Chin. Ann. Math.
30B(3), 2009, 239–250
DOI: 10.1007/s11401-008-0225-4

Chinese Annals of Mathematics, Series B

Existence of Canards under Non-generic Conditions***

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Abstract The canard phenomenon occurring in planar fast-slow systems under non-generic conditions is investigated. When the critical manifold has a non-generic fold point, by using the method of asymptotic analysis combined with the recently developed blow-up technique, the existence of a canard is established and the asymptotic expansion of the parameter for which a canard exists is obtained.

Keywords Canard, Slow manifold, Singular perturbation, Blow-up, Non-generic condition
2000 MR Subject Classification 34C23, 34C37, 34E15

1 Introduction and Statement of the Main Result

In a system of singularly perturbed ordinary differential equations, a canard is a trajectory which follows both the attracting and repelling parts of a slow manifold. Canard is a new kind of bifurcation phenomena relative to other bifurcations in singularly perturbed systems, such as Hopf bifurcation (see [24]), homoclinic bifurcation (see [19]), tori bifurcation (see [28]) and so on. Canard phenomena were first found in a study of the van der Pol equation by using the method of nonstandard analysis (see [1, 4]). Later, Eckhaus [8] and Mishchenko et al [2, 14, 21] applied classical asymptotic analysis to the study of canards. Recently, some geometric methods (see, for instance, [7, 15, 16, 25]) have also been used to analyze a variety of canards. In addition to the van der Pol system, canard phenomena have been found and investigated in varying degrees for a variety of chemical, biological and other systems (see for instance [3, 5, 12, 23, 26, 27] and the references therein). However, most of the previous works use generic conditions. Few works consider canards for non-generic conditions. In [18], Li studied the existence of multiple canard cycles for a class of planar fast-slow systems under non-generic conditions by using classical asymptotic analysis. Very recently, Maesschalck and Dumortier [20] used a geometric approach to consider canards in the following fast-slow system:

$$\begin{aligned} \dot{x} &= -y + x^{2n} + O(x^{2n+1}), \\ \dot{y} &= \varepsilon(a + x^{2n-1}) + O(\varepsilon^2), \end{aligned} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, which is a kind of degenerate case for $n > 1$.

Manuscript received June 13, 2008. Published online April 16, 2009.
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***Project supported by the National Natural Science Foundation of China (No. 10701023) and the E-Institutes of Shanghai Municipal Education Commission (No. N.E03004).

The present paper is devoted to the study of canards under another kind of non-generic conditions. When the critical manifold has a non-generic fold point, by using the method of asymptotic analysis combined with the recently developed blow-up technique, we investigate the existence of canards.

Let us consider a one-parameter family of singularly perturbed ODEs,

$$\begin{aligned}\dot{x} &= -y + f(x),\\ \dot{y} &= \varepsilon(x - \lambda),\end{aligned} \tag{1.2}$$

where $0 < \varepsilon \ll 1$, $|\lambda| \ll 1$, and the function f is of class C^k. Assume that $(0, 0)$ is a quadratic fold point of the critical manifold, that is,

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = 0, \quad f^{(4)}(0) > 0. \tag{1.3}$$

Therefore, without loss of generality, in a small neighborhood of the origin, $f(x) = x^4 + \mathcal{O}(x^5)$. The critical manifold of (1.2) consists of the attracting part S_a, the repelling part S_r and the break off point $(0, 0)$. It follows from the geometric singular perturbation theory (see [9, 13]) that outside an arbitrary small neighborhood of the origin, the manifolds S_a and S_r can be perturbed smoothly to locally invariant slow manifolds $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$ (see Figure **??**). Notice that these manifolds are not uniquely defined. An important issue is to study the dynamics of system (1.2) and, in particular, to clarify the behavior of the slow manifolds $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$ in the vicinity of the origin. In what follows, we will prove the existence of the value $\lambda^*(\varepsilon)$ of the parameter λ for which the manifolds $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$ merge in the neighborhood of the origin. The manifold obtained for the parameter value $\lambda^*(\varepsilon)$ is called the maximal canard.

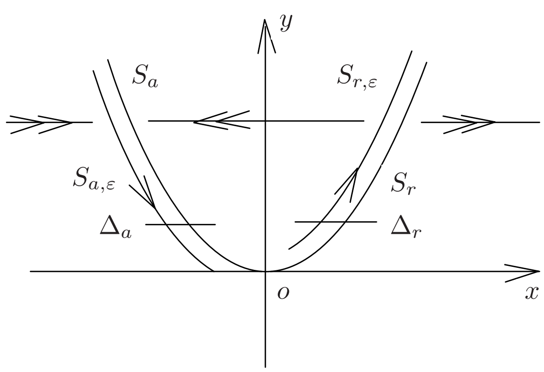

Figure 1 The critical manifold and slow manifolds

Our main result is the following theorem.

