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Dynamics of a Rational Difference Equation

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Abstract The authors investigate the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}}, \quad n = 0, 1, \cdots,$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, \cdots, x_0$ are arbitrary positive real numbers, $r = \max\{l, k, p, q\}$ is a nonnegative integer and a, b, c are positive constants. Some special cases of this equation are also studied in this paper.

Keywords Stability, Periodic solutions, Difference equations
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1 Introduction

In this paper, we deal with some properties of the solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}}, \quad n = 0, 1, \cdots, \tag{1.1}$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, \cdots, x_0$ are arbitrary positive real numbers, $r = \max\{l, k, p, q\}$ is a nonnegative integer and a, b, c are positive constants. Also, we study some special cases of equation (1.1).

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \cdots,$ $x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \quad n = 0, 1, \cdots \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ (see [15]).

A point $\overline{x} \in I$ is called an equilibrium point of equation (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \cdots, \overline{x}).$$

That is, $x_n = \overline{x}$, for $n \ge 0$, is a solution of equation (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

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Definition 1.1 (Stability) (i) *The equilibrium point* $\overline{x}$ *of equation* (1.2) *is locally stable if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) *The equilibrium point* $\overline{x}$ *of equation* (1.2) *is locally asymptotically stable if* $\overline{x}$ *is a locally stable solution of equation* (1.2) *and there exists* $\gamma > 0$, *such that for all* $x_{-k}, x_{-k+1}, \cdots, x_{-1}$, $x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) *The equilibrium point* $\overline{x}$ *of equation* (1.2) *is a global attractor if for all* $x_{-k}, x_{-k+1}, \cdots$, $x_{-1}, x_0 \in I$, *we have*

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) *The equilibrium point* $\overline{x}$ *of equation* (1.2) *is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of equation* (1.2).

(v) *The equilibrium point* $\overline{x}$ *of equation* (1.2) *is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation of equation (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \cdots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem 1.1 (see [14]) *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \{0, 1, 2, \cdots\}$. *Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + p x_n + q x_{n-k} = 0, \quad n = 0, 1, \cdots.$$

Remark 1.1 Theorem 1.1 can be easily extended to a general linear equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad n = 0, 1, \cdots, \tag{1.4}$$

where $p_1, p_2, \cdots, p_k \in \mathbb{R}$ and $k \in \{1, 2, \cdots\}$. Then equation (1.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Definition 1.2 (Fibonacci Sequence) *The sequence* $\{F_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, \cdots\}$, *i.e.,* $F_m = F_{m-1} + F_{m-2}$, $m \geq 0$, $F_{-2} = 0$, $F_{-1} = 1$, *is called Fibonacci Sequence.*

Solutions of difference equations, periodicity, stability and boundedness of solutions to abstract difference equations have been discussed by many authors, e.g., Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Elabbasy et al. [10] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

In [8], E. M. Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}.$$

Yang et al. [20] investigated the invariant intervals, the global attractivity of equilibrium points, and the asymptotic behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}}.$$

Cinar [5–7] has got the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}},$$
$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}},$$
$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Aloqeili [1] obtained the form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

For some related works see [1–20].

The paper proceeds as follows. In Section 2 we show that when $3a < (b + c)$, the equilibrium point of equation (1.1) is locally asymptotically stable. In Section 3 we prove that the equilibrium point of equation (1.1) is a global attractor. In Section 4 we give the solutions of some special cases of equation (1.1) and give numerical examples of each case. The solutions obtained are plotted in (n, x_n)-plane by using Matlab 6.5.

2 Local Stability of Equation (1.1)

In this section, we investigate the local stability character of the solutions of equation (1.1). equation (1.1) has a unique positive equilibrium point and is given by

$$\overline{x} = \frac{a\overline{x}^2}{b\overline{x} + c\overline{x}}.$$

If $a \neq b+c$, then the unique equilibrium point is $\overline{x}=0$.

