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Some Aspects of $L^r_{w_1}(G) \cap L(p, q, w_2\mathrm{d}\mu)(G)$

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Abstract Let G be a locally compact Abelian group with Haar measure. The authors discuss some basic properties of $L^r_{w_1}(G) \cap L(p, q, w_2\mathrm{d}\mu)(G)$ spaces. Then the necessary conditions for compact embeddings of the spaces $L^r_{w_1}(R^d) \cap L(p, q, w_2\mathrm{d}\mu)(R^d)$ are showed.

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1 Introduction and Preliminaries

Throughtout this work, G will denote a non-compact and non-discrete locally compact Abelian group with Haar measure μ. Also, we will use Beurling's weight function, i.e., a measurable, locally bounded function on G satisfying $w(x) \geq 1$ and $w(x+y) \leq w(x)w(y)$ for all $x, y \in G$. For two weight functions w_1 and w_2, we write $w_1 \prec w_2$ if there exists $C > 0$, such that $w_1(x) \leq Cw_2(x)$ for all $x \in G$. We write $w_1 \approx w_2$ if and only if $w_1 \prec w_2$ and $w_2 \prec w_1$. Certain well-known terms such as Banach module, Banach ideal, translation and character invariance, compact embedding will be used frequently in the sequel; their definitions may be found, e.g., in [3, 4, 10, 12]. For $1 \leq r < \infty$, we set weighted Lebesgue spaces as

$$L^r_w(G) = \{f \mid fw \in L^r(G)\},$$

which are Banach spaces under the naturel norm

$$\|f\|_{r,w} = \Big\{ \int_G |f(x)|^r w^r(x)\mathrm{d}\mu(x) \Big\}^{\frac{1}{r}}.$$

Recall that one has $L^r_{w_1}(G) \subset L^r_{w_2}(G)$ if and only if $w_2 \prec w_1$ (see [6, 8]). The Lorentz spaces over weighted measure spaces $L(p, q, w\mathrm{d}\mu)$ are defined and discussed in [5, 13]. Instead of Haar measure μ, let us take the measure as $w\mathrm{d}\mu$. Then the distribution function of f which is considered complex-valued measurable and defined on the measure space $(G, w\mathrm{d}\mu)$ is

$$\lambda_{f,w}(y) = w\{x \in G : |f(x)| > y\} = \int_{\{x\in G:|f(x)|>y\}} w(x)\mathrm{d}\mu(x), \quad y \geq 0.$$

The nonnegative rearrangement of f is given by

$$f^*_w(t) = \inf\{y > 0 : \lambda_{f,w}(y) \leq t\} = \sup\{y > 0 : \lambda_{f,w}(y) > t\}, \quad t \geq 0,$$

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where we assume that $\inf\phi = \infty$ and $\sup\phi = 0$. Also the average function of f on $(0,\infty)$ is given by

$$f_w^{**}(t) = \frac{1}{t}\int_0^t f_w^*(s)\mathrm{d}s.$$

Note that $\lambda_{f,w}(\cdot)$, $f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are nonincreasing and right continuous functions. The weighted Lorentz space $L(p,q,w\mathrm{d}\mu)$ is the collection of all the functions f such that $\|f\|_{p,q,w}^* < \infty$, where

$$\|f\|_{p,q,w}^* = \Big(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f_w^*(t)]^q\mathrm{d}t\Big)^{\frac{1}{q}}, \quad 0<p,q<\infty,$$
$$\|f\|_{p,\infty,w}^* = \sup_{t>0} t^{\frac{1}{p}} f_w^*(t), \quad 0<p<q=\infty.$$

In general, however, $\|\cdot\|_{p,q,w}^*$ is not a norm since the Minkowski inequality may fail. But by replacing f_w^* with f_w^{**} in the above definition, we get that $L(p,q,w\mathrm{d}\mu)$ is a Banach space, with the norm $\|\cdot\|_{p,q,w}$ defined by

$$\|f\|_{p,q,w} = \Big(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f_w^{**}(t)]^q\mathrm{d}t\Big)^{\frac{1}{q}}, \quad 0<p,q<\infty,$$
$$\|f\|_{p,\infty,w} = \sup_{t>0} t^{\frac{1}{p}} f_w^{**}(t), \quad 0<p<q=\infty.$$