Theorem 1.1 *Assume that* (1.3) *holds. There exist* $\varepsilon_0 > 0$ *and* $\lambda^*(= \mathcal{O}(\varepsilon^{\frac{1}{3}}))$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *and* $\lambda = \lambda^*$*, system* (1.2) *has a maximal canard.*

2 Blow-up Analysis

The recently developed blow-up method (see [7, 15]) is essentially a clever coordinate transformation by which the degenerate equilibrium is "blown-up" to a two-sphere. In certain directions transverse to the sphere and even on the sphere, one gains enough hyperbolicity to allow a complete analysis by standard techniques. The technique is a generalization of the well known blow-up methods for degenerate equilibria of planar vector field (see [6]).

Consider the extended system

$$\begin{aligned} \dot{x} &= -y + x^4 + \mathcal{O}(x^5), \\ \dot{y} &= \varepsilon(x - \lambda), \\ \dot{\varepsilon} &= 0, \\ \dot{\lambda} &= 0. \end{aligned} \tag{2.1}$$

The linearization of system (2.1) at the origin has fourfold zero eigenvalue while the linearization at the other points of the critical manifold has a triple zero eigenvalue and one negative (resp. positive) eigenvalue for $x < 0$ (resp. $x > 0$). Therefore, the quadratic fold point $(0, 0, 0)$ is a more degenerate equilibrium point of system (2.1).

To system (2.1) we apply the blow-up transformation $\Phi : B = S^2 \times [-\mu, \mu] \times [0, \rho] \to \mathbb{R}^4$:

$$x = \overline{r}\,\overline{x}, \quad y = \overline{r}^4\overline{y}, \quad \varepsilon = \overline{r}^6\overline{\varepsilon}, \quad \lambda = \overline{r}\overline{\lambda}, \tag{2.2}$$

where μ and ρ are chosen such that system (1.2) is described by the extended system (2.1) in the region $\Phi(B)$. Denote by $\overline{X}$ the blown-up vector field. It is easy to check that there are four equilibria p_a, p_r, q_{in} and q_{out} in the invariant circle $\overline{\lambda} = \overline{r} = \overline{\varepsilon} = 0$, where p_a and p_r correspond to the two branches of the critical manifold, q_{in} corresponds to the incoming critical fibre, and q_{out} corresponds to the outgoing critical fibre. We need two charts to describe all the dynamics, namely the usual rescaling chart K_2 defined by

$$x = r_2 x_2, \quad y = r_2^4 y_2, \quad \varepsilon = r_2^6, \quad \lambda = r_2\lambda_2, \tag{2.3}$$

and the chart K_1 defined by

$$x = r_1 x_1, \quad y = r_1^4, \quad \varepsilon = r_1^6\varepsilon_1, \quad \lambda = r_1\lambda_1, \tag{2.4}$$

with coordinates $(x_1, r_1, \varepsilon_1, \lambda_1) \in \mathbb{R}^4$ and $(x_2, y_2, r_2, \lambda_2) \in \mathbb{R}^4$. Loosely speaking, K_2 describes a neighborhood of the upper half-sphere defined by $\varepsilon_1 > 0$, and K_1 describes a neighborhood of the equator of S^2 defined by $y_2 > 0$.

Lemma 2.1 *Let κ_{12} denote the change of coordinates from K_1 to K_2. Then for $\varepsilon_1 > 0$, κ_{12} is given by*

$$x_2 = x_1\varepsilon_1^{-\frac{1}{6}}, \quad y_2 = \varepsilon_1^{-\frac{2}{3}}, \quad r_2 = r_1\varepsilon_1^{\frac{1}{6}}, \quad \lambda_2 = \lambda_1\varepsilon_1^{-\frac{1}{6}}, \tag{2.5}$$

and for $y_2 > 0$, $\kappa_{21} = \kappa_{12}^{-1}$ is given by

$$x_1 = x_2 y_2^{-\frac{1}{4}}, \quad r_1 = r_2 y_2^{\frac{1}{4}}, \quad \varepsilon_1 = y_2^{-\frac{3}{2}}, \quad \lambda_1 = \lambda_2 y_2^{-\frac{1}{4}}. \tag{2.6}$$

Proof Straightforward computations from (2.3)–(2.4).

2.1 Analysis of the dynamics in the chart K_2

The dynamics of the blown-up vector field $\overline{X}$ in a neighborhood of the upper half-sphere is studied in the chart K_2. Using the blow-up change (2.3) to desingularize the origin of system (2.1), we obtain

$$\begin{aligned} x_2' &= -y_2 + x_2^4 + \sigma r_2 x_2^5 + \mathcal{O}(r_2^2), \\ y_2' &= x_2 - \lambda_2, \end{aligned} \tag{2.7}$$

where the prime denotes the differentiation with respect to t_2, $t_2 \equiv r_2^3 t$, and $\sigma = \frac{f^{(5)}(0)}{5!}$.