Let $f:(0,\infty)^4 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w,z)=\frac{auv}{bw+cz}. \tag{2.1}$$

Therefore it follows that

$$\begin{aligned}
f_u(u,v,w,z) &= \frac{av}{(bw+cz)},\\
f_v(u,v,w,z) &= \frac{au}{(bw+cz)},\\
f_w(u,v,w,z) &= \frac{-bauv}{(bw+cz)^2},\\
f_z(u,v,w,z) &= \frac{-cauv}{(bw+cz)^2}.
\end{aligned}$$

We see that

$$\begin{aligned}
f_u(\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{a}{(b+c)},\\
f_v(\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{a}{(b+c)},\\
f_w(\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{-ab}{(b+c)^2},\\
f_z(\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{-ac}{(b+c)^2}.
\end{aligned}$$

The linearized equation of equation (1.1) about $\overline{x}$ is

$$y_{n+1}+\frac{a}{(b+c)}y_{n-l}+\frac{a}{(b+c)}y_{n-k}-\frac{ab}{(b+c)^2}y_{n-p}-\frac{ac}{(b+c)^2}y_{n-q}=0. \tag{2.2}$$

Theorem 2.1 *Assume that*

$$3a<(b+c).$$

Then the equilibrium point of equation (1.1) *is locally asymptotically stable.*

Proof It follows by Theorem 1.1 that equation (2.2) is asymptotically stable if

$$\left|\frac{a}{(b+c)}\right|+\left|\frac{a}{(b+c)}\right|+\left|\frac{ab}{(b+c)^2}\right|+\left|\frac{ac}{(b+c)^2}\right|<1,$$

or

$$\frac{2a}{(b+c)}+\frac{a}{(b+c)}<1,$$

and so

$$3a<b+c.$$

This completes the proof.

3 Global Attractor of the Equilibrium Point of Equation (1.1)

In this section, we investigate the global attractivity character of solutions of equation (1.1). We give the following theorem which is a minor modification of [15, Theorem A.0.2].

Theorem 3.1 *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b]^{k+1} \to [a, b]$$

is a continuous function satisfying the following properties:

(i) *$f(x_1, x_2, \cdots, x_{k+1})$ is non-decreasing in any two components (for example x_t, x_y) for each x_r ($r \neq t, y$) in $[a, b]$ and non-increasing in the remaining components for each x_t, x_y in $[a, b]$,*

(ii) *$m = M$ once $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(M, M, \cdots, M, m, M, \cdots, M, m, M, \cdots, M, M),$$
$$M = f(m, m, \cdots, m, M, m, \cdots, m, M, m, \cdots, m, m).$$

Then equation (1.2) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of equation (1.2) converges to $\overline{x}$.

Proof Set

$$m_0 = a, \quad M_0 = b,$$

and for each $i = 1, 2, \cdots$, set

$$M_i = f(m_{i-1}, m_{i-1}, \cdots, m_{i-1}, M_{i-1}, m_{i-1}, \cdots, m_{i-1}, M_{i-1}, m_{i-1}, \cdots, m_{i-1}, m_{i-1}),$$
$$m_i = f(M_{i-1}, M_{i-1}, \cdots, M_{i-1}, m_{i-1}, M_{i-1}, \cdots, M_{i-1}, m_{i-1}, M_{i-1}, \cdots, M_{i-1}, M_{i-1}).$$

Now observe that for each $i \geq 0$,

$$a = m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0 = b,$$

and

$$m_i \leq x_p \leq M_i \quad \text{for } p \geq (k+1)i + 1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then

$$M \geq \limsup_{i\to\infty} x_i \geq \liminf_{i\to\infty} x_i \geq m,$$

and by the continuity of f,

$$m = f(M, M, \cdots, M, m, M, \cdots, M, m, M, \cdots, M, M),$$
$$M = f(m, m, \cdots, m, M, m, \cdots, m, M, m, \cdots, m, m).$$

In view of (ii),

$$m = M = \overline{x},$$

from which the result follows.

Theorem 3.2 *The equilibrium point* $\overline{x}$ *of equation* (1.1) *is a global attractor.*

Proof Let r, s be nonnegative real numbers and assume that $f : [r, s]^4 \rightarrow [r, s]$ is a function defined by equation (2.1). Then we can easily see that the function $f(u, v, w, z)$ increases in u, v and decreases in w, z.

