:

$$1 \le i, j, k, \cdots \le n.$$

Let $\widetilde{M} = T^{1,0}M \setminus \{0\}$ denote $T^{1,0}M$ without the zero section. $\{\partial_i, \dot{\partial}_j = \frac{\partial}{\partial v^j}\}$ gives a local holomorphic frame field of the holomorphic tangent bundle $T^{1,0}\widetilde{M}$ of $\widetilde{M}$.

Definition 2.1 *A complex Finsler metric on M is an upper continuous function F : $T^{1,0}M \to [0, +\infty)$ which satisfies the following conditions:*

(1) $G = F^2(z, v) \in C^\infty(\widetilde{M})$, *that is, G is smooth in $\widetilde{M}$;*

(2) $G(z, v) \ge 0$, *where the equality holds if and only if $v = 0$;*

(3) $G(z, \lambda v) = |\lambda|^2 G(z, v)$ *for all $(z, v) \in T^{1,0}M$ and $\lambda \in C^* = C \setminus \{0\}$.*

The pair (M, G) is called a complex Finsler manifold. A complex Finsler metric F is said to be strongly pseudoconvex if the complex v-Hessian

$$(G_{i\overline{j}}) := (\dot{\partial}_i \dot{\partial}_{\overline{j}} G), \tag{2.1}$$

where $\dot{\partial}_i = \frac{\partial}{\partial v^i}$, $\dot{\partial}_{\overline{j}} = \frac{\partial}{\partial \overline{v}^j}$ of G is positively definite on $\widetilde{M}$. In particular, if $G(z, v) = h_{i\overline{j}}(z) v^i \overline{v}^j$ is a Hermitian metric on M, then $G(z, v)$ defines a strongly pseudoconvex Finsler metric. Here and from now on, the lower indices of G always mean to take derivatives, i.e., $G_{i\overline{j}k} := \dot{\partial}_i \dot{\partial}_{\overline{j}} \dot{\partial}_k G$. And the indices after ";" are the derivatives with respect to z, i.e., $G_{i;j} := \dot{\partial}_i \partial_j G$.

The Chern-Finsler connection associated to a strongly pseudoconvex Finsler metric $G = F^2$ is defined as follows (see [1]). By setting

$$N^i_k := G^{i\overline{j}} G_{\overline{j};k} \tag{2.2}$$

with $(G^{i\overline{j}})$ the inverse of $(G_{k\overline{l}})$, the horizontal vectors and vertical covectors can be defined by

$$\frac{\delta}{\delta z^k} := \frac{\partial}{\partial z^k} - N^i_k \frac{\partial}{\partial v^i}, \quad \delta v^k := \mathrm{d} v^k + N^k_i \mathrm{d} z^i. \tag{2.3}$$

Then we have the following horizontal and vertical decomposition

$$T_{\mathbb{C}}\widetilde{M} = \mathcal{H} \oplus \overline{\mathcal{H}} \oplus \mathcal{V} \oplus \overline{\mathcal{V}}, \quad T^*_{\mathbb{C}}\widetilde{M} = \mathcal{H}^* \oplus \overline{\mathcal{H}^*} \oplus \mathcal{V}^* \oplus \overline{\mathcal{V}^*}, \tag{2.4}$$

where $\mathcal{H} = \mathrm{span}\{\frac{\delta}{\delta z^i}\}$, $\mathcal{V} = \mathrm{span}\{\frac{\partial}{\partial v^i}\}$ and $\mathcal{H}^* = \mathrm{span}\{\mathrm{d}z^i\}$, $\mathcal{V}^* = \mathrm{span}\{\delta v^i\}$. A direct computation will give $\frac{\delta G}{\delta z^i} = 0$. The complex Finsler F can induce a Hermitian metric on the holomorphic subbundle $\mathcal{V}$. Namely, for any $X = X^i\dot{\partial}_i, Y = Y^i\dot{\partial}_i \in \mathcal{V}_{(z,v)}$, their inner product is defined as

$$\langle X, Y\rangle_{(z,v)} := (\partial\overline{\partial}G)(X, \overline{Y}) = G_{i\overline{k}}(z, v)X^i\overline{Y}^k.$$