Setting $r_2 = \lambda_2 = 0$ in (2.7), we have

$$\begin{aligned} x_2' &= -y_2 + x_2^4, \\ y_2' &= x_2. \end{aligned} \tag{2.8}$$

In what follows, we analyze the global dynamics of system (2.8). It follows from [11, Theorem 4.8] that the origin is a center of (2.8). To investigate the dynamics of the singular points at infinity, we make the Poincaré transformation:

$$z = \frac{1}{x_2}, \quad u = \frac{y_2}{x_2}.$$

From (2.8) it follows that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{z^3}[(1+u^2)z^3 - u], \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{z^2}(uz^3 - 1). \tag{2.9}$$

By rescaling the time $\mathrm{d}\tau = \frac{1}{z^3}\mathrm{d}t$, system (2.9) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = (1+u^2)z^3 - u, \quad \frac{\mathrm{d}z}{\mathrm{d}\tau} = z(uz^3 - 1). \tag{2.10}$$

System (2.10) has an integral line $z = 0$, and the origin is a stable node.

Now we investigate the infinite singular point on y_2-axis. By the transformation

$$z = \frac{1}{y_2}, \quad v = -\frac{x_2}{y_2},$$

system (2.8) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -zv, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{z^3}(z^3 + v^2 z^3 - v^4).$$

Letting $\mathrm{d}\tau = \frac{1}{z^3}\mathrm{d}t$, from the above equations we have

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = z^4 v, \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = z^3 + v^2 z^3 - v^4. \tag{2.11}$$

In order to investigate the dynamics of (2.11) in the neighborhood of $(0, 0)$, we make the transformation

$$z = w^3, \quad v = v.$$

From (2.11), we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = 3w^2 v, \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = w + wv^2 - v^4. \tag{2.12}$$

It follows from [11, Theorem 5.5, Chapter 1] that the origin is an equilibrium of saddle type for system (2.12). Hence, we have the global phase portrait of (2.8) as shown in Figure 2.

Lemma 2.2 *System* (2.8) *has a unique separatrix denoted by* γ_2 *homoclinic to the infinite singular point* p *on the compactified Poincaré disc.*

Proof From the above analysis and by noting that the trajectories of (2.8) are symmetric about the y-axis, the result follows.

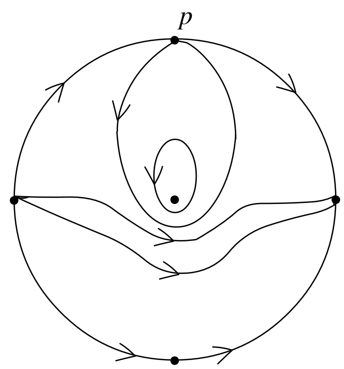

Figure 2 The global phase portrait of (2.8)

2.2 Analysis of the dynamics in the chart K_1

The chart K_1 is used to analyze the dynamics of the blown-up vector field $\overline{X}$ in a neighborhood of the equator. Substituting (2.4) into system (2.1), we obtain in K_1,

$$\begin{aligned} x_1' &= -1 + x_1^4 - \frac{1}{4}\varepsilon_1 x_1(x_1 - \lambda_1) + \mathcal{O}(r_1), \\ r_1' &= \frac{1}{4} r_1 \varepsilon_1 (x_1 - \lambda_1), \\ \varepsilon_1' &= -\frac{3}{2}\varepsilon_1^2 (x_1 - \lambda_1), \\ \lambda_1' &= -\frac{1}{4}\lambda_1 \varepsilon_1 (x_1 - \lambda_1), \end{aligned} \tag{2.13}$$

where the prime denotes the derivative with respect to a rescaled time variable t_1, $t_1 \equiv r_1^3 t$.

The hyperplane $r_1 = 0$, $\varepsilon_1 = 0$, $\lambda_1 = 0$ is invariant for the flow of system (2.13), and the invariant line $l_1 := \{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\}$ contains two equilibria $p_a = (-1, 0, 0, 0)$ and $p_r = (1, 0, 0, 0)$.

The dynamics in the invariant plane $\varepsilon_1 = \lambda_1 = 0$ is governed by

$$\begin{aligned} x_1' &= -1 + x_1^4 + \mathcal{O}(r_1), \\ r_1' &= 0. \end{aligned} \tag{2.14}$$

For r_1 small enough, it follows from the Implicit Function Theorem that system (2.14) has two curves $S_{a,1}$ and $S_{r,1}$ of equilibria emanating from p_a and p_r, respectively. The dynamics in the invariant plane $r_1 = \lambda_1 = 0$ is governed by

$$\begin{aligned} x_1' &= -1 + x_1^4 - \frac{1}{4}\varepsilon_1 x_1^2, \\ \varepsilon_1' &= -\frac{3}{2}\varepsilon_1^2 x_1. \end{aligned} \tag{2.15}$$

Obviously, there exist an attracting one-dimensional center manifold $N_{a,1}$ at p_a and a repelling one-dimensional center manifold $N_{r,1}$ at p_r. Define

$$D_1 \equiv \{(x_1, r_1, \varepsilon_1, \lambda_1) : -2 < x_1 < 2,\ 0 \le r_1 \le \rho,\ 0 \le \varepsilon_1 \le \delta,\ -\mu < \lambda_1 < \mu\},$$

where ρ, δ and μ are small positive constants.