Suppose that (m, M) is a solution of the system

$$m = f(m, m, M, M) \quad \text{and} \quad M = f(M, M, m, m).$$

Then from equation (1.1), we see that

$$\begin{aligned} m &= \frac{am^2}{bM + cM}, & M &= \frac{aM^2}{bm + cm}, \\ (b + c)mM &= am^2, & (b + c)Mm &= aM^2, \end{aligned}$$

so

$$M = m.$$

It follows from Theorem 3.1 that $\overline{x}$ is a global attractor of equation (1.1) and then the proof is completed.

4 Special Cases of Equation (1.1)

Case 1 In this case, we study the following special case of equation (1.1)

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-1}}, \tag{4.1}$$

where the initial conditions x_{-1}, x_0 are arbitrary positive real numbers.

Theorem 4.1 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation* (4.1). *Then for* $n = 0, 1, \cdots,$

$$x_n = \frac{hk}{F_{n-1}k + F_{n-2}h},$$

where $x_{-1} = k$, $x_0 = h$.

Proof For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$, $n - 2$. That is,

$$x_{n-2} = \frac{hk}{F_{n-3}k + F_{n-4}h}, \quad x_{n-1} = \frac{hk}{F_{n-2}k + F_{n-3}h}.$$

Now, it follows from equation (4.1) that

$$\begin{aligned} x_n &= \frac{x_{n-1}x_{n-2}}{x_{n-1} + x_{n-2}} \\ &= \frac{(\frac{hk}{F_{n-3}k+F_{n-4}h})(\frac{hk}{F_{n-2}k+F_{n-3}h})}{(\frac{hk}{F_{n-3}k+F_{n-4}h} + \frac{hk}{F_{n-2}k+F_{n-3}h})} \\ &= \frac{(\frac{hk}{F_{n-3}k+F_{n-4}h})(\frac{1}{F_{n-2}k+F_{n-3}h})}{(\frac{1}{F_{n-3}k+F_{n-4}h} + \frac{1}{F_{n-2}k+F_{n-3}h})} \\ &= \frac{hk}{(F_{n-2}k + F_{n-3}h + F_{n-3}k + F_{n-4}h)} \\ &= \frac{hk}{(F_{n-2}h + F_{n-1}k)}. \end{aligned}$$

Hence, the proof is completed.

Lemma 4.1 *Every positive solution of equation* (4.1) *is bounded and* $\lim_{n\to\infty} x_n = 0$.

Proof It follows from equation (4.1) that

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-1}} \le \frac{x_n x_{n-1}}{x_{n-1}} = x_n,$$

or

$$x_{n+1} \le x_n.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so is bounded from above by $M = \max\{x_{-1}, x_0\}$.

For $x_{-1} = 5$, $x_0 = 9$, the solution of equation (4.1) will take the form $\{3.214286, 2.368421, 1.363636, 0.8653846, 0.5294118, \cdots\}$, this solution is stable and $\lim_{n\to\infty} x_n = 0$ (see Figure 1).

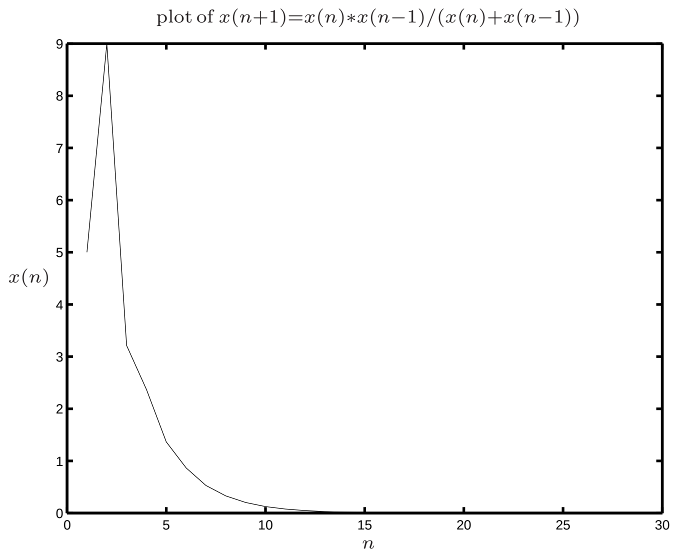

Figure 1 plot of $x(n+1) = x(n) * x(n-1)/(x(n) + x(n-1))$

Case 2 In this case, we study the following special case of equation (1.1)

