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# The Generalized Prime Number Theorem for Automorphic $L$-Functions

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**Abstract** Let $\pi$ and $\pi'$ be automorphic irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{Q}_{\mathbb{A}})$ and $\mathrm{GL}_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively, and $L(s,\pi\times\widetilde{\pi}')$ be the Rankin-Selberg $L$-function attached to $\pi$ and $\pi'$. Without assuming the Generalized Ramanujan Conjecture (GRC), the author gives the generalized prime number theorem for $L(s,\pi\times\widetilde{\pi}')$ when $\pi\cong\pi'$. The result generalizes the corresponding result of Liu and Ye in 2007.

**Keywords** Perron's formula, Prime number theorem, Rankin-Selberg $L$-functions
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## 1 Introduction

To each irreducible unitary cuspidal representation $\pi$ of $\mathrm{GL}_m(\mathbb{Q}_{\mathbb{A}})$, one can attach a global $L$-function which is given by products for local factors for $\sigma>1$ as in [4]:

$$L(s,\pi)=\prod_p L_p(s,\pi_p)=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s}, \tag{1.1}$$

where

$$L(s,\pi_p)=\prod_{j=1}^{m}\Big(1-\frac{\alpha_\pi(p,j)}{p^s}\Big)^{-1}.$$

The complete $L$-function is defined by

$$\Phi(s,\pi)=L_\infty(s,\pi_\infty)L(s,\pi),$$

where

$$L_\infty(s,\pi_\infty)=\prod_{j=1}^{m}\Gamma_{\mathbb{R}}(s+\mu_\pi(j))$$

is the Archimedean local factor. Here $\Gamma_{\mathbb{R}}(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$, $\alpha_\pi(p,j)$ and $\mu_\pi(j)$, $j=1,\cdots,m$, are complex numbers associated with $\pi_p$ and $\pi_\infty$, respectively, according to the Langlands correspondence.

To link $L(s,\pi)$ with primes, we take logarithmic differentiation in (1.1). Then for $\sigma>1$, we have

$$\frac{L'(s,\pi)}{L(s,\pi)}=-\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)}{n^s},$$

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where $\Lambda(n)$ is the von Mangoldt function, and

$$a_\pi(p^k) = \sum_{1\le j\le m} \alpha_\pi(p,j)^k.$$

If $\pi'$ is an automorphic irreducible cuspidal representation of $\mathrm{GL}_{m'}(\mathbb{Q}_\mathbb{A})$, we define $L(s,\pi')$, $\alpha_{\pi'}(p,i)$, $\mu_{\pi'}(i)$ and $a_{\pi'}(p^k)$ likewise for $i=1,\cdots,m'$. If $\pi$ and $\pi'$ are equivalent, then $m=m'$ and $\{\alpha_\pi(p,j)\}=\{\alpha_{\pi'}(p,i)\}$ for any $p$. Hence $a_\pi(n)=a_{\pi'}(n)$ for any $n=p^k$, when $\pi\cong\pi'$. The prime number theorem for Rankin-Selberg $L$-function $L(s,\pi\times\widetilde{\pi}')$ concerns the asymptotic behavior of the function $\sum\limits_{n\le x}\Lambda(n)a_\pi(n)\overline{a}_{\pi'}(n)$, and the main theorem of Liu and Ye [11] asserts that

$$\sum_{n\le x}\Lambda(n)a_\pi(n)\overline{a}_{\pi'}(n)=\begin{cases}\dfrac{x^{1+\tau_0}}{1+\mathrm{i}\tau_0}+O\{x\exp(-c\sqrt{\log x}\,)\}, & \text{if } \pi'\cong\pi\otimes|\det|^{\mathrm{i}\tau_0} \text{ for some } \tau_0\in\mathbb{R},\\ O\{x\exp(-c\sqrt{\log x}\,)\}, & \text{if } \pi'\ncong\pi\otimes|\det|^{\mathrm{i}\tau} \text{ for any } \tau\in\mathbb{R},\end{cases}$$

under the condition that at least one of $\pi$ and $\pi'$ is self-contragredient.

