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On Regular Power-Substitution**

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Abstract The necessary and sufficient conditions under which a ring satisfies regular power-substitution are investigated. It is shown that a ring R satisfies regular power-substitution if and only if $a\overline{\sim}b$ in R implies that there exist $n \in \mathbb{N}$ and a $U \in \mathrm{GL}_n(R)$ such that $aU = Ub$ if and only if for any regular $x \in R$ there exist $m, n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $x^m I_n = x^m U x^m$, where $a\overline{\sim}b$ means that there exists $x, y, z \in R$ such that $a = ybx$, $b = xaz$ and $x = xyx = xzx$. It is proved that every directly finite simple ring satisfies regular power-substitution. Some applications for stably free R-modules are also obtained.

Keywords Regular power-substitution, Regular power-cancellation, Stably free module
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1 Introduction

Let R be an associative ring with identity. We say that R satisfies power-substitution in case $aR+bR = R$ implies that there exist $n \in \mathbb{N}$ and $Y \in M_n(R)$ such that $aI_n + bY \in \mathrm{GL}_n(R)$. If R satisfies power-substitution, then R satisfies power-cancellation, i.e., $R \oplus B \cong R \oplus C \Longrightarrow B^n \cong C^n$ for some $n \in \mathbb{N}$. Many authors have studied power-substitution such as [1–4, 9, 11, 12]. We introduce, in this article, a new class of partially power cancellations, i.e., regular power-substitution. A ring R is said to satisfy regular power-substitution in case for any regular $x \in R$ there exist $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $xI_n = xUx$. Many classes of rings of interest satisfy regular power-substitution. For instance: (1) all Abelian ring, including all commutative rings, (2) all domains, (3) all rings satisfying power-substitution, including all rings having stable range one, and hence all unit-regular rings, all strongly π-regular rings and all unit π-regular rings, (4) all directly finite simple rings (see Theorem 3.3). But there exist many rings satisfying regular power-substitution which do not belong to the proceeding classes, e.g., $M_2(\mathbb{Z})$. From these, we see that many known classes of rings satisfy regular power-substitution.

In this article, many characterizations of regular power-substitution are obtained. We prove that R satisfies regular power-substitution if and only if $a\overline{\sim}b$ implies that there exist $n \in \mathbb{N}$ and a $U \in \mathrm{GL}_n(R)$ such that $aU = Ub$ if and only if for any regular $x \in R$ there exist $m, n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $x^m I_n = x^m U x^m$. Further, we prove that every directly finite simple ring satisfies regular power-substitution. Some applications for stably free R-modules are obtained

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as well. These extend many known results on power-substitution of modules.

Throughout this paper, all rings are associative with identity and all modules are right modules. We say that an element $x \in R$ is regular provided that $x = xyx$ for a $y \in R$. Let $M_n(R)$ be the ring of $n \times n$ matrices over R. $M_n(R)$ has identity I_n, and its group of units is the general linear group $\mathrm{GL}_n(R)$. We always use $\mathbb{N}$ to denote the set of all natural numbers.

2 Pseudo-similarity

We begin with a simple characterization of such rings.

Theorem 2.1 *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies regular power-substitution;*

(2) *Whenever $ax + b = 1$ with $ba = 0$, there exist $n \in \mathbb{N}$ and $Y \in M_n(R)$ such that $aI_n + bY \in \mathrm{GL}_n(R)$.*

Proof (1) $\Rightarrow$ (2) Suppose that $ax + b = 1$ with $ba = 0$. Then $axa = a$. So we can find $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $aI_n = aUa$. Set $E = aU$. Then $aI_n = EU$ and $E = E^2$. So we see that $EUx + bI_n = I_n$, and then $EUx(I_n - E) + b(I_n - E) = I_n - E$. It follows that $aI_n + b(I_n - E)U^{-1} = EU^{-1} + b(I_n - E)U^{-1} = (I_n - EUx(I_n - E))U^{-1}$. Set $Y = (I_n - E)U^{-1}$. We have $aI_n + bY \in \mathrm{GL}_n(R)$, as required.