$$x_{n+1} = \frac{x_{n-1} x_{n-2}}{x_{n-1} + x_{n-2}}, \tag{4.2}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers.

Theorem 4.2 *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (4.2). *Then* $x_1 = \frac{rk}{k+r}$, *for* $n = 1, 2, \cdots$,

$$x_{n+1} = \frac{hkr}{d_{n-4}hk + d_{n-3}kr + d_{n-2}hr},$$

where $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$, $\{d_m\}_{m=0}^{\infty} = \{1, 2, 2, 3, 4, 5, 7, 9, \cdots\}$, *i.e.*, $d_m = d_{m-2} + d_{m-3}$, $m \ge 0$, $d_{-3} = 0$, $d_{-2} = 1$, $d_{-1} = 1$.

Proof For $n = 1, 2, 3$ the result holds; then suppose that our assumption holds for $n-1$, $n-2$, $n-3$ where $n > 3$. That is,

$$x_{n-2} = \frac{hkr}{d_{n-7}hk + d_{n-6}kr + d_{n-5}hr}, \quad x_{n-1} = \frac{hkr}{d_{n-6}hk + d_{n-5}kr + d_{n-4}hr}.$$

Now, it follows from equation (4.2) that

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-1}x_{n-2}}{x_{n-1}+x_{n-2}} \\
&= \frac{\left(\frac{hkr}{d_{n-7}hk+d_{n-6}kr+d_{n-5}hr}\right)\left(\frac{hkr}{d_{n-6}hk+d_{n-5}kr+d_{n-4}hr}\right)}{\left(\frac{hkr}{d_{n-7}hk+d_{n-6}kr+d_{n-5}hr}+\frac{hkr}{d_{n-6}hk+d_{n-5}kr+d_{n-4}hr}\right)} \\
&= \frac{\left(\frac{hkr}{d_{n-7}hk+d_{n-6}kr+d_{n-5}hr}\right)\left(\frac{1}{d_{n-6}hk+d_{n-5}kr+d_{n-4}hr}\right)}{\left(\frac{1}{d_{n-7}hk+d_{n-6}kr+d_{n-5}hr}+\frac{1}{d_{n-6}hk+d_{n-5}kr+d_{n-4}hr}\right)} \\
&= \frac{hkr}{(d_{n-7}hk+d_{n-6}kr+d_{n-5}hr+d_{n-6}hk+d_{n-5}kr+d_{n-4}hr)} \\
&= \frac{hkr}{(d_{n-7}+d_{n-6})hk+(d_{n-6}+d_{n-5})kr+(d_{n-5}+d_{n-4})hr} \\
&= \frac{hkr}{d_{n-4}hk+d_{n-3}kr+d_{n-2}hr}.
\end{aligned}$$

Hence, the proof is completed.

Lemma 4.2 *Every positive solution of equation* (4.2) *is bounded and* $\lim_{n\to\infty} x_n = 0$.

Proof It follows from equation (4.2) that

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_{n-1}+x_{n-2}} \le \frac{x_{n-1}x_{n-2}}{x_{n-2}} = x_{n-1}$$

or

$$x_{n+1} \le x_{n-1}.$$

Then the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-2}, x_{-1}, x_0\}$.

Let $x_{-2} = 5$, $x_{-1} = 9$, $x_0 = 12$. Then the solution will be $\{3.214286, 5.142857, 2.535211, 1.978022, 1.698113, 1.111111, 0.9137055, 0.6716418, 0.5013927, \cdots\}$ (see Figure 2).