Lemma 2.3 *The constants ρ, δ and μ can be chosen sufficiently small such that the following assertions hold for system* (2.13).

(i) *There exists an attracting three-dimensional C^k-center manifold $M_{a,1}$ at p_a that contains the curve of equilibria $S_{a,1}$ and the center manifold $N_{a,1}$. In D_1, the manifold $M_{a,1}$ is given as a graph $x_1 = h_a(r_1, \varepsilon_1, \lambda_1)$ with*

$$h_a = -1 - \frac{1}{22}\varepsilon_1 + \frac{\sigma}{4}r_1 + \frac{3\sigma}{484}\varepsilon_1 r_1 - \frac{4}{121}\varepsilon_1\lambda_1 - \frac{1}{968}\varepsilon_1^2 + \frac{3\sigma^2}{32}r_1^2 + o(2). \tag{2.16}$$

(ii) *There exists an repelling three-dimensional C^k-center manifold $M_{r,1}$ at p_r that contains the curve of equilibria $S_{r,1}$ and the center manifold $N_{r,1}$. In D_1 the manifold $M_{r,1}$ is given as a graph $x_1 = h_r(r_1, \varepsilon_1, \lambda_1)$ with*

$$h_r = 1 + \frac{1}{22}\varepsilon_1 - \frac{\sigma}{4}r_1 - \frac{3\sigma}{484}\varepsilon_1 r_1 - \frac{4}{121}\varepsilon_1\lambda_1 + \frac{1}{968}\varepsilon_1^2 - \frac{3\sigma^2}{32}r_1^2 + o(2), \tag{2.17}$$

where $o(2) = o(\varepsilon_1^2, r_1^2, \lambda_1^2, \varepsilon_1 r_1, \varepsilon_1\lambda_1, r_1\lambda_1)$.

Proof The assertions follow from the center manifold theory (see [10, 17] for instance), where (2.16) and (2.17) can be obtained by using the method of [17].

We define the sections:

$$\begin{aligned}
\Delta_{a,1}^{\text{in}} &\equiv \{(x_1, r_1, \varepsilon_1, \lambda_1) \in D_1 : r_1 = \rho,\ |1 + x_1| < \beta\},\\
\Delta_{a,1}^{\text{out}} &\equiv \{(x_1, r_1, \varepsilon_1, \lambda_1) \in D_1 : \varepsilon_1 = \delta,\ |1 + x_1| < \beta\},\\
\Delta_{r,1}^{\text{in}} &\equiv \{(x_1, r_1, \varepsilon_1, \lambda_1) \in D_1 : \varepsilon_1 = \delta,\ |1 - x_1| < \beta\},\\
\Delta_{r,1}^{\text{out}} &\equiv \{(x_1, r_1, \varepsilon_1, \lambda_1) \in D_1 : r_1 = \rho,\ |1 - x_1| < \beta\},
\end{aligned}$$

where $\beta > 0$ is small enough. For $\lambda_1 = 0$, Figure 3 shows the geometry of K_1.

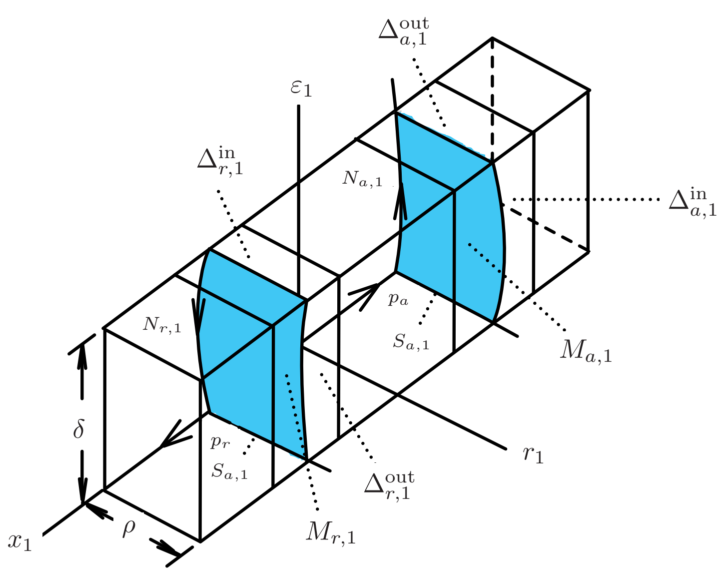

Figure 3 Geometry of K_1 for $\lambda_1 = 0$

Remark 2.1 Note that system (2.8) under the transformation (2.6) corresponds to (2.15) in K_1. The infinite singular point p is "blown-up" into two hyperbolic points p_a and p_r. Therefore, to investigate the dynamics in the neighborhood of p for system (2.8), we only need to analyze the dynamics in the neighborhoods of p_a and p_r for system (2.15) in K_1. From the global phase portrait of (2.8) and Lemma 2.3, we know that, for $y_2 > 0$, $\kappa_{21}(\gamma_2)$ corresponds to the center manifolds $N_{a,1}$ and $N_{r,1}$; that is, γ_2 is a heteroclinic orbit connecting the singular points p_a and p_r in the blown-up vector field $\overline{X}$, which also implies that the center manifolds $N_{a,1}$ and $N_{r,1}$ are unique.