(2) $\Rightarrow$ (1) Given any regular $x \in R$, there exists $y \in R$ such that $x = xyx$ and $y = yxy$. Since $yx + (1 - yx) = 1$ and $(1 - yx)y = 0$, we can find $n \in \mathbb{N}$ and $Y \in M_n(R)$ such that $yI_n + (1 - yx)Y = U \in \mathrm{GL}_n(R)$. Therefore $xI_n = x(yI_n + (1 - yx)Y)x = xUx$, as asserted.

Further, we claim that a ring R satisfies regular power-substitution if and only if for any regular $a, b \in R$, $aR + bR = R$ implies that there exist $n \in \mathbb{N}$ and $Y \in M_n(R)$ such that $aI_n + bY \in \mathrm{GL}_n(R)$. Clearly, every ring satisfying power-substitution satisfies regular power-substitution.

Corollary 2.1 *Let R be a ring, and let $e = e^2 \in R$. If R satisfies regular power-substitution, then so does eRe.*

Proof Suppose that $ax+b = e$ with $a, x, b \in eRe$ and $ba = 0$. Then we have $(a+1-e)(x+1-e)+b = 1$. Clearly, $b(a+1-e) = 0$. Since R satisfies regular power-substitution, there exist $n \in \mathbb{N}$ and $Y \in M_n(R)$ such that $(a+1-e)I_n + bY \in \mathrm{GL}_n(R)$. We infer that $U((a+1-e)I_n + bY) = ((a + 1 - e)I_n + bY)U = I_n$. Clearly, $(1 - e)U = (1 - e)I_n$. Hence $eUe = Ue$. Thus we have $(eUe)(aI_n + b(eYe)) = (aI_n + b(eYe))(eUe) = eI_n$. Hence, $aI_n + b(eYe) \in \mathrm{GL}_n(eRe)$. It follows from Theorem 2.1 that eRe satisfies regular power-substitution.

Let M be an R-R-bimodule. Then the module extension of R by M is the ring $R \boxtimes M$ with the usual addition and multiplication defined by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$, $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. For module extensions of regular power-substitution, we prove that a ring R satisfies regular power-substitution if and only if so does $R \boxtimes M$. Further, we see that a ring R satisfies regular power-substitution ring if and only if so does the ring of all $n \times n$ lower (upper) triangular matrices over R.

Recall that $a \overset{\sim}{\frown} b$ with $a, b \in R$ if there exist some $x, y, z \in R$ such that $a = ybx, b = xaz$ and $x = xyx = xzx$. Guralnick and Lanski showed that a ring R satisfies internal cancellation if and only if pseudo-similarity is equivalent to similarity in R. In the sequel, we observe that regular power-substitution can be characterized by pseudo-similarity.

Theorem 2.2 *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies regular power-substitution;*

(2) *Whenever $a \overset{\sim}{\frown} b$, there exist $n \in \mathbb{N}$ and a $U \in \mathrm{GL}_n(R)$ such that $aU = Ub$.*

Proof (1) $\Rightarrow$ (2) Suppose that $a \overset{\sim}{\frown} b$. Obviously, we have $x, y \in R$ such that $a = xby$, $b = yax$, $x = xyx$ and $y = yxy$. As R satisfies regular power-substitution, we can find $n \in \mathbb{N}$ and $V \in \mathrm{GL}_n(R)$ such that $yI_n = yVy$. Set $U = (I_n - xyI_n - Vy)V(I_n - yxI_n - yV)$. We verify $U^{-1} = (I_n - yxI_n - yV)V^{-1}(I_n - xyI_n - Vy)$. Furthermore, we have $aI_n = UbU^{-1}$, as required.

(2) $\Rightarrow$ (1) Given any regular $x \in R$, we have a $y \in R$ such that $x = xyx$ and $y = yxy$. Since $xy = x(yx)y$ and $yx = y(xy)x$, we see that $xy \overset{\sim}{\frown} yx$. So there are $n \in \mathbb{N}$ and $V \in \mathrm{GL}_n(R)$ such that $(xy)I_n = V(yx)V^{-1}$. Hence $(1-xy)I_n = V(1-yx)V^{-1}$. Set $A = (1-xy)V(1-yx)$ and $B = (1-yx)V^{-1}(1-xy)$. Then $(1-xy)I_n = AB$ and $(1-yx)I_n = BA$ with $A \in (1-xy)M_n(R)(1-yx)$ and $B \in (1-yx)M_n(R)(1-xy)$. We deduce that $\psi : ((1-xy)R)^n \cong ((1-yx)R)^n$. Clearly, $R^n = (yxR)^n \oplus ((1-yx)R)^n = (xyR)^n \oplus ((1-xy)R)$ with $\phi^* : (xyR)^n = (xR)^n \cong (yxR)^n$. Define $U \in \mathrm{End}_R(R^n)$ so that U restricts to ϕ^* and U restricts to ψ. Then $xI_n = xUx$ with $U \in \mathrm{GL}_n(R)$, as desired.