If $1<p\le\infty$ and $1\le q\le\infty$, then

$$\|f\|_{p,q,w}^* \le \|f\|_{p,q,w} \le \frac{p}{p-1}\|f\|_{p,q,w}^*,$$

where the first inequality is an immediate consequence of the fact that $f_w^* \le f_w^{**}$. The second follows from the Hardy inequality.

The plan of the paper is as follows. In Section 2, we will define the intersection of weighted Lebesgue and weighted Lorentz spaces and give some unmentioned properties of these spaces. Then the compact embedding of this intersected spaces will be discussed in Section 3.

2 Some Results in $L_{w_1}^r(G)\cap L(p,q,w_2\mathrm{d}\mu)(G)$

For $1\le r\le\infty$ and $0<p,q\le\infty$, we will write the intersection of weighted Lebesgue and weighted Lorentz spaces $L_{w_1}^r(G)\cap L(p,q,w_2\mathrm{d}\mu)(G)$ as $B_{r,p,q}^{w_1,w_2}(G)$. If we equip this space with the sum norm

$$\|\cdot\|_{r,p,q}^{w_1,w_2} = \|\cdot\|_{r,w_1} + \|\cdot\|_{p,q,w_2}, \tag{2.1}$$

then it is easy to see that $(B_{r,p,q}^{w_1,w_2}(G), \|\cdot\|_{r,p,q}^{w_1,w_2})$ is a normed space. Now we will give some properties of these spaces without their complete proofs.

Theorem 2.1 $(B_{r,p,q}^{w_1,w_2}(G), \|\cdot\|_{r,p,q}^{w_1,w_2})$ *is a Banach space for* $1\le r\le\infty$ *and* $p=q=1$, $p=q=\infty$ *or* $1<p<\infty$, $1\le q\le\infty$.

Proof Let (f_n) be a $(B_{r,p,q}^{w_1,w_2}(G), \|\cdot\|_{r,p,q}^{w_1,w_2})$ Cauchy sequence. Clearly (f_n) is a Cauchy sequence in both $L_{w_1}^r(G)$ and $L(p,q,w_2\mathrm{d}\mu)(G)$. Therefore, (f_n) converges to some $f\in L_{w_1}^r(G)$ and $g\in L(p,q,w_2\mathrm{d}\mu)(G)$. To prove the theorem, we need to show that $f=g(\mu-\text{a.e.})$. Since the convergence in both $L_{w_1}^r(G)$ and $L(p,q,w_2\mathrm{d}\mu)(G)$ implies the convergence in measure, we get $f=g(\mu-\text{a.e.})$ (see [10, 16]).

Theorem 2.2 *The space* $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|^{w_1,w_2}_{r,p,q})$ *is translation invariant and the function* $f \to L_x f$ *is continuous from* $B^{w_1,w_2}_{r,p,q}(G)$ *to* $B^{w_1,w_2}_{r,p,q}(G)$ *for all* $x \in G$*, where* $L_x f(\cdot) = f(\cdot - x)$.

Proof Let us take any $x \in G$ and $f \in B^{w_1,w_2}_{r,p,q}(G)$. Since $\lambda_{L_x f, w_2}(y) \le w_2(x)\lambda_{f,w_2}(y)$ for all $y \ge 0$, we have $\|L_x f\|_{p,q,w_2} \le (w_2(x))^{\frac{1}{p}}\|f\|_{p,q,w_2}$ (see [5]). With the inequality $\|L_x f\|_{r,w_1} \le w_1(x)\|f\|_{r,w_1}$, we get

$$\begin{aligned}\|L_x f\|^{w_1,w_2}_{r,p,q} &= \|L_x f\|_{r,w_1} + \|L_x f\|_{p,q,w_2}\\ &\le w_1(x)\|f\|_{r,w_1} + (w_2(x))^{\frac{1}{p}}\|f\|_{p,q,w_2}\\ &\le \max\{w_1(x), (w_2(x))^{\frac{1}{p}}\}\|f\|^{w_1,w_2}_{r,p,q}.\end{aligned}$$

Also, the continuity from $B^{w_1,w_2}_{r,p,q}(G)$ to $B^{w_1,w_2}_{r,p,q}(G)$ follows from the linearity of L_x for all $x \in G$.