The following result gives an estimate of the separatrix.

Lemma 2.4 *The separatrix γ_2 in K_2 can be parametrized with $(x_2, y_2) = (x_2, \varphi_0(x_2))$ satisfying $\alpha(x_2) \le \varphi_0(x_2) \le \beta(x_2)$, where*

$$\alpha(x_2) = \begin{cases} x_2^4 - 0.25, & |x_2| \ge 1, \\ x_2^4 + 0.3x_2^2 + 0.2|x_2| - 0.75, & |x_2| < 1, \end{cases}$$
$$\beta(x_2) = \begin{cases} x_2^4 - (4x_2^2 + 6|x_2|)^{-1}, & |x_2| \ge 1, \\ x_2^4 + 0.4x_2^2 - 0.5, & |x_2| < 1. \end{cases}$$

Proof In view of symmetry, we only consider the case of $x_2 \le 0$.
For $x_2 \le -1$,

$$\frac{x_2}{-\alpha(x_2) + x_2^4} - \alpha'(x_2) = 4x_2(1 - x_2^2) \ge 0,$$
$$\frac{x_2}{-\beta(x_2) + x_2^4} - \beta'(x_2) = -\frac{48x_2^6 - 144x_2^5 + 108x_2^4 + 4x_2 - 3}{2x_2^2(2x_2 - 3)^2} < 0.$$

For $-1 < x_2 \le 0$,

$$\frac{x_2}{-\alpha(x_2) + x_2^4} - \alpha'(x_2) = -\frac{160x_2^5 - 240x_2^4 - 584x_2^3 - 36x_2^2 + 158x_2 + 45}{4x_2^2 - 6x_2 - 15} > 0,$$
$$\frac{x_2}{-\beta(x_2) + x_2^4} - \beta'(x_2) = -\frac{120x_2^5 - 80x_2^4 - 282x_2^3 - 18x_2^2 + 59x_2 + 15}{5(6x_2^2 - 4x_2 - 15)} < 0.$$

As $x_2 \to -\infty$, we consider the auxiliary functions $\alpha(x_2)$ and $\beta(x_2)$ in K_1. Denote by α_2 (resp. β_2) the curve determined by $y_2 = \alpha(x_2)$ (resp. $y_2 = \beta(x_2)$) in K_2. From (2.6) we have in K_1,

$$\kappa_{21}(\alpha_2) : 1 = x_1^4 - \frac{1}{4}\varepsilon_1^{\frac{2}{3}},$$
$$\kappa_{21}(\beta_2) : 1 = x_1^4 - \frac{\varepsilon_1}{4x_1^2 - 6x_1\varepsilon_1^{\frac{1}{6}}}.$$

It follows from (2.16) that the center manifold $N_{a,1}$ can be represented as

$$x_1 = -1 - \frac{1}{22}\varepsilon_1 + \mathcal{O}(\varepsilon_1^2).$$

Clearly, both $\kappa_{21}(\alpha_2)$ and $\kappa_{21}(\beta_2)$ pass through the point p_a. For $\varepsilon_1 > 0$ small enough, the center manifold $N_{a,1}$ lies between the curves $\kappa_{21}(\alpha_2)$ and $\kappa_{21}(\beta_2)$. Thus, we complete the proof of Lemma 2.4.

3 Proof of Theorem 1.1

For $j = a, r$, let $\Delta_j \equiv \{(x, \rho^4), x \in I_j\}$ be a section of S_j, where I_a and I_r are suitable intervals containing $-\rho$ and ρ, respectively (see Figure **??**). To show the existence of parameter λ for which $S_{a,\varepsilon}$ is connected to $S_{r,\varepsilon}$, we need to extend $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$ down to $x = 0$, which will be studied in K_1 and K_2.

Noting the maps $\Pi_a : \Delta_a \to \Delta^{\rm in}_{a,1}$ and $\Pi_r : \Delta^{\rm out}_{r,1} \to \Delta_r$, we only need to consider the positive and negative trajectories emanating from the center manifolds $M_{a,1}$ and $M_{r,1}$, respectively. Denote by $A(x_1, r_1, \delta, \lambda_1)$ the first intersection point of $\Delta^{\rm out}_{a,1}$ and the positive trajectory emanating from $M_{a,1}$. Then from (2.16) and (2.5) we have

$$x_1^4 = 1 + \frac{2}{11}\delta + \mathcal{O}(\delta^2) + (\sigma\delta^{-\frac{1}{6}} + \mathcal{O}(\delta^{\frac{5}{6}}))r_2 + \mathcal{O}(r_2^2\delta^{-\frac{1}{3}}). \tag{3.1}$$