Recall that an element $a \in R$ is strongly π-regular if there exist $n \in \mathbb{N}$ and $x \in R$ such that $a^n = a^{n+1}x$, $ax = xa$ and $x = xax$. We say that the solution $x \in R$ is a Drazin inverse of a.

Corollary 2.2 *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies regular power-substitution;*

(2) *Whenever $ab, ba \in R$ are strongly π-regular, there exist $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $(ab)^d U = U(ba)^d$.*

Proof (2) $\Rightarrow$ (1) Given any regular $x \in R$, there exists $y \in R$ such that $x = xyx$ and $y = yxy$. Obviously, $xy, yx \in R$ both have Drazin inverses, so we have $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $(xy)^d I_n = U(yx)^d U^{-1}$. That is, $xyI_n = UyxU^{-1}$. Analogously to the consideration in Theorem 2.2, we have $V \in \mathrm{GL}_n(R)$ such that $xI_n = xVx$, as required.

(1) $\Rightarrow$ (2) Suppose that ab and ba have Drazin inverses. One easily checks that $(ab)^d = a(ba)^d(ba)^d b$, $(ba)^d = (ba)^d b(ab)^d a$ and $(ba)^d ba(ba)^d b = (ba)^d b$. So $(ba)^d \overset{\sim}{\frown} (ab)^d$. Therefore we complete the proof by Theorem 2.2.

Theorem 2.3 *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies regular power-substitution;*

(2) *For any idempotents $e, f \in R$, $eR \cong fR$ implies that there exist $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $eU = Uf$;*

(3) *For any idempotents $e, f \in R$, $eR \cong fR$ implies that there exist $n \in \mathbb{N}$ and $U, V \in$*

$\mathrm{GL}_n(R)$ *such that* $eI_n = UfV$.

Proof (1) $\Rightarrow$ (2) For any idempotents $e, f \in R$, $eR \cong fR$ implies that $e \overset{\sim}{\approx} f$. So there exist $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $eI_n = UfU^{-1}$ by Theorem 2.2.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Given any regular $x \in R$, there is a $y \in R$ such that $x = xyx$ and $y = yxy$. Clearly, $xy \overset{\sim}{\approx} yx$. So we can find $n \in \mathbb{N}$ and $U, V \in \mathrm{GL}_n(R)$ such that $eI_n = VfU^{-1}$. Set $Y = VfV^{-1}$, and $W = I_n - eI_n + Y$. Then $Ye = VfV^{-1}e = VfV^{-1}VfU^{-1} = VfU^{-1} = eI_n$; $Y = VfV^{-1} = Vf(U^{-1}U)V^{-1} = eUV^{-1}$. So $eY = Y$. It is easy to check that $W^{-1} = I_n + eI_n - Y$. Since $eVf = Vf$, we deduce that $WVf = (I_n - eI_n + Y)Vf = Vf - eVf + YVf = Vf$. Also we have $eWV = e(I_n - eI_n + Y)V = eYV = eVf = Vf$. Thus, $eWU = WUf$. That is, $eI_n = WUf(WU)^{-1}$. Similarly to the consideration in Theorem 2.2, we show that $xI_n = xWx$ for some $W \in \mathrm{GL}_n(R)$. Therefore R satisfies regular power-substitution.

Corollary 2.3 *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies regular power-substitution;*

(2) *For any regular $a, b \in R$, $aR \cong bR$ implies that there exist $n \in \mathbb{N}$ and $U, V \in \mathrm{GL}_n(R)$ such that $aU = Vb$.*

Proof (2) $\Rightarrow$ (1) Clear from Theorem 2.3.