Theorem 2.3 *The space* $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|^{w_1,w_2}_{r,p,q})$ *is strongly character invariant and the function* $f \to M_t f$ *is continuous from* $B^{w_1,w_2}_{r,p,q}(G)$ *to* $B^{w_1,w_2}_{r,p,q}(G)$ *for all* $t \in \widehat{G}$.

Proof Let us take any $t \in \widehat{G}$ and $f \in B^{w_1,w_2}_{r,p,q}(G)$. Since

$$\begin{aligned}\lambda_{M_t f, w_2}(y) &= w_2\{x \in G : |M_t f(x)| > y\}\\ &= w_2\{x \in G : |\langle x, t\rangle f(x)| > y\}\\ &= w_2\{x \in G : |f(x)| > y\} = \lambda_{f,w_2}(y),\end{aligned}$$

we get $(M_t f)^*_{w_2} = f^*_{w_2}$ and $(M_t f)^{**}_{w_2} = f^{**}_{w_2}$, so $\|M_t f\|_{p,q,w_2} = \|f\|_{p,q,w_2}$. By the equality $\|M_t f\|_{r,w_1} = \|f\|_{r,w_1}$, we have $\|M_t f\|^{w_1,w_2}_{r,p,q} = \|f\|^{w_1,w_2}_{r,p,q}$.

Theorem 2.4 *For every* $f \in B^{w_1,w_2}_{r,p,q}(G)$*, the function* $x \to L_x f$*,* $G \to B^{w_1,w_2}_{r,p,q}(G)$ *is continuous where* $1 < p < \infty$*,* $1 \le q < \infty$ *and* $1 \le r < \infty$.

Proof We know that, for $1 \le r < \infty$, $x \to L_x f$ is continuous in $L^r_{w_1}(G)$ (see [8]). Also the continuity of $x \to L_x f$ in $L(p,q,w_2\mathrm{d}\mu)(G)$ was shown for $1 < p < \infty$, $1 \le q < \infty$ in [5]. Thus the proof is easily completed by combining the two results mentioned above.

Theorem 2.5 *If the weight functions are constant, then* $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|^{w_1,w_2}_{r,p,q})$ *is a homogeneous Banach space.*

Proof Let the weight functions w_1 and w_2 be constant. Therefore, the spaces $L^r_{w_1}(G)$ and $L(p,q,\ w_2\mathrm{d}\mu)(G)$ become strongly translation invariant spaces. Then by Theorem 2.4, we get the result.

Theorem 2.6 *If* $w_0 \succ w_1$ *and* $w_0 \succ w_2$*, then* $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|^{w_1,w_2}_{r,p,q})$ *is a Banach* $L^1_{w_0}(G)$*-module.*

Proof Let $w_0 \succ w_1$ and $w_0 \succ w_2$. Then we know that $L^1_{w_0}(G) \subset L^1_{w_1}(G)$ and $L^1_{w_0}(G) \subset L^1_{w_2}(G)$. Therefore for any $f \in L^1_{w_0}(G)$, there exist $c_1, c_2 > 0$ such that $\|f\|_{1,w_1} \le c_1\|f\|_{1,w_0}$ and $\|f\|_{1,w_2} \le c_2\|f\|_{1,w_0}$. Since $L(p,q,w_2\mathrm{d}\mu)(G)$ is a Banach $L^1_{w_2}(G)$-module for $1 < p < \infty$, $q \ne \infty$ (see [5]) and $L^r_{w_1}(G)$ is a Banach $L^1_{w_1}(G)$-module for $1 \le r < \infty$, we have