By eliminating the time variable t_2, it follows from (2.7) that

$$(-y_2 + x_2^4 + \sigma r_2 x_2^5 + \mathcal{O}(r_2^2))\frac{\mathrm{d}y_2}{\mathrm{d}x_2} = x_2 - \lambda_2. \tag{3.2}$$

We seek solutions of (3.2) of the form

$$y_2 = \varphi_0(x_2) + \sum_{i=1}^{\infty}\varphi_i(x_2)r_2^i. \tag{3.3}$$

Let

$$\lambda_2 = \sum_{i=1}^{\infty}\lambda_{2,i}r_2^i. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we obtain

$$\begin{aligned} &\mathcal{O}(1) \text{ order}: \ \varphi_0'(x_2) = \frac{x_2}{x_2^4 - \varphi_0(x_2)}, \\ &\mathcal{O}(r_2) \text{ order}: \ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x_2} = \frac{x_2\varphi_1}{x_2^4 - \varphi_0(x_2)} - \frac{\sigma x_2^6}{(x_2^4 - \varphi_0(x_2))^2} - \frac{\lambda_{2,1}}{x_2^4 - \varphi_0(x_2)}. \end{aligned} \tag{3.5}$$

To obtain the initial condition of (3.5), applying the transformation (2.5) to (3.3) we have

$$1 = \varphi_0(x_1\delta^{-\frac{1}{6}})\delta^{\frac{2}{3}} + \varphi_1(x_1\delta^{-\frac{1}{6}})\delta^{\frac{2}{3}}r_2 + \cdots. \tag{3.6}$$

Comparing the coefficients of r_2 in (3.1) and (3.6), we obtain

$$\varphi_1(x_1\delta^{-\frac{1}{6}}) = \delta^{-\frac{2}{3}}(\sigma\delta^{-\frac{1}{6}} + \mathcal{O}(\delta^{\frac{5}{6}})). \tag{3.7}$$

Therefore, we obtain the solution to (3.5) with the initial condition (3.7):

$$\begin{aligned} \varphi_1(x_2) = {} & \delta^{-\frac{2}{3}}(\sigma\delta^{-\frac{1}{6}} + \mathcal{O}(\delta^{\frac{5}{6}}))\mathrm{e}^{Q(x_2) - Q(x_1\delta^{-\frac{1}{6}})} \\ & - \int_{x_1\delta^{-\frac{1}{6}}}^{x_2} \mathrm{e}^{Q(x_2)-Q(s)}\Big[\frac{\sigma s^6}{(s^4 - \varphi_0(s))^2} + \frac{\lambda_{2,1}}{s^4 - \varphi_0(s)}\Big]\mathrm{d}s, \quad x_2 < 0, \end{aligned}$$

where $Q(x_2) = \int_0^{x_2}\frac{\xi}{(\xi^4 - \varphi_0(\xi))^2}\mathrm{d}\xi$. Taking $\delta \to 0$, we get

$$\varphi_1(x_2) = -\int_{-\infty}^{x_2}\mathrm{e}^{Q(x_2)-Q(s)}\Big[\frac{\sigma s^6}{(s^4 - \varphi_0(s))^2} + \frac{\lambda_{2,1}}{s^4 - \varphi_0(s)}\Big]\mathrm{d}s, \quad x_2 < 0. \tag{3.8}$$

In a similar way, considering the negative trajectory emanating from $M_{r,1}$, we can obtain

$$\varphi_1(x_2) = \int_{x_2}^{+\infty} \mathrm{e}^{Q(x_2)-Q(s)}\Big[\frac{\sigma s^6}{(s^4-\varphi_0(s))^2} + \frac{\lambda_{2,1}}{s^4-\varphi_0(s)}\Big]\mathrm{d}s, \quad x_2 \geq 0. \tag{3.9}$$

By the continuity of $\varphi_1(x_2)$ at $x_2 = 0$, it follows from (3.8)–(3.9) that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-Q(s)}\Big[\frac{\sigma s^6}{(s^4-\varphi_0(s))^2} - \frac{\lambda_{2,1}}{s^4-\varphi_0(s)}\Big]\mathrm{d}s = 0. \tag{3.10}$$

Noting that the function $\frac{1}{x_2^4-\varphi_0(x_2)}$ is of at most algebraic growth for $x_2 \to \pm\infty$, we have

$$\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\Delta(s)}}{s^4-\varphi_0(s)}\mathrm{d}s < +\infty, \quad \int_{-\infty}^{+\infty} \frac{s^6\mathrm{e}^{-\Delta(s)}}{(s^4-\varphi_0(s))^2}\mathrm{d}s < +\infty.$$

Therefore, $\lambda_{2,1}$ is uniquely determined from (3.10). Thus, there exists a unique $\lambda_{2,1} \in \mathbb{R}$ such that (3.5) has a continuous solution $\varphi_1(x_2)$ defined on $\mathbb{R}$. The function $\varphi_1(x_2)$ is of at most algebraic growth for $x_2 \to \pm\infty$, which follows from the following lemma.