(1) $\Rightarrow$ (2) Suppose that $aR \cong bR$ with regular $a, b \in R$. Since a and b are regular, we have idempotents $e, f \in R$ such that $aR = eR$ and $bR = fR$. Hence $eR \cong fR$. It follows by Theorem 2.3 that there exist $s \in \mathbb{N}$ and $V_1 \in \mathrm{GL}_s(R)$ such that $eI_s = V_1 f V_1^{-1}$. From $aR = eR$, we have an $x \in R$ such that $ax = e$. Since $(xe)a(xe) = xe$ and $a(xe) = e$, we may assume that $x \in R$ is regular. So we have $t \in \mathbb{N}$ and $W \in \mathrm{GL}_t(R)$ such that $xI_t = xWx$. Set $E = xW$. Then $xI_t = EW^{-1}$ and $E = E^2$. Clearly, we have $y \in R$ such that $a = ey$. It follows from $xy + (1 - xy) = 1$ that $EW^{-1}y + (1 - xy)I_t = I_t$; hence, $xI_t + (1 - xy)(I_t - E)W^{-1} = (I_t - EW^{-1}y(I_t - E))W^{-1}$. Set $V_2 = (I_t - EW^{-1}y(I_t - E))W^{-1}$. As $axy = a$, we deduce that $eI_t = axI_t = aV_2$. Likewise, we have $m \in \mathbb{N}$ and $V_2 \in \mathrm{GL}_m(R)$ such that $fI_m = bV_3$. Set $U = \mathrm{diag}(V_2, \cdots, V_2)_{sm}\mathrm{diag}(V_1, \cdots, V_1)_{tm}\mathrm{diag}(V_3, \cdots, V_3)_{st}^{-1}$ and $V = \mathrm{diag}(V_1, \cdots, V_1)_{tm}$. Then we see that $U, V \in \mathrm{GL}_{stm}(R)$ and $aU = Vb$, as required.

3 Power-Cancellation

Now we investigate power-cancellation of modules over regular power-substitution.

Theorem 3.1 *Let P be a right R-module. Then the following are equivalent:*

(1) $\mathrm{End}_R(P)$ *satisfies regular power-substitution;*

(2) *Whenever $P \cong A \oplus B \cong A \oplus C$, there exists $n \in \mathbb{N}$ such that $B^n \cong C^n$.*

Proof (1) $\Rightarrow$ (2) Suppose that $P \cong A \oplus B \cong A \oplus C$. Then we have right R-module decompositions $P = A_1 \oplus B' = A_2 \oplus C'$ such that $A_1 \cong A \cong A_2, B' \cong B$ and $C' \cong C$. Let $e : A = A_1 \oplus B' \to A_1 \to A_1 \oplus B' = A$ be given by $e(a_1, b') = a_1$ for any $a_1 \in A_1$, $b' \in B'$ and $f : A = A_2 \oplus C' \to A_2 \to A_2 \oplus C' = A$ be given by $f(a_2, c') = a_2$ for any $a_2 \in A_2$,

$c' \in C'$. Assume that $\psi : A_1 \cong A_2$. Let $x : A = A_1 \oplus B' \to A_1 \cong A_2 \to A_2 \oplus C' = A$ be given by $e(a_1, b') = \psi(a_1)$ for any $a_1 \in A_1$, $b' \in B'$ and $y : A = A_2 \oplus C' \to A_2 \to A_1 \oplus B' = A$ be given by $y(a_2, c') = \psi^{-1}(a_2)$ for any $a_2 \in A_2$, $c' \in C'$. Clearly, $e = yfx$, $f = xey$ and $x = xyx$. Hence $e\overline{\sim}f$. In view of Theorem 2.3, there exist $n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $eI_n = UfU^{-1}$, and then $(1-e)I_n = U(1-f)U^{-1}$. Define right R-module homomorphism $\phi : ((1-e)P)^n \to ((1-f)P)^n$ by

$$\phi\left((1-e)I_n\begin{pmatrix}p_1\\ \vdots\\ p_n\end{pmatrix}\right) = U^{-1}\left((1-e)I_n\begin{pmatrix}p_1\\ \vdots\\ p_n\end{pmatrix}\right).$$

Clearly, ϕ is well defined. In addition, ϕ is injective. Given any $(1-f)I_n\begin{pmatrix}q_1\\ \vdots\\ q_n\end{pmatrix} \in ((1-e)P)^n$, we have $(1-e)I_nU\begin{pmatrix}q_1\\ \vdots\\ q_n\end{pmatrix} \in ((1-e)P)^n$ such that

$$\phi\left((1-e)I_nU\begin{pmatrix}q_1\\ \vdots\\ q_n\end{pmatrix}\right) = (1-f)I_n\begin{pmatrix}q_1\\ \vdots\\ q_n\end{pmatrix}.$$

Therefore we show that $B^n \cong (B')^n \cong (C')^n \cong C^n$, as required.