$$\begin{aligned}\|f * g\|^{w_1,w_2}_{r,p,q} &= \|f * g\|_{r,w_1} + \|f * g\|_{p,q,w_2}\\ &\le \|f\|_{r,w_1}\|g\|_{1,w_1} + \|f\|_{p,q,w_2}\|g\|_{1,w_2}\\ &\le \|f\|_{r,w_1}c_1\|g\|_{1,w_0} + \|f\|_{p,q,w_2}c_2\|g\|_{1,w_0}\\ &\le \|f\|^{w_1,w_2}_{r,p,q}\max\{c_1, c_2\}\|g\|_{1,w_0}\end{aligned}$$

for any $f \in B^{w_1,w_2}_{r,p,q}(G)$ and $g \in L^1_{w_0}(G)$. If we define a new norm $|||\cdot|||$ on $L^1_{w_0}(G)$ such that $|||\cdot||| = \max\{c_1, c_2\}\|\cdot\|_{1,w_0}$, then this norm is equivalent to the norm $\|\cdot\|_{1,w_0}$ on $L^1_{w_0}(G)$. So $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|^{w_1,w_2}_{r,p,q})$ is a Banach $(L^1_{w_0}(G), |||\cdot|||)$-module.

Theorem 2.7 *If $1 < p < \infty$, $q \neq \infty$ and $w_2 \succ w_1$ or $w_1 \succ w_2$, then the space $B^{w_1,w_2}_{1,p,q}(G)$ is a Banach algebra.*

Proof Assume that $w_1 \succ w_2$. Then we know that $L^1_{w_1}(G) \subset L^1_{w_2}(G)$ and for any $f \in L^1_{w_1}(G)$, there exists $c > 0$ such that $\|f\|_{1,w_2} \leq c\|f\|_{1,w_1}$. If we define a new function on $B^{w_1,w_2}_{1,p,q}(G)$ such that $|||\cdot|||^{w_1,w_2}_{1,p,q} = \max\{1, c\}\|\cdot\|^{w_1,w_2}_{1,p,q}$, then it is easy to see that it is a norm. Also, these two norms on $B^{w_1,w_2}_{1,p,q}(G)$ are equivalent and the identity map $i : (B^{w_1,w_2}_{1,p,q}(G), \|\cdot\|^{w_1,w_2}_{1,p,q}) \to (B^{w_1,w_2}_{1,p,q}(G), |||\cdot|||^{w_1,w_2}_{1,p,q})$ is a homeomorphism on $B^{w_1,w_2}_{1,p,q}(G)$. Now take any $f, g \in B^{w_1,w_2}_{1,p,q}(G)$. Since $L^1_{w_1}(G)$ is a Beurling algebra and the space $L(p, q, w_2\mathrm{d}\mu)(G)$ is a Banach $L^1_{w_2}(G)$-module for $1 < p < \infty$, $q \neq \infty$, we get

$$\begin{aligned}
|||f * g|||^{w_1,w_2}_{r,p,q} &= \max\{1, c\}\|f * g\|^{w_1,w_2}_{r,p,q} \\
&= \max\{1, c\}(\|f * g\|_{1,w_1} + \|f * g\|_{p,q,w_2}) \\
&\leq \max\{1, c\}(\|f\|_{1,w_1}\|g\|_{1,w_1} + \|f\|_{1,w_2}\|g\|_{p,q,w_2}) \\
&\leq \max\{1, c\}(\max\{1, c\}\|f\|_{1,w_1}\{\|g\|_{1,w_1} + \|g\|_{p,q,w_2}\}) \\
&\leq \max\{1, c\}\|f\|_{1,w_1} \max\{1, c\}\|g\|^{w_1,w_2}_{r,p,q} \leq |||f|||^{w_1,w_2}_{r,p,q}|||g|||^{w_1,w_2}_{r,p,q}.
\end{aligned}$$

Similarly, if $w_2 \succ w_1$, then the same way may be followed.