In this paper, we will show a generalized prime number theorem for a special case of the Rankin-Selberg $L$-function $L(s,\pi\times\widetilde{\pi}')$. Consider

$$\frac{L^{(k)}(s,\pi\times\widetilde{\pi}')}{L(s,\pi\times\widetilde{\pi}')}=(-1)^k\sum_{n=1}^{\infty}\frac{\rho_{\pi\times\widetilde{\pi}'}(n)}{n^s},\quad \sigma>1, \tag{1.2}$$

where $k$ is a positive integer, and $\rho_{\pi\times\widetilde{\pi}'}(n)$ is a complex number attached to $\pi$ and $\pi'$.

By modifying the argument of Liu and Ye [11], we are able to prove the following result.

**Theorem 1.1** *Let $\pi$ and $\widetilde{\pi}$ be automorphic irreducible cuspidal representations of* $\mathrm{GL}_m(\mathbb{Q}_\mathbb{A})$. *Assume that $\pi$ is self-contragredient: $\pi\cong\widetilde{\pi}$. Then*

$$\sum_{n\le x}\rho_{\pi\times\widetilde{\pi}}(n)=(k\log^{k-1}x+a_{1,k}\log^{k-2}x+\cdots+a_{k-1,k})x+O\{x\exp(-c\sqrt{\log x}\,)\}, \tag{1.3}$$

*where the complex constants $a_{j,k}$ $(j=1,\cdots,k-1)$ are computable.*

Note that [11, Lemma 5.1] is a special case of our theorem with $k=1$.

## 2 Rankin-Selberg $L$-Functions

Let $\pi$ and $\pi'$ be automorphic irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{Q}_\mathbb{A})$ and $\mathrm{GL}_{m'}(\mathbb{Q}_\mathbb{A})$, respectively, over $Q$ with unitary central characters. One can obtain the Rankin-Selberg $L$-functions $L(s,\pi\times\widetilde{\pi}')$ attached to $\pi$ and $\pi'$, which are developed by Jacquet, Piatetski-Shapiro and Shalika [7] and Shahidi [17]. This $L$-function is given by local factors

$$L(s,\pi\times\widetilde{\pi}')=\prod_p L_p(s,\pi_p\times\widetilde{\pi}'_p), \tag{2.1}$$

where

$$L_p(s,\pi_p\times\widetilde{\pi}'_p)=\prod_{j=1}^{m}\prod_{k=1}^{m'}\Big(1-\frac{\alpha_\pi(p,j)\overline{\alpha}_{\pi'}(p,k)}{p^s}\Big)^{-1}.$$

The complete $L$-function is defined by

$$\Phi(s,\pi\times\widetilde{\pi}')=L_\infty(s,\pi_\infty\times\widetilde{\pi}'_\infty)L(s,\pi\times\widetilde{\pi}')$$

with the Archimedean local factor

$$L_\infty(s,\pi_\infty\times\widetilde{\pi}'_\infty)=\prod_{j=1}^{m}\prod_{k=1}^{m'}\Gamma_{\mathbb{R}}(s+\mu_{\pi\times\widetilde{\pi}'}(j,k)),$$

where the complex numbers $\mu_{\pi\times\widetilde{\pi}'}(j,k)$ satisfy the trivial bound

$$\mathrm{Re}(\mu_{\pi\times\widetilde{\pi}'}(j,k))>-1. \tag{2.2}$$

Now we review some properties of the $L$-functions $L(s,\pi\times\widetilde{\pi}')$ and $\Phi(s,\pi\times\widetilde{\pi}')$, which we will use for our proofs.

