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Pseudo-Anosov Mapping Classes and Their Representations by Products of Two Dehn Twists

Chaohui ZHANG*

Abstract Let $\widetilde{S}$ be a Riemann surface of analytically finite type (p, n) with $3p-3+n > 0$. Let $a \in \widetilde{S}$ and $S = \widetilde{S} - \{a\}$. In this article, the author studies those pseudo-Anosov maps on S that are isotopic to the identity on $\widetilde{S}$ and can be represented by products of Dehn twists. It is also proved that for any pseudo-Anosov map f of S isotopic to the identity on $\widetilde{S}$, there are infinitely many pseudo-Anosov maps F on $S - \{b\} = \widetilde{S} - \{a, b\}$, where b is a point on S, such that F is isotopic to f on S as b is filled in.

Keywords Riemann surface, Pseudo-Anosov map, Dehn twist, Teichmüller space, Bers fiber space
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1 Introduction

According to Thurston [14], an orientation-preserving homeomorphism f of a Riemann surface is called pseudo-Anosov if there exists a pair $(\mathcal{F}_+, \mathcal{F}_-)$ of transverse measured foliations on the surface with $f(\mathcal{F}_+) = \lambda\mathcal{F}_+$ and $f(\mathcal{F}_-) = \frac{1}{\lambda}\mathcal{F}_-$ for some $\lambda > 1$ (see also [5, 12–13]).

Let $\widetilde{S}$ be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. Fix a point $a \in \widetilde{S}$. Then $S = \widetilde{S} - \{a\}$ is of type $(p, n+1)$. Let $\mathcal{F}$ be the set of maps on S that fix a and are isotopic to the identity on $\widetilde{S}$. By [2, Theorem 10] (see also [4, Theorems 4.2 and 4.3]), $\mathcal{F}$ is isomorphic to the Fuchsian group G that uniformizes $\widetilde{S}$ under a universal covering $\varrho : \mathbb{H} = \{z : \mathrm{Im} z > 0\} \to \widetilde{S}$. Throughout this paper, we write $f = g^*$ if $g \in G$ corresponds to $f \in \mathcal{F}$ under the isomorphism. In [7, Theorem 2], Kra showed that $g^* \in \mathcal{F}$ is pseudo-Anosov if and only if g is an essential hyperbolic element of G; that is, its axis c projects to a filling geodesic $\widetilde{c}$ in the sense that the complement $\widetilde{S} - \{\widetilde{c}\}$ consists of disks and once punctured disks. Let $\mathcal{F}_0 \subset \mathcal{F}$ denote the subset consisting of elements g^* for essential hyperbolic elements of $g \in G$.

It is obvious that if α_1, α_2 are simple closed geodesics on S that are trivial on $\widetilde{S}$, then any products

$$\prod_i t_1^{r_i} \circ t_2^{-s_i}, \quad r_i, s_i \in \mathbb{Z}^+ - \{0\}, \tag{1.1}$$

where t_i is the Dehn twist along α_i, are in $\mathcal{F}$. That is, (1.1) is of form g^* for some $g \in G$. On the other hand, certain elements $g^* \in \mathcal{F}$ are isotopic to products of Dehn twists along two filling simple closed geodesics α_1 and α_2 on S that are also nontrivial on $\widetilde{S}$.

The main purpose of this paper is to investigate elements in $\mathcal{F}_0$ that are isotopic to a map of form (1.1).

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*Department of Mathematics, Morehouse College, Atlanta, GA 30314, USA. E-mail: czhang@morehouse.edu

Theorem 1.1 *There exist infinitely many pseudo-Anosov maps $f = g^* \in \mathcal{F}_0$ that can not be isotopic to any products of Dehn twists along two simple curves that are trivial on $\widetilde{S}$. Furthermore, if $\widetilde{S}$ contains at least one puncture, there exist infinitely many pseudo-Anosov maps $f = g^* \in \mathcal{F}_0$ that are isotopic to products of Dehn twists along two simple curves that are trivial on $\widetilde{S}$.*

Now we assume that $f = g^* \in \mathcal{F}_0$ is isotopic to a product (1.1) for α_i being nontrivial on $\widetilde{S}$. Let $\widetilde{\alpha}_i$ denote the geodesic homotopic to α_i on $\widetilde{S}$. The Dehn twist $t_{\widetilde{\alpha}_i}$ can be lifted to a mapping $\tau_i : \mathbb{H} \to \mathbb{H}$ so that $\tau_i^* = t_i$. The map τ_i determines a collection $\mathcal{U}_i$ of disjoint maximal half-planes D_i each of which is invariant under τ_i. Let H_i be the complement of $\mathcal{U}_i$ in $\mathbb{H}$. Then $\tau_i|_{H_i}$ is the identity. See Section 4 for an illustration.

Theorem 1.2 *There exist infinitely many elements $f = g^* \in \mathcal{F}_0$ that are isotopic to products* (1.1) *with $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ being nontrivial. Furthermore, if $f = g^* \in \mathcal{F}_0$ is isotopic to a product* (1.1)*, then either α_1 and α_2 are trivial on $\widetilde{S}$, or $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are nontrivial. In later case, we let τ_i denote the lift of $t_{\widetilde{\alpha}_i}$ so that $\tau_i^* = t_i$. Then the following two conditions hold:*

(1) *The pair $\{\alpha_1, \alpha_2\}$ fills S, $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$, and thus $t_{\widetilde{\alpha}_1} = t_{\widetilde{\alpha}_2}$ and $\sum_i (r_i - s_i) = 0$;*

(2) *There are maximal elements $D_1 \in \mathcal{U}_1$ and $D_2 \in \mathcal{U}_2$ such that $D_1 \cap D_2 \neq \emptyset$, $\partial D_1 \cap \partial D_2 = \emptyset$, and the axis of g lies in $D_1 \cap D_2$.*

Denote by $\mathcal{L}$ the set of pseudo-Anosov maps on S obtained from products of Dehn twists along two filling simple closed geodesics. Let $a' \in S$ and $\dot{S} = S - \{a'\}$. In [16, Theorem 1.2], we showed that for any element $f \in \mathcal{L}$, there exist infinitely many pseudo-Anosov maps F on $\dot{S} = S - \{a'\}$ isotopic to f on S as a' is filled in.