Theorem 2.8 *The space $B^{w_1,w_2}_{r,p,q}(G)$ has a bounded approximate identity with compact support for $1 < p < \infty$, $1 \leq q < \infty$.*

Proof Let K be a compact neighbourhood of the identity of G. Then $w_i(y) \leq A$ for all $y \in K$ and $i = 1, 2$. Let F be the family of all neighbourhoods of the identity contained in K. For $U, V \in F$, define $V \prec U$ if $U \subset V$. Then, clearly $(F, \prec)$ is a directed set. For every $U \in F$, there exists a positive continuous function h_U on G such that $\int_G h_U(x)\mathrm{d}\lambda(x) = 1$ and the support of h_U is contained in U. If we consider the net $\{h_U\}_{U\in F}$, then we have $\|h_U\|_{1,w_i} \leq A$ for each $U \in F$. Therefore we find a bounded approximate identity for $L^1_{w_i}(G)$ for $i = 1, 2$. It is shown in [14] that this bounded approximate identity is also a bounded approximate identity for $L^r_{w_1}(G)$. In [5], it was showed that this bounded approximate identity is also a bounded approximate identity for $L(p, q, w_2\mathrm{d}\mu)(G)$ for $1 < p < \infty$, $1 \leq q < \infty$. So $B^{w_1,w_2}_{r,p,q}(G)$ possesses a bounded approximate identity.

The next theorem follows from Theorems 2.6 and 2.8.

Theorem 2.9 *The space $B^{w_1,w_2}_{r,p,q}(G)$ is an essential Banach $L^1_{w_0}(G)$-module, if $w_0 \succ w_1$, $w_0 \succ w_2$ and $1 < p < \infty$, $1 \leq q < \infty$.*

Theorem 2.10 *Let w_1, w_2, w_3 and w_4 be weight functions on G and $1 \leq r \leq \infty$, $0 < p, q < \infty$. Then*

(1) *$B^{w_1,w_2}_{r,p,q}(G) \subset B^{w_3,w_4}_{r,p,q}(G)$ if and only if there exists a constant $c > 0$ such that $\|f\|^{w_3,w_4}_{r,p,q} \leq c\|f\|^{w_1,w_2}_{r,p,q}$ for all $f \in B^{w_1,w_2}_{r,p,q}(G)$,*

(2) *$B^{w_1,w_2}_{r,p,q_1}(G) \subset B^{w_3,w_2}_{r,p,q_2}(G)$ if $w_1 \succ w_3$ and $0 < q_1 \leq q_2 \leq \infty$.*

Proof (1) Let $B^{w_1,w_2}_{r,p,q}(G) \subset B^{w_3,w_4}_{r,p,q}(G)$. We define a norm $\|\cdot\|$ on $B^{w_1,w_2}_{r,p,q}(G)$ such that

$\|f\| = \|f\|^{w_1,w_2}_{r,p,q} + \|f\|^{w_3,w_4}_{r,p,q}$. It is easy to show that $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|)$ is a Banach space. Since the unit function i from $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|^{w_1,w_2}_{r,p,q})$ onto $(B^{w_1,w_2}_{r,p,q}(G), \|\cdot\|)$ is continuous, i is a homeomorphism by the closed graph theorem. This shows that the norms $\|\cdot\|$ and $\|\cdot\|^{w_1,w_2}_{r,p,q}$ are equivalent. Then there is a constant $c > 0$ such that $\|f\| \le c\|f\|^{w_1,w_2}_{r,p,q}$. This gives the inequality

$$\|f\|^{w_3,w_4}_{r,p,q} \le \|f\| \le c\|f\|^{w_1,w_2}_{r,p,q}.$$

Conversely, if $\|f\|^{w_3,w_4}_{r,p,q} \le c\|f\|^{w_1,w_2}_{r,p,q}$ for all $f \in B^{w_1,w_2}_{r,p,q}(G)$, then the inclusion $B^{w_1,w_2}_{r,p,q}(G) \subset B^{w_3,w_4}_{r,p,q}(G)$ is easy to see.