**Proposition 2.1** (see [8]) *The Euler product for $L(s,\pi\times\widetilde{\pi}')$ in (2.1) converges absolutely for $\sigma>1$.*

**Proposition 2.2** (see [17–20]) *The complete $L$-function $\Phi(s,\pi\times\widetilde{\pi}')$ has an analytic continuation to the entire complex plane and satisfies the functional equation*

$$\Phi(s,\pi\times\widetilde{\pi}')=\varepsilon(s,\pi\times\widetilde{\pi}')\Phi(1-s,\pi\times\widetilde{\pi}')$$

*with*

$$\varepsilon(s,\pi\times\widetilde{\pi}')=\tau(\pi\times\widetilde{\pi}')Q_{\pi\times\widetilde{\pi}'}^{-s},$$

*where $Q_{\pi\times\widetilde{\pi}'}>0$ and $\tau(\pi\times\widetilde{\pi}')=\pm Q_{\pi\times\widetilde{\pi}'}^{\frac{1}{2}}$.*

**Proposition 2.3** (see [8, 9]) *Denote $\alpha(g)=|\det(g)|$. When $\pi'\ncong\pi\otimes|\det|^{\mathrm{i}\tau}$ for any $\tau\in\mathbb{R}$, $\Phi(s,\pi\times\widetilde{\pi}')$ is holomorphic. When $m=m'$ and $\pi'\cong\pi\otimes|\det|^{\mathrm{i}\tau_0}$ for some $\tau_0\in\mathbb{R}$, the only poles of $\Phi(s,\pi\times\widetilde{\pi}')$ are simple poles at $s=\mathrm{i}\tau_0$ and $s=1+\mathrm{i}\tau_0$ coming from $L(s,\pi\times\widetilde{\pi}')$.*

**Proposition 2.4** (see [3]) *$\Phi(s,\pi\times\widetilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in vertical strips.*

**Proposition 2.5** (see [2, 16, 17]) *$\Phi(s,\pi\times\widetilde{\pi}')$ and $L(s,\pi\times\widetilde{\pi}')$ are non-zero in $\sigma\ge 1$. Furthermore, it is zero-free in the region*

$$\sigma\ge 1-\frac{c_3}{\log(Q_{\pi\times\widetilde{\pi}'}(|t|+c_4))},\quad |t|\ge 1 \tag{2.3}$$

*and at most one exceptional zero in the region*

$$\sigma\ge 1-\frac{c_3}{\log(Q_{\pi\times\widetilde{\pi}'}c_4)},\quad |t|\le 1 \tag{2.4}$$

*for some effectively computable positive constants $c_3$ and $c_4$, if at least one of $\pi$ and $\pi'$ is self-contragredient.*

In addition to the above Propositions 2.1–2.5, we will also need to use a region $\mathbf{C}(m,m')$, defined as the complex plane $\mathbf{C}$ with the discs

$$|s-2n+\mu_{\pi\times\widetilde{\pi}'}(j,k)|<\frac{1}{8mm'},\quad n\le 0,\ 1\le j\le m,\ 1\le k\le m'$$

excluded. For $j=1,\cdots,m$ and $k=1,\cdots,m'$, denote by $\beta(j,k)$ the fractional part of $\mathrm{Re}(\mu_{\pi\times\widetilde{\pi}'}(j,k))$. In addition, let $\beta(0,0)=0$ and $\beta(m,m')=1$. Then all $\beta(j,k)\in[0,1]$, and hence there exist $\beta(j_1,k_1)$ and $\beta(j_2,k_2)$, such that $\beta(j_2,k_2)-\beta(j_1,k_1)\ge\frac{1}{3mm'}$ and there is no $\beta(j,k)$ lying between $\beta(j_1,k_1)$ and $\beta(j_2,k_2)$. It follows that the strip $S_0=\{s:\beta(j_1,k_1)+\frac{1}{8mm'}\le\sigma\le\beta(j_2,k_2)-\frac{1}{8mm'}\}$ is contained in $\mathbf{C}(m,m')$. Consequently, for all $n=0,-1,-2,\cdots$, the strips

$$S_n=\Big\{s:n+\beta(j_1,k_1)+\frac{1}{8mm'}\le\sigma\le n+\beta(j_2,k_2)-\frac{1}{8mm'}\Big\}\tag{2.5}$$

are subsets of $\mathbf{C}(m,m')$. This structure of $\mathbf{C}(m,m')$ will be used later.