Unfortunately, by [6, Corollary 1.3], we know that not every pseudo-Anosov map on S is in $\mathcal{L}$. Also, it is not clear whether every element of $\mathcal{F}_0$ is in $\mathcal{L}$. In contrast, $\mathcal{F}_0 \cap \mathcal{L}$ contains infinitely many elements. A question arises as to whether there exist pseudo-Anosov maps F on $\dot{S}$ isotopic to a given map f in $\mathcal{F}_0 - \mathcal{L}$ on S (if the set is not empty). Our last result is the following:

Theorem 1.3 *Let $\widetilde{S}$ be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. For any $f \in \mathcal{F}_0$, there exist infinitely many pseudo-Anosov maps F on $\dot{S}$ that are isotopic to f on S as a' is filled in.*

Here we recall the main theorem of Imayoshi, Ito and Yamamoto [8]. Denote $M = \widetilde{S} \times \widetilde{S}$, $\vec{a} = \{a, a'\}$, and $\Delta = \{(x, y) \in M : x \neq y\}$. Since F is isotopic to the identity on $\widetilde{S}$, there is an isotopy $H : \widetilde{S} \times I \to \widetilde{S}$ such that $H(\cdot, 0) = F$ and $H(\cdot, 1) = \mathrm{id}$. Then $s_1 = F(a, t)$ and $s_2 = F(a', t)$, where $1 \leq t \leq 1$, are closed curve on $\widetilde{S}$, which define a pure braids $[b_F]$ represented by $b_F = (s_1, s_2)$ in the fundamental group $\pi_1(M - \Delta, \vec{a})$. By Theorem 1.3 and the main theorem of [8], we obtain infinitely many essential pure braids $[b_F]$ so that s_1 and s_2 are nontrivial and nonparallel.

2 Notation and Background

In this section, we review some basic facts on Teichmüller theory (see [1–3, 7] for more details). Denote by $L_\infty(\mathbb{H}, G)$ the space of measurable functions on the hyperbolic plane $\mathbb{H}$ satisfying

$$(\mu \circ g)(z) \cdot \frac{\overline{g'(z)}}{g'(z)} = \mu(z) \quad \text{for all } g \in G.$$

Let $M(G) \subset L_\infty(\mathbb{H}, G)$ denote the unit ball. For each element $\mu \in M(G)$, there exists a quasiconformal mapping $w^\mu : \mathbb{C} \to \mathbb{C}$ that fixes $0, 1$ and also satisfies these properties: (i) $w^\mu G(w^\mu)^{-1}$ is a group of Möbius transformations, (ii) w^μ is conformal on $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$, (iii) the Beltrami coefficient $\frac{\partial_{\bar{z}} w^\mu(z)}{\partial_z w^\mu(z)}$ of w^μ on $\mathbb{H}$ is $\mu(z)$, (iv) for any fixed $z \in \mathbb{C}$, the function

$$M(G) \ni \mu \longmapsto w^\mu(z) \in \mathbb{C}$$

is holomorphic (see [1]). For μ, ν in $M(G)$, we set $\mu \sim \nu$ if and only if

$$w^\mu \circ g \circ (w^\mu)^{-1} = w^\nu \circ g \circ (w^\nu)^{-1} \quad \text{for each } g \in G.$$

The Teichmüller space $T(\widetilde{S})$ is defined as the quotient $M(G)/\sim$ equipped with the quotient structure. The equivalence class of $\mu \in M(G)$ is denoted by $[\mu]$. $T(\widetilde{S})$ is a complex manifold with dimension $3p - 3 + n$. The Bers fiber space $F(\widetilde{S})$ over $T(\widetilde{S})$ is defined by the total space

$$F(\widetilde{S}) = \{([\mu], z) : [\mu] \in T(\widetilde{S}),\ z \in w^\mu(\mathbb{H})\}.$$

The projection $\pi : F(\widetilde{S}) \to T(\widetilde{S})$ that sends a point $([\mu], z)$ to $[\mu]$ is holomorphic. Bers [2, Theorem 9] states that there is an isomorphism $\varphi : F(\widetilde{S}) \to T(S)$ making the following diagram commutative:

$$\begin{array}{ccc} F(\widetilde{S}) & \xrightarrow{\ \varphi\ } & T(S) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\eta_a} \\ T(\widetilde{S}) & \xrightarrow{\ \mathrm{id}\ } & T(\widetilde{S}) \end{array} \tag{2.1}$$

where $\eta_a : T(S) \to T(\widetilde{S})$ is defined by forgetting the puncture a. The group of isotopy classes of selfmaps of $\widetilde{S}$ is called the mapping class group of $\widetilde{S}$ and is denoted by $\mathrm{Mod}_{\widetilde{S}}$. Let $\theta \in \mathrm{Mod}_{\widetilde{S}}$ and w be a representative of θ. Then w can be lifted to an automorphism $\widehat{w} : \mathbb{H} \to \mathbb{H}$ under the universal covering $\varrho : \mathbb{H} \to \widetilde{S}$. Let mod $\widetilde{S}$ denote the group that consists of equivalence classes $[\widehat{w}]$ of $\widehat{w}$, where two lifts $\widehat{w}$ and $\widehat{w}' : \mathbb{H} \to \mathbb{H}$ of w are considered equivalent (we write $\widehat{w} \sim \widehat{w}'$) if they induce the same automorphism by conjugation on G. $\widehat{w}$ naturally extends to $\partial\mathbb{H}$, and $\widehat{w}|_{\partial\mathbb{H}} = \widehat{w}'|_{\partial\mathbb{H}}$ if and only if $\widehat{w} \sim \widehat{w}'$.

The group mod $\widetilde{S}$ acts on $F(\widetilde{S})$ in a fiber preserving way, and the group G, which is isomorphic to the fundamental group $\pi_1(\widetilde{S}, a)$, can be regarded as a normal subgroup of mod $\widetilde{S}$ so that mod $\widetilde{S}/G \cong \mathrm{Mod}_{\widetilde{S}}$. Let Mod_S^a be the subgroup of Mod_S that consists of mapping classes on S fixing a. From [2, Theorem 10], the group mod $\widetilde{S}$ is isomorphic to Mod_S^a under the isomorphism $\varphi^* : \text{mod } \widetilde{S} \to \mathrm{Mod}_S^a$, defined as

$$\text{mod } \widetilde{S} \ni [\widehat{w}] \overset{\varphi^*}{\longmapsto} \widehat{\omega}^* = \varphi \circ [\widehat{w}] \circ \varphi^{-1} \in \mathrm{Mod}_S^a. \tag{2.2}$$

An element $\theta \in \mathrm{Mod}_S^a$ is called a reducible mapping class if there is a curve system $\mathcal{C} = \{c_1, \cdots, c_s\}$ of independent simple closed geodesics on S with $f(\{c_1, \cdots, c_s\}) = \{c_1, \cdots, c_s\}$ for certain representative f in θ. There is a smallest positive integer K such that f^K maps each loop in $\mathcal{C}$ to itself and the restriction of f^K to each component of $S - \{c_1, \cdots, c_s\}$ is either the identity or a pseudo-Anosov map.