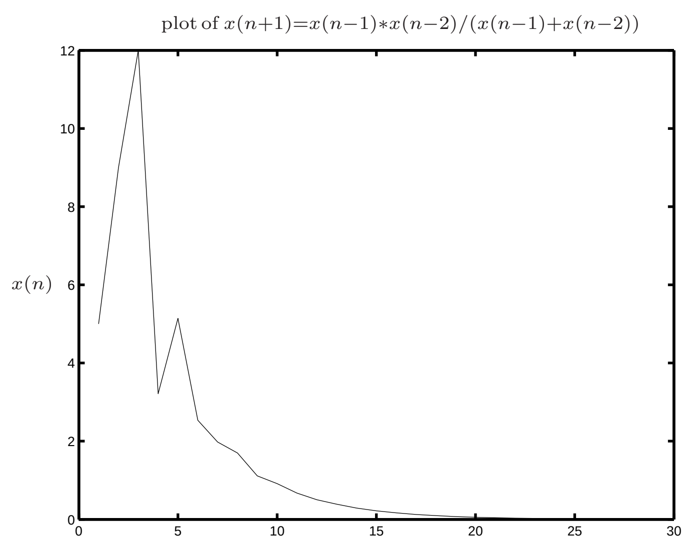

Figure 2 plot of $x(n+1) = x(n-1) * x(n-2)/(x(n-1) + x(n-2))$

The following cases can be treated similarly.

Case 3 Let $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$. Then the solution of the sequence

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n + x_{n-2}} \tag{4.3}$$

is given by

$$x_{2n} = \frac{h\prod\limits_{i=0}^{n-1}(F_{2i-1}h + F_{2i}r)}{\prod\limits_{i=0}^{n-1}(F_{2i}h + F_{2i+1}r)}, \quad x_{2n+1} = \frac{kr\prod\limits_{i=0}^{n-1}(F_{2i}h + F_{2i+1}r)}{\prod\limits_{i=0}^{n}(F_{2i-1}h + F_{2i}r)},$$

where $n = 0, 1, \cdots$, which is bounded and $\lim\limits_{n\to\infty} x_n = 0$.

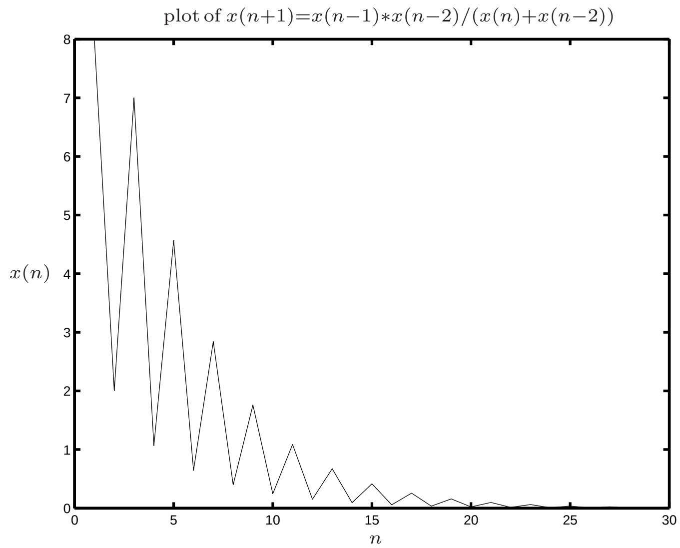

Figure 3 plot of $x(n+1) = x(n-1) * x(n-2)/(x(n) + x(n-2))$

Figure 3 shows the solution when $x_{-2} = 8$, $x_{-1} = 2$, $x_0 = 7$.