(2) $\Rightarrow$ (1) Given any regular $x \in \mathrm{End}_R(P)$, we have a $y \in \mathrm{End}_R(P)$ such that $x = xyx$ and $y = yxy$. Obviously, we have right R-module decompositions $P = (xy)P \oplus (1-xy)P = (yxP) \oplus (1-yx)P$ with $\psi : (xy)P \cong (yx)P$. Thus we have $n \in \mathbb{N}$ such that $\phi : ((1-xy)P)^n \cong ((1-yx)P)^n$. Denote by $\psi^* : ((xy)P)^n \cong ((yx)P)^n$ the corresponding homomorphism of ψ. Define $U \in \mathrm{End}_R(P^n)$ so that U restricts to ψ^* and U restricts to ϕ. Then $xI_n = xUx$ with $U \in \mathrm{GL}_n(\mathrm{End}_R(P))$, as desired.

A projective right R-module P is called stably free in case there is a free module F of finite dimension such that the direct sum $P \oplus F$ is free (see [9]). We say that P is power free in case P^s is free for a positive integer s. An interesting problem is when a stably free module is power free. If R is a right Noetherian ring or a commutative ring, then every stably free module is power free (see [9, Theorems 5.10 and 5.11]). We now observe the following fact.

Corollary 3.1 *If $M_n(R)$ satisfies regular power-substitution for all $n \in \mathbb{N}$, then every stably free right R-module is power free.*

Proof A theorem of Gabel guarantees that every non-finitely generated projective right R-module is always free, so it suffices to consider finitely generated projective right R-modules. Let P be a right R-module with $P \oplus R^m \cong R^n$. Assume that $n \geq m$. Then $R^n \cong P \oplus R^m \cong R^{n-m} \oplus R^m$. Clearly, $\mathrm{End}_R(R^n) \cong M_n(R)$ satisfies regular power-substitution for all $n \in \mathbb{N}$. It follows from Theorem 3.1 that $P^s \cong R^{s(n-m)}$ is free for some $s \in \mathbb{N}$.

Assume that $n < m$. Then we have $R^n \cong P \oplus R^{m-n} \oplus R^n \cong 0 \oplus R^n$. Inasmuch as $\mathrm{End}_R(R^n)$ satisfies regular power-substitution for all $n \in \mathbb{N}$, from Theorem 3.1, we can find $s \in \mathbb{N}$ such that $P^s \oplus R^{s(m-n)} \cong 0$, a contradiction. Therefore we conclude that P is power free, as asserted.

Recall that a ring R is directly finite provided that for any $x, y \in R$, $xy = 1$ if and only if $yx = 1$. Obviously, every ring satisfying regular power-substitution is directly finite. Further, we observe the following interesting fact.

Theorem 3.2 *Every directly finite simple ring satisfies regular power-substitution.*

Proof Let R be a directly finite simple ring. Given $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then there exist idempotents $e, f \in R$ such that $B_1 = eR$ and $B_2 = fR$. If $B_1 = 0$, then $R \cong R \oplus B_2$, hence, $B_2 = 0$. Likewise, $B_2 = 0$ implies $B_1 = 0$. Thus, we may assume that $B_1 \neq 0$ and $B_2 \neq 0$. Since R is simple, $ReR = RfR = R$. Thus, we have some s_i, t_i $(1 \leq i \leq n)$ such that $1 = \sum\limits_{i=1}^{n} s_i e t_i$. Construct a map $\varphi : n(eR) \to R$ given by $\varphi(er_1, \cdots, er_n) = \sum\limits_{i=1}^{n} s_i e r_i$ for any $(er_1, \cdots, er_n) \in n(eR)$. Obviously, φ is an R-epimorphism, and so $R \oplus \operatorname{Ker}\varphi \overset{\theta}{\cong} n(eR)$. Hence, $A_1 \lesssim^{\oplus} R \lesssim^{\oplus} n(eR)$. Likewise, $A_2 \lesssim^{\oplus} R \lesssim^{\oplus} n(fR)$.