We assume that θ is reducible and projects to a pseudo-Anosov mapping class $\widetilde{\theta}$ on $\widetilde{S}$ that is induced by a map w. By [15, Lemmas 3.1 and 3.2], the curve system $\mathcal{C}$ consists of only one curve c_1 that bounds a twice punctured disk enclosing a and another puncture of $\widetilde{S}$, which is equivalent to that c_1, becomes a trivial loop on $\widetilde{S}$. If we denote by $[\widehat{w}]$ the element of mod $\widetilde{S}$

corresponding to θ, then $\widehat{w}:\overline{\mathbb{H}}\to\overline{\mathbb{H}}$ fixes a parabolic fixed point of G. Conversely, every element $[\widehat{w}]$ fixing the fixed point of a parabolic element of G corresponds to a reducible mapping class in Mod_S^a which is reduced by a single closed geodesic that is trivial on $\widetilde{S}$.

The natural projection $\eta_a^*:\mathrm{Mod}_S^a\to\mathrm{Mod}_{\widetilde{S}}$ induced by (2.1) makes the diagram commutative

$$\begin{array}{ccc} \mathrm{mod}\ \widetilde{S} & \xrightarrow{\varphi^*} & \mathrm{Mod}_S^a \\ {\scriptstyle\pi^*}\big\downarrow & & {\scriptstyle\eta_a^*}\big\downarrow \\ \mathrm{Mod}_{\widetilde{S}} & \xrightarrow{\mathrm{id}} & \mathrm{Mod}_{\widetilde{S}} \end{array} \tag{2.3}$$

Thus the kernel $\ker(\eta_a^*)$ of $\eta_a^*:\mathrm{Mod}_S^a\to\mathrm{Mod}_{\widetilde{S}}$ is $G^*=\varphi^*(G)$. For every element $g\in G$, $g^*=\varphi^*(g)$ defines a mapping class on S that projects to the trivial mapping class on $\widetilde{S}$. Conversely, any mapping class on S that projects to the trivial mapping class is of the form g^* for some $g\in G$.

Let g be a simple hyperbolic element with $c\subset\mathbb{H}$ its axis, which means that $\widetilde{c}=\varrho(c)$ is a simple closed geodesic on $\widetilde{S}$. [7, Theorem 2] and [11] show that g^* is induced by a spin $t_2^{-1}\circ t_1$, where t_1 and t_2 are the Dehn twists along boundary geodesics $\{\alpha_1,\alpha_2\}$ of an a-punctured cylinder P so that they are both homotopic to $\widetilde{c}$ on $\widetilde{S}$. If $g\in G$ is parabolic, g^* is represented by a Dehn twist along a loop α that bounds a twice punctured disk $\Delta\subset S$ enclosing a and another puncture of $\widetilde{S}$ corresponding to the conjugacy class of g in G. Since every essential hyperbolic element g is written as a word generated by simple hyperbolic and parabolic elements of G, the pseudo-Anosov class g^* can be represented as a word generated by spins and Dehn twists.

3 Proof of Theorem 1.1

It suffices to show that there are (infinitely many) essential hyperbolic elements g of G, so that g^* can not be isotopic to a finite product (1.1), where α_i bounds a twice punctured disk Δ_i that encloses a for $i=1,2$.

Let $a=x_1,x_2,\cdots,x_n$ $(n\geq 2)$ denote the punctures of S. Thus $\widetilde{S}=S\cup\{a\}$ has punctures $x_2,\cdots,x_n$. For every $i=2,\cdots,n$, let

$$S_i=S\cup\{x_i\}\quad\text{and}\quad\widetilde{S}_i=\widetilde{S}\cup\{x_i\}=S_i\cup\{a\}.$$

Let G_i denote the Fuchsian group that uniformizes $\widetilde{S}_i$. G_i acts on the Bers fiber space $F(\widetilde{S}_i)$ fiber wise, and is regarded as a normal subgroup of mod $\widetilde{S}_i$. Let $\varphi_i:F(\widetilde{S}_i)\to T(S_i)$ denote the Bers isomorphism. φ_i induces a group isomorphism φ_i^* of mod $\widetilde{S}_i$ onto $\mathrm{Mod}_{S_i}^a$ by conjugation.

Let Mod_S^{a,x_i} be the subgroup of Mod_S consisting of mapping classes fixing both a and x_i, and $\eta_i^*:\mathrm{Mod}_S^{a,x_i}\to\mathrm{Mod}_{S_i}^a$ the natural projection defined by forgetting the puncture x_i. We fix an isomorphism $\pi_1(\widetilde{S}_i,a)\cong G_i$ as well as the isomorphism $\pi_1(\widetilde{S},a)\cong G$. Clearly, there exists a naturally defined projection ξ_i of $\pi_1(\widetilde{S},a)$ onto $\pi_1(\widetilde{S}_i,a)$ by forgetting the puncture x_i. Then we obtain a projection $\zeta_i:G\to G_i$ making the diagram

$$\begin{array}{ccc} G & \xrightarrow{\zeta_i} & G_i \\ {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ \pi_1(\widetilde{S},a) & \xrightarrow{\xi_i} & \pi_1(\widetilde{S}_i,a) \end{array} \tag{3.1}$$

commutative.

Lemma 3.1 *Let G and G_i be regarded as normal subgroups of* mod $\widetilde{S}$ *and* mod $\widetilde{S}_i$*, respectively. With notations above, for every $i = 2, \cdots, n$, the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\varphi^*} & \mathrm{Mod}_S^{a,x_i} \\ \zeta_i \big\downarrow & & \eta_i^* \big\downarrow \\ G_i & \xrightarrow{\varphi_i^*} & \mathrm{Mod}_{S_i}^{a} \end{array} \tag{3.2}$$

commutes.

Proof Fix a set of generators of G. Let $g \in G$. Without loss of generality, we assume that g is one of the generators of G. The general case is handled by the properties of group homomorphisms. Then g is either parabolic or simple hyperbolic. Let $\widetilde{c} \in \pi_1(\widetilde{S}, a)$ be a loop that corresponds to g under the fixed isomorphism.