Lemma 3.1 *Suppose that the continuous function $g(x)$ defined on $\mathbb{R}$ is of at most algebraic growth for $x_2 \to \pm\infty$; that is, there exists a $\nu > 0$ such that $\sup\limits_{x\in\mathbb{R}}\{|g(x)| \cdot |x|^{-\nu}\} < +\infty$ holds. Then*

$$\sup_{x\in\mathbb{R}^+}\Big\{\Big|\int_x^{+\infty} \mathrm{e}^{Q(x)-Q(s)}g(s)\mathrm{d}s\Big| \cdot |x|^{-\nu}\Big\} < +\infty,$$
$$\sup_{x\in\mathbb{R}^-}\Big\{\Big|\int_{-\infty}^{x} \mathrm{e}^{Q(x)-Q(s)}g(s)\mathrm{d}s\Big| \cdot |x|^{-\nu}\Big\} < +\infty.$$

Proof From the definition of $Q(x)$, we have

$$Q(0) = Q'(0) = 0, \quad Q''(x) > 0, \quad x \in \mathbb{R}.$$

Then

$$\begin{aligned} |x|^{-\nu}\int_x^{+\infty} \mathrm{e}^{Q(x)-Q(s)}|g(s)|\mathrm{d}s &\leq \int_x^{+\infty} \mathrm{e}^{Q(x)-Q(s)}|g(s)| \cdot |s|^{-\nu}\mathrm{d}s \\ &\leq \sup_{x\in\mathbb{R}^+}\{|g(x)| \cdot |x|^{-\nu}\}\int_x^{+\infty} \mathrm{e}^{Q(x)-Q(s)}\mathrm{d}s \\ &\leq \sup_{x\in\mathbb{R}^+}\{|g(x)| \cdot |x|^{-\nu}\}\int_0^{+\infty} \mathrm{e}^{-Q(s)}\mathrm{d}s \\ &< +\infty. \end{aligned}$$

This proves the first inequality, and the second one can be proved in the same way.

Let us continue the proof of Theorem 1.1. Setting

$$y_2 = \varphi_0(x_2) + \varphi_1(x_2)r_2 + zr_2^2, \quad \lambda_2 = \lambda_{2,1}r_2 + \hbar r_2^2$$

in (3.2), we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}x_2} = \frac{x_2 z}{(x_2^4-\varphi_0(x_2))^2} + \Lambda(x_2,\hbar) + \Xi(x_2,z,r_2)r_2, \tag{3.11}$$

where

$$\Lambda(x_2,\hbar)=\frac{(\varphi_1(x_2)-\sigma x_2^5)(x_2\varphi_1(x_2)-\sigma x_2^6-(x_2^4-\varphi_0(x_2))\lambda_{2,1})}{(x_2^4-\varphi_0(x_2))^3}-\frac{\hbar}{x_2^4-\varphi_0(x_2)},$$

and the function Ξ is of at most algebraic growth for $x_2\to\pm\infty$ provided that z is of at most algebraic growth for $x_2\to\pm\infty$.

Define a Banach space:

$$C_z\equiv\Big\{z\in C(\mathbb{R},\mathbb{R}):\sup_{x_2\in\mathbb{R}}\{|z(x_2)|\cdot|x_2|^{-\ell}\}<+\infty\Big\},$$

where ℓ is a positive constant. In view of (3.11), we introduce the operator $\mathscr{T}$ as follows:

$$\mathscr{T}z(x_2)=\begin{cases}-\displaystyle\int_{x_2}^{+\infty}\mathrm{e}^{Q(x_2)-Q(s)}[\Lambda(s,\hbar)+\Xi(s,z(s),r_2)r_2]\mathrm{d}s, & x_2\ge 0,\\ \displaystyle\int_{-\infty}^{x_2}\mathrm{e}^{Q(x_2)-Q(s)}[\Lambda(s,\hbar)+\Xi(s,z(s),r_2)r_2]\mathrm{d}s, & x_2<0.\end{cases}$$

From the continuity of $\mathscr{T}z(x_2)$ at $x_2=0$, we get

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-Q(s)}[\Lambda(s,\hbar)+\Xi(s,z(s),r_2)r_2]\mathrm{d}s=0. \tag{3.12}$$

For any $z\in C_z$, the functions Λ and Ξ are of at most algebraic growth for $x_2\to\pm\infty$. Hence, for each $z\in C_z$, there exists a unique $\hbar\in\mathbb{R}$ such that (3.12) holds.

Finally, with the help of Lemma 3.1 it is not difficult to check that $\mathscr{T}$ is a contracting map from C_z to itself. Thus, we have proved that for $\lambda_2=\lambda_{2,1}r_2+\hbar r_2^2$, (2.7) has a trajectory which is of at most algebraic growth for $x_2\to\pm\infty$ near $\varphi_0(x_2)$. This trajectory connects $M_{a,1}$ to $M_{r,1}$ in K_1, which implies the existence of a maximal canard. By the transformation (2.3), we have $\lambda^*=\lambda_{2,1}r_2^2+\hbar r_2^3=\mathcal{O}(\varepsilon^{\frac{1}{3}})$. The proof of Theorem 1.1 is completed.