Firstly, we give a lemma which is the expansion of [12, Lemma 4.1(e) and Lemma 4.2].

**Lemma 2.1** *Let $s=\sigma+\mathrm{i}\tau$. Assume $m=m'$ and $\pi'\cong\pi\otimes|\det|^{\mathrm{i}\tau_0}$ for some nonzero $\tau_0\in\mathbb{R}$.*

(i) *If $-2\le\sigma\le 2$, then for $|T|>2$, there exists a $\tau$ with $T\le\tau\le T+1$, such that*

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\log L(\sigma\pm\mathrm{i}\tau,\pi\times\widetilde{\pi}')\ll\log^{k+1}(Q_{\pi\times\widetilde{\pi}'}|\tau|).$$

(ii) *If $s$ is in some strip $S_n$ as in (2.5) with $n\le -2$, then*

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\log L(\sigma\pm\mathrm{i}\tau,\pi\times\widetilde{\pi}')\ll 1.$$

The proof of this lemma is similar to that of Liu and Ye [11] as we mentioned above, so we omit the proof.

**Lemma 2.2** *Assume that $n$ is an integer and $f$ is a meromorphic function on the complex plane. Then $\frac{f^{(n)}}{f}$ could be expressed as the differential polynomial of $\frac{f'}{f}$.*

The lemma is [21, Lemma 1.8].

## 3 A Weighted Generalized Prime Number Theorem

Now we prove a weighted generalized prime number theorem.

**Theorem 3.1** *Let $\pi$ be a self-contragredient automorphic irreducible cuspidal representation of $\mathrm{GL}_m$ over $\mathbb{Q}$. Then*

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\rho_{\pi\times\widetilde{\pi}}(n)=\Big(\frac{k}{2}\log^{k-1}x+b_{1,k}\log^{k-2}x+\cdots+b_{k-1,k}\Big)x+O\{x\exp(-c\sqrt{\log x}\,)\},$$

*where the constants $b_{j,k}$ $(j=1,\cdots,k-1)$ are computable.*

**Proof** By Proposition 2.1, we have

$$J(s):=(-1)^k\frac{L^{(k)}(s,\pi\times\widetilde{\pi})}{L(s,\pi\times\widetilde{\pi})}=\sum_{n=1}^{\infty}\frac{\rho_{\pi\times\widetilde{\pi}}(n)}{n^s}$$

for $\sigma > 1$. Note that

$$\frac{1}{2\pi\mathrm{i}}\int_{(b)}\frac{y^s}{s(s+1)}\mathrm{d}s = \begin{cases} 1-\dfrac{1}{y}, & \text{if } y \geq 1, \\ 0, & \text{if } 0 < y < 1, \end{cases}$$

where $(b)$ means the line $\sigma = b > 0$. Taking $b = 1 + \frac{1}{\log x}$, we have

$$\sum_{n\leq x}\Big(1-\frac{n}{x}\Big)\rho_{\pi\times\widetilde{\pi}}(n) = \frac{1}{2\pi\mathrm{i}}\int_{(b)} J(s)\frac{x^s}{s(s+1)}\mathrm{d}s = \frac{1}{2\pi\mathrm{i}}\Big(\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}\infty}^{b-\mathrm{i}T} + \int_{b+\mathrm{i}T}^{b+\mathrm{i}\infty}\Big).$$

The last two integrals are clearly bounded by

$$\int_T^\infty \frac{x}{t^2}\mathrm{d}t = \frac{x}{T}.$$

Thus

$$\sum_{n\leq x}\Big(1-\frac{n}{x}\Big)\rho_{\pi\times\widetilde{\pi}}(n) = \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} J(s)\frac{x^s}{s(s+1)}\mathrm{d}s + O\Big(\frac{x}{T}\Big).$$