If g is parabolic, then $\widetilde{c}$ goes around x_i, $\zeta_i(g)$ is trivial, and hence $\varphi_i^* \circ \zeta_i(g)$ is trivial. On the other hand, by [7, Theorem 2] and [11], g^* is the Dehn twist along the loop c bounding a twice punctured disk Δ that encloses a and x_i. As x_i is filled in, c shrinks and thus g^* becomes a trivial loop, which means that $\eta_i^*(g^*)$ is trivial. This shows that

$$\eta_i^*(g^*) = \varphi_i^* \circ \zeta_i(g)$$

if g is a parabolic element corresponding to x_i.

If g is a parabolic element that corresponds to x_j $(i \neq j)$ $\zeta_i(g) \in G_i$ corresponds to x_j (as a puncture of $\widetilde{S}_i$). So $\varphi_i^* \circ \zeta_i(g)$ is the Dehn twist along the loop c that bounds a twice punctured disk Δ enclosing a and x_j. On the other hand, g^* is the Dehn twist along $\partial\Delta$. Since x_i lies outside of Δ, as x_i is filled in, Δ does not vanish. So $\eta_i^*(g^*)$ is also the Dehn twist along $\partial\Delta$. So in this case, we again have

$$\eta_i^*(g^*) = \varphi_i^* \circ \zeta_i(g).$$

If g is simple hyperbolic, the argument is similar to the above. Instead of having a twice punctured disk Δ, g^* is a spin defined by an a-punctured cylinder $P \subset S$ that does not contain any other punctures of $\widetilde{S}$. Thus $\eta_i^*(g^*)$ is the spin defined by $P \subset S_i$. If we follow the other path, we see that $\zeta_i(g) \in G_i$ is also simple hyperbolic, and it is easy to check that $\varphi_i^* \circ \zeta_i(g)$ is the spin determined by P.

Now we claim that there are infinitely many essential hyperbolic elements g of G such that $\xi_i(g) \in G_i$ are also essential elements. One example is demonstrated below, from which one can generate infinitely many essential elements by taking powers of generators or permuting generators.

Let $\overline{S}$ denote the compactification of S, $\widetilde{c}_1, \widetilde{c}_2$ be two simple closed geodesics on $\overline{S}$ so that $\widetilde{c}_1$ and $\widetilde{c}_2$ go through a and $\{\widetilde{c}_1, \widetilde{c}_2\}$ fills $\overline{S}$, and $Q_1, \cdots, Q_k$ be the disk components of $\overline{S} - \{\widetilde{c}_1, \widetilde{c}_2\}$. They are all polygons whose boundaries are geodesic segments (some of which may be identical). We assume that all the points $x_2, \cdots, x_n$ lie in Q_1, say, and $a = x_1$ is a vertex of Q_1. In Q_1, we can take a parabolic basis $\widetilde{e}_1, \cdots, \widetilde{e}_n \in \pi_1(\widetilde{S}, a)$. That is, $\widetilde{e}_i$ is a loop representative that starts from a, goes around x_i exactly once in the clockwise direction, and then return to a.

Note that $\widetilde{c}_1$ and $\widetilde{c}_2$ also represent two nontrivial elements in $\pi_1(\widetilde{S}, a)$. For $1 \leq i \leq n$, let $T_i \in G$ be the elements that correspond to $\widetilde{e}_i$, and h_1, h_2 the elements corresponding to $\widetilde{c}_1, \widetilde{c}_2$, respectively, under the isomorphism $\pi_1(\widetilde{S}, a) \cong G$. Note that $T_i \in G$ are parabolic, while $h_1, h_2 \in G$ are hyperbolic.

We define

$$g = T_2 \circ (T_2 \circ T_3) \cdots \circ (T_2 \circ \cdots \circ T_n) \circ h_2 \circ h_1. \tag{3.3}$$

Consider the curve

$$\widetilde{\lambda} = \widetilde{c}_1 \cdot \widetilde{c}_2 \cdot (\widetilde{e}_n \cdots \widetilde{e}_2) \cdots (\widetilde{e}_3 \cdot \widetilde{e}_2) \cdot \widetilde{e}_2.$$

See Figure 1. We notice that the complement of $\widetilde{\lambda}$ on Q_1 consists of one disk and $n-1$ once punctured disks (Figure 1 above shows the portion of the curve $\widetilde{\lambda}$ in Q_1), and the complement of $\widetilde{\lambda}$ in $\widetilde{S} - Q_1$ is determined by $\overline{S} - \{\widetilde{c}_1, \widetilde{c}_2\}$ that consists of disks only. It follows that $\widetilde{\lambda}$ is a filling curve. Now the axis of g projects to a geodesic homotopic to the filling curve $\widetilde{\lambda}$ on $\widetilde{S}$.

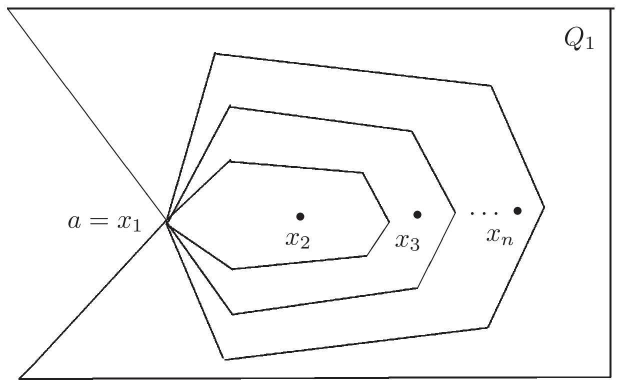

Figure 1 The portion of the curve $\widetilde{\lambda}$ in Q_1

It follows that the element g defined in (3.3) is essential hyperbolic. Moreover, as x_i is filled in, the homomorphism $\pi_1(\widetilde{S}, a) \to \pi_1(\widetilde{S}_i, a)$ only kills the loop $\widetilde{e}_i$. The image loop still fills $\widetilde{S}_i$, which says that $\zeta_i(g)$ corresponds to the element that fills $\widetilde{S}_i$. Therefore, $\zeta_i(g)$ is an essential hyperbolic element of G_i. Finally, we need

Lemma 3.2 *Let g be defined in* (3.3)*. Then $\eta_i^*(g^*) \in \mathrm{Mod}_{S_i}^a$ represents a pseudo-Anosov mapping class.*

Proof We know that $\zeta_i(g)$ is an essential hyperbolic element of G_i. Hence, by [7, Theorem 2], $\varphi_i^* \circ \zeta_i(g)$ is pseudo-Anosov. By Lemma 3.1, $\eta_i^*(g^*) = \varphi_i^* \circ \zeta_i(g)$. We conclude that $\eta_i^*(g^*) \in \mathrm{Mod}_{S_i}^a$ is a pseudo-Anosov class.