Remark 3.1 If the critical manifold of (1.2) has a higher order degenerate fold point, that is, instead of the condition (1.3) we assume

$$f(0)=f'(0)=\cdots=f^{(m-1)}(0)=0,\quad f^{(m)}(0)>0,$$

where $m>4$ is an even number, then we can similarly obtain the existence of canards and the asymptotic expansion of canard value.

Remark 3.2 Applying our approach to the following system (the case of generic folds):

$$\begin{aligned}x'&=-yh_1(x,y,\lambda,\varepsilon)+x^2h_2(x,y,\lambda,\varepsilon)+\varepsilon h_3(x,y,\lambda,\varepsilon),\\ y'&=\varepsilon(xh_4(x,y,\lambda,\varepsilon)-\lambda h_5(x,y,\lambda,\varepsilon)+yh_6(x,y,\lambda,\varepsilon)),\end{aligned}$$

where

$$\begin{aligned}&h_3(x,y,\lambda,\varepsilon)=\mathcal{O}(x,y,\lambda,\varepsilon),\\ &h_j(x,y,\lambda,\varepsilon)=1+\mathcal{O}(x,y,\lambda,\varepsilon),\quad j=1,2,4,5,\end{aligned}$$

we can obtain the same results as [15], and we can easily calculate the asymptotic expansion of arbitrary order for the canard value λ.

Remark 3.3 It is a pity that we can not obtain the exact coefficients in the asymptotic expansion of the canard value, because we do not know the exact formula of $\varphi_0(x_2)$. However, with the aid of the auxiliary functions $\alpha(x_2)$ and $\beta(x_2)$, we can estimate the range of the canard value. See the following example.

Example 3.1 Consider the system (see [18])

$$\begin{aligned} \varepsilon\frac{\mathrm{d}x}{\mathrm{d}t} &= y - \frac{1}{2}x^4 + \frac{3}{5}x^5 - \frac{5}{7}x^7, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= a - x. \end{aligned} \tag{3.13}$$

By the transformation

$$\widetilde{x} = -\frac{1}{\sqrt[3]{2}}x, \quad \widetilde{y} = \frac{1}{\sqrt[3]{2}}y, \quad \lambda = -\frac{1}{\sqrt[3]{2}}a, \quad \tau = \frac{t}{\varepsilon},$$

system (3.13) can be transformed into the form

$$\begin{aligned} \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau} &= -\widetilde{y} + \widetilde{x}^4 - \frac{6}{5}\widetilde{x}^5 - \frac{20}{7}\widetilde{x}^7, \\ \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\tau} &= \varepsilon(\widetilde{x} - \lambda). \end{aligned} \tag{3.14}$$

It follows from Theorem 1.1 that there exists a λ^* such that for $\lambda = \lambda^*$ system (3.14) has a canard, where λ^* has the asymptotic expansion $\lambda^* = \omega\varepsilon^{\frac{1}{3}} + \mathcal{O}(\varepsilon^{\frac{1}{2}})$. Therefore, for $a = a^* = -\sqrt[3]{2}\,\omega\varepsilon^{\frac{1}{3}} + \mathcal{O}(\varepsilon^{\frac{1}{2}})$, system (3.13) has a canard. In (3.10), substituting $\varphi_0(x_2)$ with the auxiliary functions $\alpha(x_2)$ and $\beta(x_2)$, respectively, we get $0.1187 < -\sqrt[3]{2}\,\omega < 0.2969$. Take $\varepsilon = 0.05$. Then a^* lies approximately between 0.04373 and 0.10938. Figure 4 shows the canard obtained by numerical simulation for $\varepsilon = 0.05$, $a = 0.05568$.

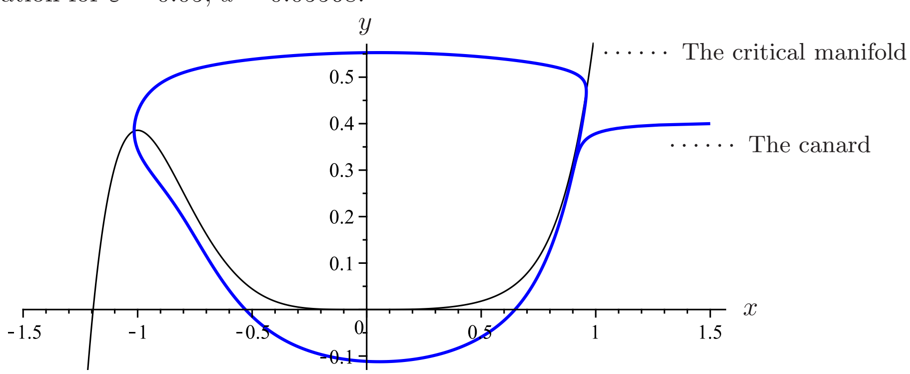

Figure 4 For $\varepsilon = 0.05$, $a = 0.05568$, the canard obtained by numerical simulation

Acknowledgement The authors thank the referees for their valuable comments and suggestions.

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