Write $n(eR) \cong A_1 \oplus D$ and $n(fR) \cong A_2 \oplus E$. Then $(n+1)(eR) \cong eR \oplus (A_1 \oplus D) \cong fR \oplus (A_2 \oplus D) \cong fR \oplus (A_1 \oplus D) \cong fR \oplus n(eR)$. Likewise, $(n+1)(fR) \cong eR \oplus n(fR)$. For any $m \geq 2n$, write $m = 2n + p$, where $p \geq 0$. Then $m(eR) \cong n(eR) \oplus (n+p)(eR) \cong n(eR) \oplus (n+p)(fR) \cong n(eR) \oplus n(fR) \oplus p(fR) \cong (n+n)(fR) \oplus p(fR) \cong m(fR)$. Thus, there exists $m \in \mathbb{N}$ such that $mB_1 \cong mB_2$. Therefore R satisfies regular power-substitution by Theorem 3.1.

A ring R is an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus\limits_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \Big(\bigoplus\limits_{i \in I} A_i'\Big)$. The class of exchange rings is very large. It includes all regular rings, all π-regular rings, all strongly π-regular rings, all semiperfect rings, all left or right continuous rings, all clean rings, all unit C^*-algebras of real rank zero and all right semi-Artinian rings (see [5–7]). We easily see that every exchange ring satisfying regular power-substitution satisfies power-substitution. Immediately, we deduce that every directly finite simple exchange ring satisfies power-cancellation. This provides a large class of such exchange rings.

Corollary 3.2 *Let R be a simple ring. If $M_n(R)$ is directly finite for all $n \in \mathbb{N}$, then every stably free right R-module is power free.*

Proof Since R is simple, so is $M_n(R)$. Thus, each $M_n(R)$ is directly finite, simple ring. According to Theorem 3.2, $M_n(R)$ satisfies regular power-substitution. Therefore we complete the proof by Corollary 3.1.

4 Unit-Regularity

Recall that an element $x \in R$ is unit π-regular provided that there exist an $m \in \mathbb{N}$ and a $u \in U(R)$ such that $x^m = x^m u x^m$. In [3], the author investigated power-substitution by means of unit π-regularity. The main purpose of this section is to extend the corresponding results on power-substitution to regular power-substitution by virtue of unit π-regularity.

Lemma 4.1 *Let R be a ring. If for any regular $x, y \in R$, there exist $n \in \mathbb{N}$ and $A \in M_n(R)$ such that $xI_n - A$ is unit-regular and $I_n - yA \in \mathrm{GL}_n(R)$, then R satisfies regular power-substitution.*

Proof Let $x \in R$ be regular. Then we have a $y \in R$ such that $x = xyx$ and $y = yxy$. So there is a positive integer n and a matrix $A \in M_n(R)$ such that $yI_n - A = W$ is unit-regular and $I_n - xA \in \mathrm{GL}_n(R)$. Thus $xW + (1 - xy)I_n = I_n - x(yI_n - W) \in \mathrm{GL}_n(R)$, hence, $xI_n + (1 - xy)W^{-1} = U \in \mathrm{GL}_n(R)$. Therefore $xI_n = xyxI_n = xyU$, and then $xI_n = xU^{-1}x$, as asserted.

Theorem 4.1 *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies regular power-substitution;*

(2) *For any regular $x \in R$, there exist $m, n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $x^m I_n = x^m U x^m$.*

Proof (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (1) Given any regular $x, y \in R$, there are $m, n \in \mathbb{N}$ and $U \in \mathrm{GL}_n(R)$ such that $x^m I_n = x^m U x^m$. Set

$$A = \begin{pmatrix} 0_n & \cdots & 0_n & 0_n \\ I_n & \cdots & 0_n & 0_n \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & I_n & 0_n \end{pmatrix}, \quad B = \begin{pmatrix} I_n & xI_n & \cdots & x^{m-1}I_n \\ 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & & \vdots \\ 0_n & 0_n & \cdots & I_n \end{pmatrix},$$

$$C = \begin{pmatrix} y^{m-1}I_n & \cdots & yI_n & I_n \\ I_n & \cdots & 0_n & 0_n \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & I_n & 0_n \end{pmatrix} \in M_{mn}(R).$$