Choose a real number $a$ with $-2 < a < -1$, such that the vertical line $\sigma = a$ is contained in the strip $S_{-2} \subset \mathbf{C}(m, m')$; this is guaranteed by the structure of $\mathbf{C}(m, m')$. Without loss of generality, let $T > 0$ be a large number, such that $T$ and $-T$ can be taken as the $\tau$ in Lemma 2.1(i). Now we consider the contour

$$\begin{aligned} &C_1: \ b \geq \sigma \geq a, \ t = -T, \\ &C_2: \ \sigma = a, \ -T \leq t \leq T, \\ &C_3: \ a \leq \sigma \leq b, \ t = T. \end{aligned}$$

Note that three poles $s = 1, 0, -1$, some trivial zeros, and certain nontrivial zeros $\rho = \beta + \mathrm{i}\gamma$ of $L(\pi \times \widetilde{\pi})$ are passed by the shifting of the contour. Also note that $s = 1$ is a pole of order $k$ and $s = 0$ is of order $k + 1$. The trivial zeros can be determined by Proposition 2.2 and (2.2): $s = -\mu_{\pi\times\widetilde{\pi}}(j,k)$ with $a < -\mathrm{Re}(\mu_{\pi\times\widetilde{\pi}}(j,k)) < 1$ and $s = -2-\mu_{\pi\times\widetilde{\pi}}(j,k)$ with $a + 2 < -\mathrm{Re}(\mu_{\pi\times\widetilde{\pi}}(j,k)) < 1$. Here we have used $-2 < a < -1$. Then we have

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} J(s)\frac{x^s}{s(s+1)}\mathrm{d}s = {} & \frac{1}{2\pi\mathrm{i}}\Big(\int_{C_1} + \int_{C_2} + \int_{C_3}\Big) + \operatorname*{Res}_{s=1,0,-1} J(s)\frac{x^s}{s(s+1)} \\ & + \sum_{a<-\mathrm{Re}(\mu_{\pi\times\widetilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\widetilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)} \\ & + \sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\widetilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\widetilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)} \\ & + \sum_{|\gamma|\leq T}\operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s(s+1)}. \end{aligned} \tag{3.1}$$

By Lemma 2.1(i), for any large $\tau > 0$, we can choose $T$ in $\tau < T < \tau + 1$ such that, when $-2 \leq \sigma \leq 2$,

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\log L(\sigma \pm \mathrm{i}T, \pi\times\widetilde{\pi}) \ll \log^{k+1}(Q_{\pi\times\widetilde{\pi}}T).$$

Furthermore, using Lemma 2.2, we can get the following estimate

$$J(\sigma \pm \mathrm{i}T) \ll \log^{2k}(Q_{\pi\times\widetilde{\pi}}T).$$

Hence

$$\int_{C_1} \ll \int_a^b \log^{2k}(Q_{\pi\times\widetilde{\pi}}T)\frac{x^\sigma}{T^2}\mathrm{d}\sigma \ll \frac{x\log^{2k}(Q_{\pi\times\widetilde{\pi}}T)}{T^2}.$$

The same upper bound also holds for the integral on $C_3$. By Lemma 2.1(ii), we can choose a $\sigma$ such that, when $|t| \le T$,

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\log L(\sigma \pm \mathrm{i}t, \pi\times\widetilde{\pi}) \ll 1.$$

Also using Lemma 2.2, we obtain

$$J(\sigma + \mathrm{i}t) \ll 1.$$

Therefore

$$\int_{C_2} \ll \int_{-T}^{T} \frac{x^\sigma}{(|t|+1)^2}\mathrm{d}t \ll \frac{1}{x}.$$