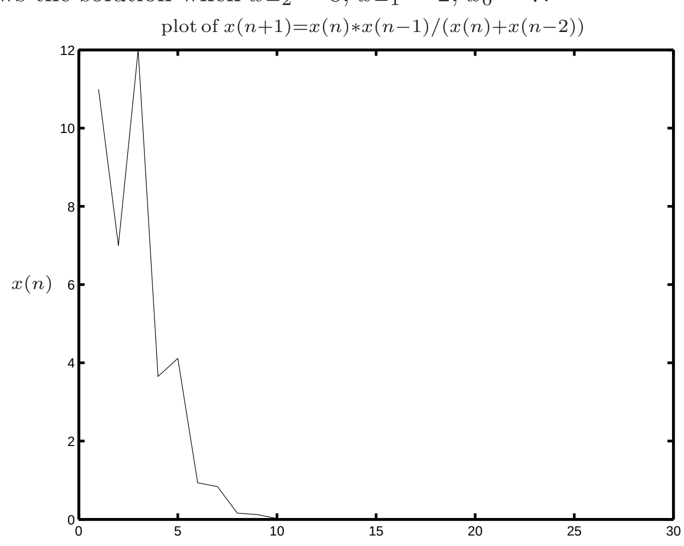

Figure 4 plot of $x(n+1) = x(n) * x(n-1)/(x(n) + x(n-2))$

Case 4 Let $x_{-2}=r$, $x_{-1}=k$, $x_0=h$. Then the solution of the sequence

$$x_{n+1}=\frac{x_{n-1}x_n}{x_n+x_{n-2}} \tag{4.4}$$

is given by

$$x_{2n-1}=\frac{kh^n}{\prod\limits_{i=1}^{n}((2i-1)h+r)},\quad x_{2n}=\frac{h^{n+1}}{\prod\limits_{i=1}^{n}(2ih+r)},$$

where $n=0,1,\cdots$, which is bounded and $\lim\limits_{n\to\infty}x_n=0$.

Figure 4 shows the solution when $x_{-2}=11$, $x_{-1}=7$, $x_0=12$.

Case 5 Let $x_{-2}=r$, $x_{-1}=k$, $x_0=h$. Then the solution of the sequence

$$x_{n+1}=\frac{x_{n-1}x_n}{x_{n-1}+x_{n-2}} \tag{4.5}$$

is given by

$$x_{2n}=\frac{h(hk)^n}{\prod\limits_{i=0}^{n-1}(((i+1)k+r)((i+1)h+k))},\quad x_{2n+1}=\frac{(hk)^{n+1}}{\prod\limits_{i=0}^{n}((i+1)k+r)\prod\limits_{i=0}^{n-1}((i+1)h+k)},$$

$n=0,1,\cdots$, which is bounded and $\lim\limits_{n\to\infty}x_n=0$.

Figure 5 shows the solution when $x_{-2}=5$, $x_{-1}=8$, $x_0=3$.

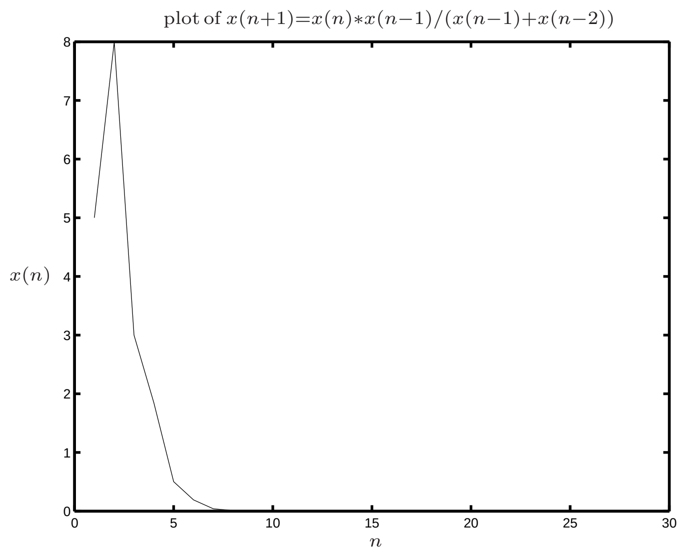

Figure 5 plot of $x(n+1)=x(n)*x(n-1)/(x(n-1)+x(n-2))$

Case 6 Let $x_{-2}=r$, $x_{-1}=k$, $x_0=h$. Then the solution of the sequence

$$x_{n+1}=\frac{x_{n-2}x_n}{x_n+x_{n-2}} \tag{4.6}$$

is given by

$$x_n=\frac{hkr}{t_{n-3}hr+t_{n-2}hk+t_{n-1}kr},\quad n=0,1,\cdots,$$

where $\{t_m\}_{m=0}^{\infty}=\{1,1,2,3,4,6,9,\cdots\}$, i.e., $t_m=t_{m-1}+t_{m-3}$, $m\ge 0$, $t_{-3}=0$, $t_{-2}=0$, $t_{-1}=1$, which is bounded and $\lim\limits_{n\to\infty}x_n=0$.