Proof of Theorem 1.1 Let g be defined in (3.3). Assume that g^* is represented by (1.1) for α_1 and α_2 being boundaries of twice punctured disks Δ_1 and Δ_2 enclosing a. Write $\alpha_1 = \partial\Delta_1$ and $\alpha_2 = \partial\Delta_2$. Let $\{a, x_i\}$ be the punctures included in Δ_1, and $\{a, x_j\}$ be the punctures included in Δ_2. If $x_i = x_j$, then on the surface $S_i = S \cup \{x_i\}$, both $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ shrink to the puncture a. It follows that the mapping class $\eta_i^*(g^*)$ is trivial, which contradicts Lemma 3.2.

We assume that $x_i \neq x_j$. Observe that as x_i is filled in, the loop α_1 shrinks to the puncture a while α_2 remains noncontractible on S_i. This means that

$$\eta_i^*\Big(\prod_i t_1^{r_i} \circ t_2^{-s_i}\Big) = \widehat{t}_2^{\,-\sum\limits_i s_i}, \tag{3.4}$$

where $\widehat{t}_2$ denotes the Dehn twist along the loop $\widetilde{\alpha}_2$ regarded as a loop on S_i. But (3.4) is a power of Dehn twist that is a special kind of reducible mapping class. This again contradicts Lemma 3.2.

Finally, by the argument of [18, Section 4], we can conclude that there exist infinitely many pseudo-Anosov maps $f = g^* \in \mathcal{F}_0$ that are isotopic to products of Dehn twists along two simple curves that are trivial on $\widetilde{S}$, if $\widetilde{S}$ contains at least one puncture. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Proof of Theorem 1.2(1) We need to construct a pair of geodesics $\{\alpha_1, \alpha_2\}$ on S so that it fills S and $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$. Then any products of form (1.1) with $\sum_i (r_i - s_i) = 0$ have the required properties.

Since $3p - 3 + n > 0$, we can take a simple closed geodesic $\widetilde{\alpha}_0 \subset \widetilde{S}$. Note that $\widetilde{\alpha}_0$ can also be viewed as a curve on S whose geodesic representative is denoted by α_1. Since $\mathcal{F}_0$ contains infinitely elements (see [7, Theorem 2]), we can pick an element $f \in \mathcal{F}_0$. By definition, f is pseudo-Anosov and is isotopic to the identity on $\widetilde{S}$. By a theorem of Masur and Minsky [10], for sufficiently large integer k, the geodesic representative α_2 of $f^k(\alpha_1)$ together with α_1 itself fills S. We must have $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = \widetilde{\alpha}_0$.

To prove the rest of the results, we let $\mathcal{I}$ denote the subset of $\mathcal{F} \cap \mathcal{L}$ that consists of elements of form (1.1) for α_1 and α_2 being boundaries of twice punctured disks $\Delta_1, \Delta_2 \subset S$ both of which enclose a. Let $\chi \in \mathcal{F} \cap (\mathcal{L} - \mathcal{I})$. There is $g \in G$ such that $\chi = g^*$. Recall that $\eta_a^* : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\widetilde{S}}$ denotes the natural projection induced by (2.1). If $\widetilde{\alpha}_1$ is contractible while $\widetilde{\alpha}_2$ is not contractible on $\widetilde{S}$, then

$$\eta_a^* \Big(\prod_i t_1^{r_i} \circ t_2^{-s_i} \Big) = t_{\widetilde{\alpha}_2}^{\sum_i -s_i}. \tag{4.1}$$

On the other hand, $\eta_a^*(g^*)$ is a trivial mapping class. This is a contradiction.

Let us now consider the case that both $\widetilde{\alpha}_i$, $i = 1, 2$, are not contractible on $\widetilde{S}$. Clearly, if $\widetilde{\alpha}_1$ is disjoint from $\widetilde{\alpha}_2$, (1.1) projects to a multi-twist that is nontrivial. This is a contradiction.

If $\widetilde{\alpha}_1$ intersects $\widetilde{\alpha}_2$, then (1.1) projects to

$$\Theta = \prod_i t_{\widetilde{\alpha}_1}^{r_i} \circ t_{\widetilde{\alpha}_2}^{-s_i}.$$

Let Σ be a system of geodesics such that one component $\widetilde{R}$ of $\widetilde{S} - \Sigma$ contains $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$. We assume that $\widetilde{R} - \Sigma$ is a union of disks, once punctured disks, and annuli with boundary loops in Σ. Then from [9], we know that $\Theta|_{\widetilde{R}}$ is pseudo-Anosov. In particular, this implies that Θ is nontrivial.

We conclude that $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$, i.e., $t_{\widetilde{\alpha}_1} = t_{\widetilde{\alpha}_2}$. Since $\chi \in \mathcal{F}$, it projects to the trivial mapping class. Hence, in order for the maps with form (1.1) to project to the identity, we must have

$$\sum_i (r_i - s_i) = 0.$$

Since g is essential, by [7, Theorem 2], g^* is pseudo-Anosov, which means that $\{\alpha_1, \alpha_2\}$ fills S. This proves Theorem 1.2(1).

Proof of Theorem 1.2(2) To prove Theorem 1.2(2), we first need to describe a lift τ_i of $t_{\widetilde{\alpha}_i}$ to the hyperbolic plane $\mathbb{H}$. Let $\widehat{\alpha}_i \subset \mathbb{H}$ be a geodesic with $\varrho(\widehat{\alpha}_i) = \widetilde{\alpha}_i$, and D_i, D_i' be the components of $\mathbb{H} - \widehat{\alpha}_i$. A lift $\tau_i : \mathbb{H} \to \mathbb{H}$ with respect to D_i can be constructed as follows. Let $g_i \in G$ be the primitive simple hyperbolic element such that $g_i(D_i) = D_i$. We assume that g_i is oriented as shown in Figure 2.