On taking $T \gg \exp(\sqrt{\log x}\,)$, finally we can get

$$\int_{C_1} + \int_{C_2} + \int_{C_3} \ll x\exp(-c\sqrt{\log x}\,). \tag{3.2}$$

The function

$$J(s)\frac{x^s}{s(s+1)}$$

has a simple pole at $s=-1$ with the residue $O(x^{-1})$, and two poles at $s=1,0$ with the order $k$ and $k+1$, respectively. The residue at $s=1$ is

$$\begin{aligned}\operatorname*{Res}_{s=1} J(s)\frac{x^s}{s(s+1)} &= \lim_{s\to1}\Big(\frac{(-1)^k(s-1)^kL^{(k)}(s,\pi\times\widetilde{\pi})x^s}{(k-1)!L(s,\pi\times\widetilde{\pi})s(s+1)}\Big)^{(k-1)} \\ &= x\Big(\sum_{j=0}^{k-1} b_{j,k}\log^{k-1-j}x\Big),\end{aligned}$$

where

$$b_{0,k} = \lim_{s\to1}\Big(\frac{(-1)^k(s-1)^kL^{(k)}(s,\pi\times\widetilde{\pi})}{(k-1)!L(s,\pi\times\widetilde{\pi})s(s+1)}\Big) = \frac{k!}{2(k-1)!} = \frac{k}{2},$$

and the other constants $b_{j,k}$ $(j=1,\cdots,k-1)$ are also computable. Similarly,

$$\begin{aligned}\operatorname*{Res}_{s=0} J(s)\frac{x^s}{s(s+1)} &= \lim_{s\to0}\Big(\frac{(-1)^ks^kL^{(k)}(s,\pi\times\widetilde{\pi})x^s}{k!L(s,\pi\times\widetilde{\pi})(s+1)}\Big)^{(k)} \\ &= \sum_{i=0}^{k} c_{i,k}\log^{k-i}x \ll \log^k x.\end{aligned}$$

Therefore

$$\operatorname*{Res}_{s=1,0,-1} J(s)\frac{x^s}{s(s+1)} = \Big(\frac{k}{2}\log^{k-1}x + b_{1,k}\log^{k-2}x + \cdots + b_{k-1,k}\Big)x + O(\log^k x). \tag{3.3}$$

Suppose that the order of a trivial zero $s = -\mu_{\pi\times\tilde{\pi}(j,k)}$ is $l$. If $l < k$, we can express $J(s)$ as

$$\frac{G(s)x^s}{(s+\mu)^l s(s+1)},$$

where $G(-\mu_{\pi\times\tilde{\pi}}(j,k)) \neq 0$. The residues at these trivial zeros can therefore be computed similarly to what we have done in (3.3). By (2.2), we know that $\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k)) \geq 1-\delta$ for some $\delta > 0$. Consequently,

$$\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{\substack{s=-\mu_{\pi\times\tilde{\pi}}(j,k) \\ l<k}} J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}\log^{l-1} x \ll x^{1-\delta}\log^{k-1} x.$$

For $l \geq k$, we get

$$J(s) = \frac{G'(s)x^s}{(s+\mu)^k s(s+1)}, \quad G'(-\mu_{\pi\times\tilde{\pi}}(j,k)) \neq 0$$

and

$$\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{\substack{s=-\mu_{\pi\times\tilde{\pi}}(j,k) \\ l\geq k}} J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}\log^{k-1} x.$$

Finally, we obtain

$$\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}\log^{k-1} x. \tag{3.4}$$

Similarly, we have the estimate

$$\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{-1-\delta}\log^{k-1} x. \tag{3.5}$$

To compute the residue corresponding to nontrivial zeros, we recall Propositions 2.4 and 2.5, and get

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} < \infty, \quad \sum_\rho \frac{1}{|\rho^i|} < \infty, \quad \sum_\rho \frac{1}{|(\rho+1)^i|} < \infty, \quad i \geq 2.$$