Then the vertical bundle $(\mathcal{V}, \widetilde{M}, \langle\cdot\,,\cdot\rangle)$ becomes a Hermitian bundle. The Chern-Finsler connection ∇ associated to F is just the Hermitian connection of $(\mathcal{V}, \widetilde{M})$. In other words, for any $X \in T^{1,0}\widetilde{M}$ and $Y, Z \in \Gamma(\mathcal{V})$, we have

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_{\overline{X}} Z\rangle. \tag{2.5}$$

In a local coordinate system, the connection 1-forms are then given by

$$\omega^i_j := G^{i\overline{l}}\partial G_{j\overline{l}} = \Gamma^i_{j;k}\mathrm{d}z^k + C^i_{jk}\delta v^k, \tag{2.6}$$

where the connection coefficients can be expressed as

$$\Gamma^i_{j;k} := G^{i\overline{l}}\frac{\delta G_{j\overline{l}}}{\delta z^k}, \quad C^i_{jk} := G^{i\overline{l}}\frac{\partial G_{j\overline{l}}}{\partial v^k}. \tag{2.7}$$

Using the isomorphism $\Phi : \mathcal{H} \to \mathcal{V}$ with $\frac{\delta}{\delta z^i} \mapsto \frac{\partial}{\partial v^i}$, one can naturally introduce a metric and a connection on $\mathcal{H}$ by

$$\langle X, Y\rangle := \langle\Phi(X), \Phi(Y)\rangle, \quad \nabla_Z X := \Phi^{-1}(\nabla_Z\Phi(X)),$$

where $X, Y \in \Gamma(\mathcal{H})$ and $Z \in T^{1,0}\widetilde{M}$. Then we will get a connection, also called the Chern-Finsler connection, on the Whitney sum $\mathcal{H} \oplus \mathcal{V}$ which is just the holomorphic tangent bundle of $\widetilde{M}$.

3 Strongly Kähler Versus Kähler

The Kähler conditions in the Finsler geometry share the same spirit with the Hermitian geometry, that is, the vanishing of some torsion. The $(2,0)$-torsion of the Chern-Finsler connection is defined by

$$\theta(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \mathcal{H} \oplus \mathcal{V}. \tag{3.1}$$

Definition 3.1 (see [1, §2.3.5]) *Let F be a complex Finsler metric, and $\chi = \frac{v^i\delta}{\delta z^i} \in \Gamma(\mathcal{H})$ be the radial horizontal field. We say that F is*

(1) *strongly Kähler, if $\theta(X, Y) = 0$ for any $X, Y \in \mathcal{H}$;*

(2) *Kähler, if $\theta(X, \chi) = 0$ for any $X \in \mathcal{H}$;*

(3) *weakly Kähler, if $\langle\theta(X, \chi), \chi\rangle = 0$ for any $X \in \mathcal{H}$.*

In a local coordinate system, the torsion θ has the form

$$\theta = (\Gamma^i_{j;k}\mathrm{d}z^j \wedge \mathrm{d}z^k + C^i_{jk}\delta v^j \wedge \mathrm{d}z^k) \otimes \frac{\delta}{\delta z^i}. \tag{3.2}$$

Hence F is strongly Kähler (resp. Kähler or weakly Kähler) if and only if

$$\Gamma^i_{j;k} = \Gamma^i_{k;j} \quad (\text{resp. } \Gamma^i_{j;k}v^k = \Gamma^i_{k;j}v^k \text{ or } G_i\Gamma^i_{j;k}v^k = G_i\Gamma^i_{k;j}v^k). \tag{3.3}$$

Strongly Kähler Finsler metrics have nice properties, such as the existence of normal coordinates, etc. Next, we will show that saying strongly Kähler is indeed equivalent to saying Kähler. The following lemma is well-known.

Lemma 3.1 $\Gamma^i_{j;k} = \dot{\partial}_j N^i_k$ *and* $\Gamma^i_{j;k}v^j = N^i_k$.