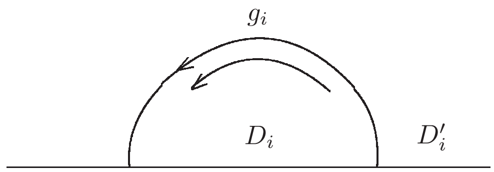

Figure 2 A maximal element for the lift τ_i of $t_{\widetilde{\alpha}_i}$

We then take an earthquake g_i-shift on D_i and leave D_i' fixed. Then we define $\tau_i : \mathbb{H} \to \mathbb{H}$ via G-invariance, which gives rise to a collection $\mathcal{U}_i$ of half planes in $\mathbb{H}$ in a partial order. In Figure 2, the arrow underneath $\widehat{\alpha}_i$ indicates the direction of the motion of τ_i on D_i. There are infinitely many disjoint maximal elements $D_i(j)$ of $\mathcal{U}_i$, each of which is invariant under τ_i (D_i is just one of them). The restriction of τ_i to the complement H_i of the union of these disjoint maximal elements is the identity. It was shown in [17] that among the preimages $\{\varrho^{-1}(\widetilde{\alpha}_i)\}$, one may choose a geodesic $\widehat{\alpha}_i$ and hence a component $D_i \in \mathcal{U}_i$, so that the lifts τ_i with respect to D_i satisfy the conditions $\tau_i^* = t_i$.

Since $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$, $\{\varrho^{-1}(\widetilde{\alpha}_1)\} = \{\varrho^{-1}(\widetilde{\alpha}_2)\}$. We see that, for any $D_1 \in \mathcal{U}_1$ and any $D_2 \in \mathcal{U}_2$, $\partial D_1 \cap \partial D_2 = \emptyset$. Suppose that there does not exist any $D_i \in \mathcal{U}_i$ such that $D_1 \cap D_2 \neq \emptyset$. Then for any $D_i \in \mathcal{U}_i$, either D_1 and D_2 are disjoint, or $D_1 \subset D_2$, or $D_2 \subset D_1$. All of these cases imply that τ_1 commutes with τ_2, which is equivalent to that τ_1^* commutes with τ_2^*. But $\tau_i^* = t_i$. We assert that t_1 commutes with t_2, which further implies that α_1 is disjoint from α_2. This contradicts the fact that $\{\alpha_1, \alpha_2\}$ fills S.

We conclude that there exist $D_i \in \mathcal{U}_i$ such that $D_1 \cap D_2 \neq \emptyset$. The pair $\{D_1, D_2\}$ is drawn in Figure 3. Clearly, $D_1 \cup D_2 = \mathbb{H}$. Denote by (U, V) and $[U, V]$ the open and the closed circular arc on $\partial\mathbb{H}$ connecting the two labeled points U and V on $\partial\mathbb{H}$ without passing through any other labeled points. Let $x \in (U, V)$. Then $\tau_1^{r_1}\tau_2^{-s_1}(x) \in (U, V)$. By induction, one showes that for any finite product

$$\zeta = \prod_i \tau_1^{r_i}\tau_2^{-s_i},$$

$\zeta(x) \in (U, V)$, and its m-th iteration $\zeta^m(x) \in (U, V)$. But we know that $\zeta^* = g^*$ for some $g \in G$. In particular, the iterations of ζ and g on the boundary circle are the same. Hence $g^m(x)$ tends to a point $x_0 \in [U, V]$ as $m \to +\infty$. By definition, x_0 is the attracting fixed point of g. If $x_0 = U$ or V, then g would share a fixed point with a simple hyperbolic element of G, which is impossible. Therefore $x_0 \in (U, V)$.

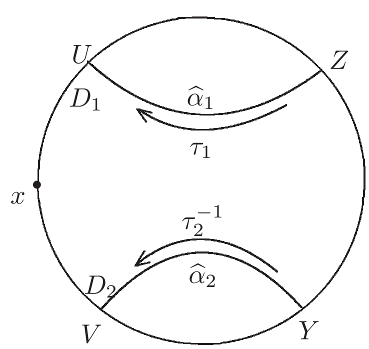

Figure 3 Stable region (U, V) and unstable region (Y, Z) for the iteration of ζ

Similarly, as $m \to -\infty$, $g^m(x)$ tends to a point $y_0 \in (Y, Z)$ that is the repelling fixed point of g. It follows that the geodesic c connecting x_0 and y_0, which is the axis of g, is completely in the interior of the region $D_1 \cap D_2$. This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Under the universal covering $\varrho : \mathbb{H} \to \widetilde{S}$, the point a determines a set $A = \{\varrho^{-1}(a)\}$ that is a discrete subset of $\mathbb{H}$ invariant under the action of G. Consider the complement $\mathbb{H} - A$. Then G keeps $\mathbb{H} - A$ invariant and ϱ restricts to a covering $\varrho : \mathbb{H} - A \to \widetilde{S}$ with the covering group G. On the other hand, $\mathbb{H} - A$ can be thought of as a Riemann surface with infinite type. Let $\varrho_1 : \mathbb{H} \to \mathbb{H} - A$ be the universal covering with a covering group G_1, which is isomorphic to the fundamental group $\pi_1(\mathbb{H} - A, \widehat{a}_0)$ for a fixed point $\widehat{a}_0 \in \mathbb{H} - A$ with $\varrho(\widehat{a}_0) = a_0$. Observe that $\pi_1(\mathbb{H} - A, \widehat{a}_0)$ is generated by infinitely many simple small loops δ_j each of which goes around a point $\widehat{a}_j \in A$ once and zero times around any other $\widehat{a}_k$ for $j \neq k$ and $\widehat{a}_k \in A$. Under the isomorphism of $\pi_1(\mathbb{H} - A, \widehat{a}_0)$ onto G_1, δ_j corresponds to a primitive parabolic element γ_j of G_1. Let $z_j \in \mathbb{R}$ be the fixed point of γ_j.

Lemma 5.1 *Let $\{u_{ij}\} \in \mathbb{H} - A$ $(i = 1, 2, \cdots)$ be a sequence such that $u_{ij} \to z_j$ nontangentially as $i \to \infty$. Then $\varrho_1(u_{ij})$ tends to the puncture a_j.*

Proof Draw a horodisk D_j at z_j that is invariant under γ_j. Then $D_j/\langle\gamma_j\rangle$ is an a_j-punctured disk conformally embedded in $\mathbb{H} - A$. Since $u_{ij} \to z_j$ nontangentially, we may assume that u_{ij} are not γ_j-equivalent. It follows that $\varrho_1(u_{ij})$ are all distinct and tend to the puncture a_j, as asserted.

By construction, there is an exact sequence of covering groups

$$1 \longrightarrow G_1 \hookrightarrow \dot{G} \longrightarrow G \longrightarrow 1. \tag{5.1}$$

This is equivalent to that $\dot{G}$ is a semi-product of G_1 and G. We need to examine the representatives f of g^* and its lifts to $\mathbb{H} - A$ under ϱ.