Just like the trivial zeros, we should pay attention to the order of the nontrivial zero $\rho$. Suppose that the order of a nontrivial zero $\rho$ is $l'$. If $l' < k$, we can express $J(s)$ as

$$J(s) = \frac{g(s)x^s}{(s-\rho)^{l'} s(s+1)},$$

where $g(\rho) \neq 0$. Consequently,

$$\begin{aligned}
\sum_{|\gamma|\leq T} \operatorname*{Res}_{\substack{s=\rho \\ l'<k}} J(s)\frac{x^s}{s(s+1)} &= \sum_{|\gamma|\leq T} \lim_{s\to\rho} \Big(\frac{g(s)x^s}{(l'-1)!s(s+1)}\Big)^{l'-1} \\
&= \sum_{|\gamma|\leq T} \lim_{s\to\rho} \Big\{ \sum_{i=0}^{l'-1} C_{l'-1}^i \Big(\frac{1}{s(s+1)}\Big)^i (g(s)x^s)^{l'-1-i} \Big\} \\
&\ll x^\beta \log^{l'-1} x \sum_{|\gamma|\leq T} \max_{0\leq i\leq l'-1} \Big| C_{l'-1}^i \Big(\frac{1}{s(s+1)}\Big)^i \Big|_{s=\rho}
\end{aligned}$$

$$\ll x^\beta \log^{k-1} x \sum_{|\gamma|\le T} \Big\{ \sum_{i=2}^{l'} \Big( \frac{1}{|\rho^i|} + \frac{1}{|(\rho+1)^i|} \Big) + \frac{1}{|\rho(\rho+1)|} \Big\}$$
$$= x^\beta \log^{k-1} x \Big( \sum_{\substack{|\gamma|\le T\\ \rho\in E}} + \sum_{\substack{|\gamma|\le T\\ \rho\notin E}} \Big) \Big\{ \sum_{i=2}^{l'} \Big( \frac{1}{|\rho^i|} + \frac{1}{|(\rho+1)^i|} \Big) + \frac{1}{|\rho(\rho+1)|} \Big\},$$

where $E$ is the set of exceptional zeros in (2.4). We have $|E| < 1$, and hence it is clear that the sum over $\rho \in E$ is far less than $x^{1-\delta} \log^k x$ for some $\delta > 0$. By (2.3), the sum over $\rho \notin E$ is far less than

$$x \exp\Big( - c_3 \frac{\log x}{2\log(Q_{\pi\times\widetilde{\pi}} T)} \Big) \Big\{ \sum_{i=2}^{l'} \Big( \frac{1}{|\rho^i|} + \frac{1}{|(\rho+1)^i|} \Big) + \frac{1}{|\rho(\rho+1)|} \Big\} \ll x \exp(-c\sqrt{\log x}\,)$$

by taking $T = \exp(\sqrt{\log x}\,) + d$ for some $d$ with $0 < d < 1$.

Similarly, for $l' \ge k$, we have

$$J(s) = \frac{g'(s) x^s}{(s-\rho)^k s(s+1)}, \quad g'(\rho) \ne 0,$$

and

$$\begin{aligned}
\sum_{|\gamma|\le T} \operatorname*{Res}_{\substack{s=\rho\\ l'\ge k}} J(s) \frac{x^s}{s(s+1)} &= \sum_{|\gamma|\le T} \lim_{s\to\rho} \Big( \frac{g(s) x^s}{(k-1)! s(s+1)} \Big)^{k-1} \\
&= \sum_{|\gamma|\le T} \lim_{s\to\rho} \Big\{ \sum_{i=0}^{k-1} C_{k-1}^i \Big( \frac{1}{s(s+1)} \Big)^i (g(s)x^s)^{k-1-i} \Big\} \\
&\ll x^\beta \log^{k-1} x \sum_{|\gamma|\le T} \max_{0\le i\le k-1} \Big| C_{k-1}^i \Big( \frac{1}{s(s+1)} \Big)^i \Big|_{s=\rho} \\
&\ll x^\beta \log^{k-1} x \sum_{|\gamma|\le T} \Big\{ \sum_{i=2}^{k} \Big( \frac{1}{|\rho^i|} + \frac{1}{|(\rho+1)^i|} \Big) + \frac{1}{|\rho(\rho+1)|} \Big\} \\
&= x^\beta \log^{k-1} x \Big( \sum_{\substack{|\gamma|\le T\\ \rho\in E}} + \sum_{\substack{|\gamma|\le T\\ \rho\notin E}} \Big) \Big\{ \sum_{i=2}^{k} \Big( \frac{1}{|\rho^i|} + \frac{1}{|(\rho+1)^i|} \Big) + \frac{1}{|\rho(\rho+1)|} \Big\} \\
&\ll x \exp(-c\sqrt{\log x}\,).
\end{aligned}$$