Proof According to (2.7), (2.3) and (2.2), one can obtain

$$\begin{aligned}
\Gamma^i_{j;k} &= G^{i\overline{l}}\delta_k G_{j\overline{l}} \\
&= G^{i\overline{l}}G_{j\overline{l};k} - G^{i\overline{l}}N^p_k G_{j\overline{l}p} \\
&= \dot{\partial}_j(G^{i\overline{l}}G_{\overline{l};k}) - \dot{\partial}_j(G^{i\overline{l}})G_{\overline{l};k} - G^{i\overline{l}}N^p_k G_{j\overline{l}p} \\
&= \dot{\partial}_j(N^i_k) + G^{i\overline{q}}G^{p\overline{l}}G_{p\overline{q}j}G_{\overline{l};k} - G^{i\overline{l}}N^p_k G_{j\overline{l}p} \\
&= \dot{\partial}_j(N^i_k) + G^{i\overline{q}}G_{p\overline{q}j}N^p_k - G^{i\overline{l}}N^p_k G_{j\overline{l}p} \\
&= \dot{\partial}_j(N^i_k).
\end{aligned}$$

Notice $N^i_k(z, \lambda v) = \lambda N^i_k(z, v)$ for any complex number λ. Then the Euler's theorem will give $v^j\Gamma^i_{j;k} = v^j\dot{\partial}_j N^i_k = N^i_k$.

Theorem 3.1 *A Kähler Finsler metric must be strongly Kähler.*

Proof By the above lemma, we see

$$\dot{\partial}_k(N^i_j v^j) = \dot{\partial}_k(N^i_j)v^j + N^i_k = \Gamma^i_{k;j}v^j + N^i_k.$$

If F is Kähler, then

$$\dot{\partial}_k(N^i_j v^j) = \Gamma^i_{j;k}v^j + N^i_k = 2N^i_k.$$

Applying Lemma 3.1 again, we have

$$\Gamma^i_{j;k} = \dot{\partial}_j(N^i_k) = \frac{1}{2}\dot{\partial}_j\dot{\partial}_k(N^i_l v^l), \tag{3.4}$$

which means that F is strongly Kähler.

This is really an unexpected result, and the Kähler conditions are now reduced to two types.

4 Kähler Versus Weakly Kähler

In this section, we will study the relation between Kähler and weakly Kähler. Being aware of the decomposition (2.4), the curvature of the Chern-Finsler connection can be divided into four parts, namely $h\overline{h}$-, $v\overline{h}$-, $h\overline{v}$- and $v\overline{v}$-curvatures (see [1]). The curvature forms $\{\Omega^i_j = \overline{\partial}\omega^i_j\}$ can then be written in the form

$$\Omega^i_j = R^i_{j;k\overline{l}}\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^l + S^i_{jk;\overline{l}}\delta v^k \wedge \mathrm{d}\overline{z}^l + P^i_{j\overline{l};k}\mathrm{d}z^k \wedge \delta\overline{v}^l + Q^i_{jk\overline{l}}\delta v^k \wedge \delta\overline{v}^l, \tag{4.1}$$

where

$$\begin{aligned}
h\overline{h}\text{-part}: &\quad R^i_{j;k\bar{l}} = -\delta_{\bar{l}}(\Gamma^i_{j;k}) - C^i_{js}\delta_{\bar{l}}(N^s_k),\\
v\overline{h}\text{-part}: &\quad S^i_{jk;\bar{l}} = -\delta_{\bar{l}}(C^i_{jk}),\\
h\overline{v}\text{-part}: &\quad P^i_{j\bar{l};k} = -\dot{\partial}_{\bar{l}}(\Gamma^i_{j;k}) - C^i_{js}\dot{\partial}_{\bar{l}}(N^s_k),\\
v\overline{v}\text{-part}: &\quad Q^i_{jk\bar{l}} = -\dot{\partial}_{\bar{l}}(C^i_{jk}).
\end{aligned}$$

The $h\overline{h}$-curvature is the Finslerian analogue of the curvature in Hermitian geometry. The other three curvatures never appear for Hermitian metrics. One can verify that F is Hermitian if and only if the $v\overline{v}$-curvature vanishes. Here we will show a result for the $h\overline{v}$-curvature. Let us first give the following

Definition 4.1 *Setting $P^i_{j\bar{l}} := P^i_{j\bar{l};k}v^k$, we call $\{P^i_{j\bar{l}}\}$ the weak $h\overline{v}$-curvature.*

Comparing with the real Finsler geometry, one can see that the weak $h\overline{v}$-curvature is an analogue of the Landsberg curvature in the complex realm. It will be an interesting non-Hermitian quantity. With this notion, we can state our result.