Hence, we have

$$B(xI_{mn} - A) = \begin{pmatrix} 0_n & \cdots & 0_n & x^m I_n \\ -I_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & xI_n & * \\ 0_n & \cdots & -I_n & * \end{pmatrix}, \quad C(I_{mn} - yA) = \begin{pmatrix} 0_n & \cdots & 0_n & I_n \\ I_n & \cdots & 0_n & * \\ -yI_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & I_n & * \end{pmatrix}.$$

Clearly, $B, C \in \mathrm{GL}_{mn}(R)$. Hence $I_{mn} - yA \in \mathrm{GL}_{mn}(R)$. Since $x^m I_n$ is unit-regular, we assume that $x^m I_n = eu$ for some $e = e^2 \in M_n(R)$ and $u \in \mathrm{GL}_n(R)$. So we have

$$\begin{aligned} xI_{mn} - A &= B^{-1} \begin{pmatrix} 0_n & \cdots & 0_n & eu^{-1} \\ -I_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & xI_n & * \\ 0_n & \cdots & -I_n & * \end{pmatrix} \\ &= B^{-1} \begin{pmatrix} e & 0_n & \cdots & 0_n \\ 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & & \vdots \\ 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & I_n \end{pmatrix} \begin{pmatrix} 0_n & \cdots & 0_n & u \\ -I_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & xI_n & * \\ 0_n & \cdots & -I_n & * \end{pmatrix}. \end{aligned}$$

Thus $xI_{mn} - A = (xI_{mn} - A)U^{-1}EB(xI_{mn} - A)$, where

$$U = \begin{pmatrix} 0_n & \cdots & 0 & u^{-1} \\ -I_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & xI_n & * \\ 0_n & \cdots & -I_n & * \end{pmatrix}, \quad E = \begin{pmatrix} e & 0_n & \cdots & 0_n \\ 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & & \vdots \\ 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & I_n \end{pmatrix}.$$

It follows from $(xI_{mn} - A)U^{-1}EB + (I_{mn} - (xI_{mn} - A)U^{-1}EB) = I_{mn}$ that $xI_{mn} - A = (xI_{mn} - A)W(xI_{mn} - A)$, where $W = U^{-1}(I_{mn} + (I_{mn} - E)B(xI_{mn} - A)U^{-1}E)^{-1}B$. That is, $xI_{mn} - A$ is unit-regular. According to Lemma 4.1, R satisfies regular power-substitution.

Corollary 4.1 *If every regular element in R is unit π-regular, then R satisfies regular power-substitution.*

Proof It immediately follows from Theorem 4.1.

Recall that an element $a \in R$ is strongly π-regular provided that there exist $n \in \mathbb{N}$ and $x \in R$ such that $a^n = a^{n+1}x$, $ax = xa$. As an immediate consequence of Corollary 4.1, we deduce that all rings in which every regular element is strongly π-regular satisfy regular power-substitution. Though every strongly π-regular ring is unit π-regular, it is worth noting that the converse is not true. Let $R = \prod\limits_{n=1}^{\infty} M_n(S)$, where S is a unit-regular ring. Then R is unit-regular. Hence, every element in R is unit π-regular. Let $a_1 = 0$ and $a_n = e_{12} + e_{23} + \cdots + e_{(n-1)n}$, where e_{ij} is the $n \times n$ matrix with 1 in the (i,j) position and 0's elsewhere $(n \geq 2)$. One easily checks that $e_{ij}e_{kl} = e_{il}$ $(j = k)$ and $e_{ij}e_{kl} = 0$ $(j \neq k)$. Thus, $a_n^n = 0$ and $a_n^{n-1} \neq 0$ $(n \in \mathbb{N})$. Choose $a = (a_1, a_2, \cdots)$. Then $aR \underset{\neq}{\supset} a^2R \underset{\neq}{\supset} \cdots$. Thus, regular element $a \in R$ is unit π-regular, while $a \in R$ is not strongly π-regular.