Finally, we obtain

$$\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s) \frac{x^s}{s(s+1)} \ll x \exp(-c\sqrt{\log x}\,). \tag{3.6}$$

Theorem 3.1 then follows by applying (3.2)–(3.6) to (3.1).

## 4 Completion of Theorem 1.1

By induction, we can easily find that $\rho_{\pi\times\widetilde{\pi}}(n)$ is a positive number. So the weight $1 - \frac{n}{x}$ can be removed from Theorem 3.1 by a standard argument of de la Vallée Poussin.

**Proof of Theorem 1.1** Let $\Psi(x)$ denote the quality on the left-hand side of (1.3). Then Theorem 3.1 states that

$$\int_1^x \Psi(t)\mathrm{d}t = \Big(\frac{k}{2}\log^{k-1} x + b_{1,k}\log^{k-2} x + \cdots + a_{k-1,k}\Big)x^2 + O\{x^2\exp(-c\sqrt{\log x}\,)\}.$$

By the Taylor expansion of $\log(1+\frac{h}{x})$, we get

$$\begin{aligned}\frac{1}{h}\int_x^{x+h}\Psi(t)\mathrm{d}t &= (k\log^{k-1} x + a_{1,k}\log^{k-2} x + \cdots + a_{k-1,k})x \\ &\quad + O(h\log^{k-1} x) + O\Big\{\frac{x^2}{h}\exp(-c\sqrt{\log x}\,)\Big\} \\ &= (k\log^{k-1} x + a_{1,k}\log^{k-2} x + \cdots + a_{k-1,k})x \\ &\quad + O\Big\{x\exp\Big(-\frac{c}{2}\sqrt{\log x}\Big)\log^{\frac{k-1}{2}} x\Big\} \\ &= (k\log^{k-1} x + a_{1,k}\log^{k-2} x + \cdots + a_{k-1,k})x \\ &\quad + O\{x\exp(-c'\sqrt{\log x}\,)\}, \end{aligned} \tag{4.1}$$

where we have chosen

$$h = x\exp\Big(-\frac{c}{2}\sqrt{\log x}\Big)(\log x)^{-\frac{k-1}{2}}$$

and $a_{i,k} = 2b_{i,k} + b_{i-1,k}(k-i)$, $i = 1, 2, \cdots, k-1$. Similarly, we get

$$\frac{1}{h}\int_{x-h}^{x}\Psi(t)\mathrm{d}t = (k\log^{k-1} x + a_{1,k}\log^{k-2} x + \cdots + a_{k-1,k})x + O\{x\exp(-c'\sqrt{\log x}\,)\}. \tag{4.2}$$

Note that the terms in $\Psi(t)$ are non-negative. Therefore, we have

$$\frac{1}{h}\int_{x-h}^{x}\Psi(t)\mathrm{d}t \le \Psi(x) \le \frac{1}{h}\int_x^{x+h}\Psi(t)\mathrm{d}t. \tag{4.3}$$

By (4.1)–(4.3), we obtain

$$\sum_{n\le x}\rho_{\pi\times\widetilde{\pi}}(n) = (k\log^{k-1} x + a_{1,k}\log^{k-2} x + \cdots + a_{k-1,k})x + O\{x\exp(-c'\sqrt{\log x}\,)\},$$

which gives Theorem 1.1.

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