Theorem 4.1 *If the weak $h\overline{v}$-curvature of the Chern-Finsler connection of a complex Finsler metric F vanishes identically, then F is Kähler if and only if it is weakly Kähler.*

Proof Assume that F is weakly Kähler, i.e.,

$$G_i(\Gamma^i_{j;k} - \Gamma^i_{k;j})v^k = 0.$$

Then the $\dot{\partial}_{\bar{l}}$ derivative will tell us

$$\begin{aligned}
G_{i\bar{l}}(\Gamma^i_{j;k} - \Gamma^i_{k;j})v^k &= -G_i(\dot{\partial}_{\bar{l}}\Gamma^i_{j;k} - \dot{\partial}_{\bar{l}}\Gamma^i_{k;j})v^k\\
&= -G_i(\dot{\partial}_{\bar{l}}\Gamma^i_{j;k})v^k + G_i\dot{\partial}_{\bar{l}}(\Gamma^i_{k;j}v^k)\\
&= -G_i(\dot{\partial}_{\bar{l}}\Gamma^i_{j;k})v^k + G_i\dot{\partial}_{\bar{l}}(N^i_j)\\
&= -G_i(\dot{\partial}_{\bar{l}}\Gamma^i_{j;k})v^k + \dot{\partial}_{\bar{l}}(G_iN^i_j) - G_{i\bar{l}}N^i_j\\
&= -G_i(\dot{\partial}_{\bar{l}}\Gamma^i_{j;k})v^k + \dot{\partial}_{\bar{l}}(G_iG^{i\overline{p}}G_{\overline{p};j}) - G_{\bar{l};j}\\
&= -G_i(\dot{\partial}_{\bar{l}}\Gamma^i_{j;k})v^k.
\end{aligned}\tag{4.2}$$

On the other hand, since

$$P^i_{j\bar{l}} = -\dot{\partial}_{\bar{l}}(\Gamma^i_{j;k})v^k - C^i_{js}\dot{\partial}_{\bar{l}}(N^s_k)v^k = 0, \tag{4.3}$$

we have

$$0 = P^i_{j\bar{l}}v^j = -\dot{\partial}_{\bar{l}}(v^j\Gamma^i_{j;k})v^k - v^jC^i_{js}\dot{\partial}_{\bar{l}}(N^s_k)v^k = -\dot{\partial}_{\bar{l}}(N^i_k)v^k. \tag{4.4}$$

Substituting (4.4) back into (4.3), we see that the right-hand side of (4.2) is in fact zero. This leads to

$$G_{i\bar{l}}(\Gamma^i_{j;k} - \Gamma^i_{k;j})v^k = 0,$$

which means that F is Kähler and hence strongly Kähler.

Corollary 4.1 *If F has vanishing $h\overline{v}$-curvature, then all the three Kähler conditions are equivalent.*

Applying the maximum principle, the vanishing of $h\overline{v}$-curvature will imply the linearity of the connection $\{N_k^i\}$. We should also remark here that the Kobayashi metrics, equivalently the Carathéodory metrics, on all strongly convex domains are weakly Kähler (see [1]). It is still open whether they are Kähler or not.

References

[1] Abate, M. and Patrizio, G., Finsler Metrics—A Global Approach, with Applications to Geometric Function Theory, Lecture Notes in Math., Vol. 1591, Springer, Berlin, 1994.

[2] Abate, M. and Patrizio, G., Kähler Finsler manifolds of constant holomorphic curvature, *Int. J. Math.*, **2**, 1997, 169–186.

[3] Bland, J. and Kalka, M., Variations of holomorphic curvature for Kähler Finsler metrics, *Cont. Math.*, **196**, 1996, 121–132.

[4] Wong, P. M., A survey of complex Finsler geometry, *Adv. Stud. in Pure Math.*, **48**, 2007, 375–433.

[5] Xiao, J., Zhong, T. and Qiu, C., Bochner technique in strongly Kähler Finsler metrics, preprint, 2008.

[6] Zhong, C. and Zhong, T., Hodge decomposition theorem on strongly Kähler Finsler manifolds, *Sci. in Chin., Ser. A*, **48**, 2006, 1696–1714.