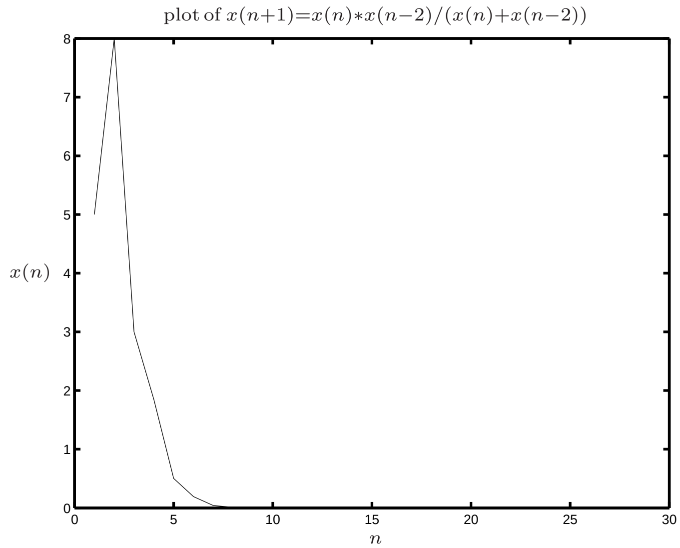

Figure 6 plot of $x(n+1) = x(n) * x(n-2)/(x(n) + x(n-2))$

Figure 6 shows the solution when $x_{-2} = 6$, $x_{-1} = 9$, $x_0 = 17$.

Case 7 Let $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$. Then the solution of the sequence

$$x_{n+1} = \frac{x_{n-2}x_n}{x_{n-1} + x_{n-2}} \tag{4.7}$$

is given by

$$x_{2n} = \frac{hkr}{(F_{n-2}k + F_{n-1}r)(F_{n-2}h + F_{n-1}k)}, \quad x_{2n+1} = \frac{hkr}{(F_{n-1}k + F_n r)(F_{n-2}h + F_{n-1}k)},$$

$n = 0, 1, \cdots$, which is bounded and $\lim_{n\to\infty} x_n = 0$.

Figure 7 shows the solution when $x_{-2} = 13$, $x_{-1} = 7$, $x_0 = 12$.

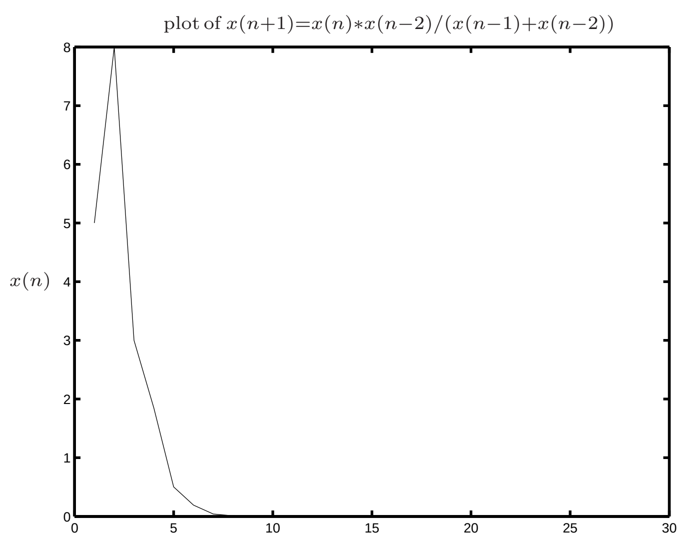

Figure 7 plot of $x(n+1) = x(n) * x(n-2)/(x(n-1) + x(n-2))$

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