Lemma 4.2 *Let R be a ring. If for any regular $x, y \in R$, there exist a positive integer n and $A \in M_n(R)$ such that $xI_n - A$ is invertible and $I_n - yA$ is unit-regular, then R satisfies regular power-substitution.*

Proof Let $x \in R$ be regular. Then we have a $y \in R$ such that $x = xyx$ and $y = yxy$. So there is a positive integer n and a matrix $A \in M_n(R)$ such that $yI_n - A = W \in \mathrm{GL}_n(R)$ and $I_n - xA \in M_n(R)$ is unit-regular. Thus $xW + (1 - xy)I_n = I_n - x(yI_n - W) \in M_n(R)$; hence, $U := xI_n + (1 - xy)W^{-1} \in M_n(R)$ is unit-regular. Write $U = EV$, where $E = E^2 \in M_n(R)$, $V \in \mathrm{GL}_n(R)$. Then $(xI_n + (1 - xy)W^{-1})y + (1 - xy)(I_n - (1 - xy)W^{-1}y) = I_n$, hence, $EVy + (1 - xy)(I_n - (1 - xy)W^{-1}y) = I_n$. This infers that $xI_n + (1 - xy)(W^{-1} + (I_n - (1 - xy)W^{-1}y)(I_n - E)V) \in \mathrm{GL}_n(R)$. Consequently, $xI_n = xV^{-1}(I_n + EVy(I_n - E))x$, as desired.

Theorem 4.2 *Let R be a ring. If for any regular $x, y \in R$, there exist positive integers m, n and a matrix $U \in \mathrm{GL}_n(R)$ such that $I_n + x^m(y^mI_n - U)$ is unit-regular, then R satisfies regular power-substitution.*

Proof Given any regular $x, y \in 1 + I$, there exist positive integers m, n and a matrix $U \in U(R)$ such that $I_n + x^m(y^mI_n - U) \in M_n(R)$ is unit-regular. Thus $y^mI_n - a_m$ is invertible,

$I_n + x^m a_m$ is unit-regular and $a_m = y^m I_n - U$. Set

$$A = \begin{pmatrix} 0_n & \cdots & 0 & a_m \\ I_n & \cdots & 0 & 0_n \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & I_n & 0_n \end{pmatrix}, \quad B = \begin{pmatrix} I_n & xI_n & \cdots & x^{m-1}I_n \\ 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & & \vdots \\ 0_n & 0_n & \cdots & I_n \end{pmatrix},$$

$$C = \begin{pmatrix} y^{m-1}I_n & \cdots & yI_n & I_n \\ I_n & \cdots & 0_n & 0_n \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & I_n & 0_n \end{pmatrix} \in M_{mn}(R).$$

Analogously to Theorem 4.1, it is easy to check that

$$B(xI_{mn} - A) = \begin{pmatrix} 0 & \cdots & 0 & x^m I_n + a_m \\ -I_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & xI_n & * \\ 0 & \cdots & -I_n & * \end{pmatrix},$$

$$C(I_{mn} - yA) = \begin{pmatrix} 0_n & \cdots & 0_n & I_n + y^m a_m \\ I_n & \cdots & 0_n & * \\ -yI_n & \cdots & 0_n & * \\ \vdots & & \vdots & \vdots \\ 0_n & \cdots & I_n & * \end{pmatrix}.$$

Since $B, C \in \mathrm{GL}_{mn}(R)$, we show that $xI_{mn} - A$ is invertible and $I_{mn} - yA$ is unit-regular. According to Lemma 4.2, we complete the proof.

Corollary 4.2 *Let R be a ring. If for any regular $x, y \in R$, there exist a positive integer m and a $u \in U(R)$ such that $1 + x^m(y^m - u) \in U(R)$, then R satisfies regular power-substitution.*

Corollary 4.3 *Let R be a unital complex C^*-algebra. If for any regular $x \in R$, there exists some $n \in \mathbb{N}$ such that x^n is the sum of a unitary and a unit, then R satisfies regular power-substitution.*

Proof For any regular $x, y \in R$, we have a unitary $v \in R$ such that $(1 + \|y\|)x^n - v$ is a unit for some $n \in \mathbb{N}$. Let $u = \frac{v}{1+\|y\|}$. Then $x^n - u \in U(R)$. As $\|v\| = 1$, we have $\|y^n u\| < 1$, hence, $1 - y^n u \in U(R)$. So $y^n - u^{-1} \in U(R)$. Let $w = x^n - u$. Then $1 - y^n(x^n - w) \in U(R)$. According to Corollary 4.2, we complete the proof.

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