Lemma 5.2 *Fix a point $\widehat{a} \in A$. Then there exists a quasiconformal map $\omega : \mathbb{H} \to \mathbb{H}$ such that the following conditions hold:*

(1) *the map ω leaves $A = \{\varrho^{-1}(a)\} \subset \mathbb{H}$ invariant,*
(2) *if ω is regarded as a selfmap of $\mathbb{H} - A$ onto itself, then $\varrho \circ \omega = f \circ \varrho$,*
(3) *the map ω commutes with every element of G,*
(4) *$\omega^{-1} \circ g(\widehat{a}) = g \circ \omega^{-1}(\widehat{a}) = \widehat{a}$.*

Proof By using topological arguments (see, for example, [7, Proposition 1]), we know that for $\widehat{a} \in A$, we can construct a quasiconformal map ω of $\mathbb{H}$ that satisfies (3) and (4). For convenience, we outline the construction as follows. Connect $\widehat{a}$ and $g(\widehat{a})$ by a geodesic segment Γ. By fattening Γ, we obtain a flat ellipse E containing $\widehat{a}$ and $g(\widehat{a})$. There is a quasiconformal map in E which sends $\widehat{a}$ to $g(\widehat{a})$ and is the identity outside of E (see [7, Lemma 1] for the construction). We then define ω via G-invariance. Evidently, ω possesses properties (3) and (4).

To see that (1) is satisfied, we choose a point $\widehat{a}' \in A$. There is an element $h \in G$ such that $h(\widehat{a}) = \widehat{a}'$. Then $\omega(\widehat{a}') = \omega \circ h(\widehat{a}) = h \circ \omega(\widehat{a}) = h \circ g^{-1}(\widehat{a}) \in A$ since G keeps A invariant. Hence (1) holds. Finally, from (4) and the construction of the Bers isomorphism (see [2, Theorem 9] or [7, Theorem 2]), we know that the map ω, if regarded as a map of $\mathbb{H} - A$ onto itself, descends to $f : S \to S$ under the restricted covering $\varrho : \mathbb{H} - A \to \widetilde{S} - \{a\} \cong S$. So (2) is satisfied.

Therefore, we can lift the map $\omega : \mathbb{H} - A \to \mathbb{H} - A$ to $\widehat{\omega} : \mathbb{H} \to \mathbb{H}$ through the covering map ϱ_1 that satisfies

$$\varrho_1 \circ \widehat{\omega} = \omega \circ \varrho_1. \tag{5.2}$$

Clearly, the composition $\varrho_0 = \varrho \circ \varrho_1 : \mathbb{H} \to \widetilde{S}$ is a universal covering with the covering group $\dot{G}$. Combining with Lemma 5.2(4) and (5.1), we compute

$$\varrho_0 \circ \widehat{\omega} = (\varrho \circ \varrho_1) \circ \widehat{\omega} = \varrho \circ \omega \circ \varrho_1 = f \circ (\varrho \circ \varrho_1) = f \circ \varrho_0,$$

which says that $\widehat{\omega}$ is a lift of f through ϱ_0. Hence, $\widehat{\omega}$ is of form $\dot{h}_1 \circ \widehat{f} \circ \dot{h}_2$ for $\dot{h}_1, \dot{h}_2 \in \dot{G}$, where $\widehat{f}$ is one of the lifts of f.

More careful investigation on the map ω yields

Lemma 5.3 *As a map of* $\mathbb{H} - A$ *onto itself,* ω *has the following properties:*

(1) *the restriction* $\omega|_{\partial\mathbb{H}}$ *is the identity,*

(2) *the action of* ω *on* A *is fixed-point free,*

(3) *for every simple hyperbolic or parabolic element* h *of* G*, the action of* $h \circ \omega$ *on* A *is also fixed-point free.*

Proof (1) is obvious since ω commutes with every element of G. Suppose that for some $\widehat{a}' \in A$ we have $\omega(\widehat{a}') = \widehat{a}'$. Choose $h \in G$ so that $h(\widehat{a}) = \widehat{a}'$. That is $\omega \circ h(\widehat{a}) = h(\widehat{a})$. Since ω commutes with each element of G, we get $h \circ \omega(\widehat{a}) = h(\widehat{a})$. By Lemma 5.2(4), $h \circ g(\widehat{a}) = h(\widehat{a})$. It follows that $g(\widehat{a}) = \widehat{a}$, contradicting the fact that g has no fixed point inside of $\mathbb{H}$. This proves (2).

To prove (3), we assume that for $\widehat{a}' \in A$, we have

$$h \circ \omega(\widehat{a}') = \widehat{a}'. \tag{5.3}$$

Choose $g_0 \in G$ so that $\widehat{a}' = g_0(\widehat{a})$. Then (5.3) becomes $h \circ \omega \circ (g_0(\widehat{a})) = g_0(\widehat{a})$. Since ω commutes with h, $h \circ g_0 \circ \omega(\widehat{a}) = g_0(\widehat{a})$, or $g_0^{-1} \circ h \circ g_0 \circ \omega(\widehat{a}) = \widehat{a}$. Set $g_0^{-1} \circ h \circ g_0 = h_0$. Then h_0 is also parabolic or simple hyperbolic, depending on whether h is parabolic or simple hyperbolic. Thus we obtain

$$h_0 \circ \omega(\widehat{a}) = \widehat{a}. \tag{5.4}$$

Now from Lemma 5.2(4), $\omega(\widehat{a}) = g(\widehat{a})$. It follows from (5.4) that

$$h_0 \circ g(\widehat{a}) = \widehat{a}. \tag{5.5}$$

Notice that $h_0 \in G$ is either parabolic or simple hyperbolic, while g is essential. We see that $h_0 \circ g \neq \mathrm{id}$. From (5.5), we conclude that $h_0 \circ g$ fixes a point inside of $\mathbb{H}$ and thus it is an elliptic Möbius transformation. This contradicts that G is a torsion free Fuchsian group. This proves (3).

Proof of Theorem 1.3 It suffices to show that there are infinitely many pseudo-Anosov mapping classes on $\dot{S}$ that are isotopic to f on S as a' is filled in.

Let $h \in G$ be any simple hyperbolic element. Consider the map $h \circ \omega : \mathbb{H} - A \to \mathbb{H} - A$. By Lemma 5.3(1), $\omega|_{\partial\mathbb{H}} = \mathrm{id}$. Hence $h \circ \omega|_{\partial\mathbb{H}}$ fixes no parabolic fixed point of G. By Lemma 5.3(3), $h \circ \omega|_A$ is fixed point free. Let $\widehat{\omega}_0 : \mathbb{H} \to \mathbb{H}$ be a lift of $h \circ w|_{\mathbb{R}}$ which satisfies

$$\varrho_1 \circ \widehat{\omega}_0 = h \circ \omega \circ \varrho_1. \tag{5.6}$$

Suppose that $\widehat{\omega}_0$ fixes some fixed point z_j of γ_j. Choose a sequence $\{u_{ij}\} \in \mathbb{H}$ that tends to the fixed point z_j of γ_j non-tangentially. By (5.6), for all u_{ij}, $i = 1, 2, \cdots$, we have

$$\varrho_1 \circ \widehat{\omega}_0(u_{ij}) = h \circ \omega \circ \varrho_1(u_{ij}). \tag{5.7}$$

Let $i \to \infty$. Then $u_{ij} \to z_j$. By continuity, we obtain

$$\varrho_1 \circ \widehat{\omega}_0(z_j) = h \circ \omega \circ \varrho_1(z_j).$$

By assumption, we have that $\widehat{\omega}_0$ fixes z_j. So $\varrho_1(z_j) = h \circ \omega \circ \varrho_1(z_j)$. By Lemma 5.1, we get $\lim_{i\to\infty} \varrho_1(u_{ij}) = a_j$; that is, $h \circ \omega$ fixes a_j. This contradicts Lemma 5.3(3).

We conclude that $\widehat{\omega}_0$ cannot fix the fixed point of any parabolic element γ_j of $\dot{G}$ that emerges from a point in the set A.

We also need to prove that $\widehat{\omega}_0$ does not fix any parabolic fixed point of $\dot{G}$ other than z_j. Suppose for the contrary, we assume that $\widehat{\omega}_0$ fixes a parabolic fixed point x of $\dot{G}$. Let $\dot{\gamma} \in \dot{G}$ be the parabolic element fixing x. From (5.1), there is a nontrivial element $\gamma \in G$ such that

$$\varrho_1 \circ \dot{\gamma}^m = \gamma^m \circ \varrho_1 \tag{5.8}$$

for any integer m. Since $\dot{\gamma} \in \dot{G}$ is parabolic, for any $u \in \mathbb{H}$, both $\dot{\gamma}^m(u)$ and $\dot{\gamma}^{-m}(u)$ tend to the fixed point x of $\dot{\gamma}$ in $\mathbb{R}$. From (5.8), we get that both $\gamma^m \varrho_1(u)$ and $\gamma^{-m}\varrho_1(u)$ tend to $\varrho_1(x)$. This implies that $\gamma \in G$ is parabolic and its fixed point is $\varrho_1(x)$. It follows that x projects (under ϱ_1) to a parabolic fixed point of G. By hypothesis, $\widehat{\omega}_0(x) = x$. We thus obtain

$$\varrho_1(x) = \varrho_1 \circ \widehat{\omega}_0(x) = h \circ \omega \circ \varrho_1(x),$$

which tells us that $h \circ \omega$ fixes $\varrho_1(x)$, a parabolic fixed point of G. By Lemma 5.3(1), we have $\omega|_{\partial\mathbb{H}} = \mathrm{id}$. We conclude that $h(\varrho_1(x)) = \varrho_1(x)$. But h is simple hyperbolic; it can not fix a parabolic fixed point of G. This contradiction proves that $\widehat{\omega}_0$ does not fix any parabolic fixed point of $\dot{G}$ other than z_j, and hence $\widehat{\omega}_0$ does not fix any parabolic fixed point of $\dot{G}$.

Now from (5.1) we know that $\varrho_0 : \mathbb{H} \to \widetilde{S}$ is a covering map with the group $\dot{G}$. To see that $\widehat{\omega}_0$ projects via ϱ_0 to the map f that represents $g^* \in \mathcal{F}_0$, we notice that $\varrho_0 = \varrho \circ \varrho_1$. From (5.1), (5.6), (5.7) and Lemma 5.2(2), one computes

$$\varrho_0 \circ \widehat{\omega}_0 = \varrho \circ \varrho_1 \circ \widehat{\omega}_0 = \varrho \circ h \circ \omega \circ \varrho_1 = \varrho \circ \omega \circ \varrho_1 = f \circ (\varrho \circ \varrho_1) = f \circ \varrho_0.$$

It follows that $\widehat{\omega}_0 \dot{G} \widehat{\omega}_0^{-1} = \dot{G}$ and $\widehat{\omega}_0$ projects to f. Moreover, its equivalence class $[\widehat{\omega}_0]$ is an element of mod S. Let $\psi : F(S) \to T(\dot{S})$ denote a Bers isomorphism. Then ψ induces an isomorphism ψ^* of mod S onto $\mathrm{Mod}_{\dot{S}}^{a'}$. By the above argument, we see that $\widehat{\omega}_0^* = \psi^*([\widehat{\omega}_0]) \in \mathrm{Mod}_{\dot{S}}^{a'}$ projects to the mapping class g^*.

Now suppose that $\widehat{\omega}_0^*$ is a reducible mapping class on $\dot{S}$ that is reduced by a curve system $\{c_1, c_2, \cdots, c_s\}$ for $s \geq 1$. By taking a suitable power, we may assume that $\widehat{\omega}_0^*$ leaves each curve in the system invariant. If $s \geq 2$, then at least one curve in the system, c_1, say, is also noncontractible on S. Let c_1' denote the corresponding curve on S. This implies that g^* leaves c_1' invariant. This contradicts the fact that g^* is pseudo-Anosov. So the only possibility is that $s = 1$ and c_1' is a trivial curve on S. That is, c_1 is a curve that is the boundary of a twice punctured disk enclosing two punctures, one of which is a'. But in this case, by [15, Lemmas 3.1 and 3.2], we have that $\widehat{\omega}_0$ fixes a parabolic fixed point of $\dot{G}$. This also contradicts the above argument.

We conclude that $\widehat{\omega}_0^* = \psi^*([\widehat{\omega}_0]) \in \mathrm{Mod}_{\dot{S}}^{a'}$ is a pseudo-Anosov element projecting to g^*. Since there are infinitely many simple hyperbolic elements in G, there are infinitely many pseudo-Anosov elements $\widehat{\omega}_0^*$ in $\mathrm{Mod}_{\dot{S}}^{a'}$. This completes the proof of Theorem 1.3.

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