(2) It is known that $L(p,q_1,w_2\mathrm{d}\mu)(G) \subset L(p,q_2,w_2\mathrm{d}\mu)(G)$, where $0 < q_1 \le q_2 \le \infty$ (see [11]). Also, since $L^r_{w_1}(G) \subset L^r_{w_3}(G)$ if $w_1 \succ w_3$, the proof is completed.

Theorem 2.11 *Let* $p = 1$, $0 < q \le 1$ *and* $1 \le r < \infty$. *Then for any* $f \in B^{w_1,w_2}_{r,1,q}(G)$, *the function* $x \to \|L_x f\|^{w_1,w_2}_{r,1,q}$ *is equivalent to the weight function* $w' = w_1 + w_2$, *i.e., there exist* $c_1(f), c_2(f) > 0$ *such that*

$$c_1(f)w'(x) \le \|L_x f\|^{w_1,w_2}_{r,1,q} \le c_2(f)w'(x). \tag{2.2}$$

Proof Let $f \in B^{w_1,w_2}_{r,1,q}(G)$. Then it is known that the function $x \to \|L_x f\|_{r,w}$ is equivalent to the weight function w, i.e., there exist $k_1(f), k_2(f) > 0$ such that

$$k_1(f)w(x) \le \|L_x f\|_{r,w} \le k_2(f)w(x) \tag{2.3}$$

for all $x \in G$ (see [8]). Also by Theorem 2.2, we write

$$\|L_x f\|_{1,q,w_2} \le w_2(x)\|f\|_{1,q,w_2} \tag{2.4}$$

and

$$\|L_x f\|_{1,q,w_2} \ge \|L_x f\|^*_{1,q,w_2} \ge \|L_x f\|^*_{1,1,w_2} = \|L_x f\|_{1,w_2}, \tag{2.5}$$

where $q \le 1$ (see [11]). By using (2.3)–(2.5), there are $s_1(f), s_2(f) > 0$ such that

$$s_1(f)w_2(x) \le \|L_x f\|_{1,w_2} \le \|L_x f\|_{1,q,w_2} \le w_2(x)\|f\|_{1,q,w_2}, \tag{2.6}$$

where $s_2(f) = \|f\|_{1,q,w_2}$. If we combine (2.3) with (2.6), then we have

$$s_1(f)w_2(x) + k_1(f)w_1(x) \le \|L_x f\|^{w_1,w_2}_{r,1,q} \le k_2(f)w_1(x) + s_2(f)w_2(x).$$

Therefore

$$c_1(f)w'(x) \le \|L_x f\|^{w_1,w_2}_{r,1,q} \le c_2(f)w'(x)$$

for all $x \in G$.

Theorem 2.12 *Let* w_1, w_2, w_3 *and* w_4 *be weight functions on* G *and* $w' = w_1 + w_2$, $w'' = w_3 + w_4$. *If* $B^{w_1,w_2}_{r,1,q}(G) \subset B^{w_3,w_4}_{r,1,q}(G)$, *then* $w' \succ w''$ *for* $p = 1$, $0 < q \le 1$. *Conversely,* $B^{w_1,w_2}_{r,1,q}(G) \subset B^{w_3,w_4}_{r,1,q}(G)$, *if* $w_1 \succ w_3$ *and* $w_2 \succ w_4$.

Proof Assume that $B^{w_1,w_2}_{r,1,q}(G) \subset B^{w_3,w_4}_{r,1,q}(G)$. By Theorem 2.11, there are $k, l > 0$ such that

$$k^{-1}w'(x) \le \|L_x f\|^{w_1,w_2}_{r,1,q} \le kw'(x) \tag{2.7}$$

and

$$l^{-1}w''(x) \le \|L_x f\|^{w_3,w_4}_{r,1,q} \le lw''(x), \tag{2.8}$$

where $w' = w_1 + w_2$ and $w'' = w_3 + w_4$. Since $B^{w_1,w_2}_{r,1,q}(G) \subset B^{w_3,w_4}_{r,1,q}(G)$ implies that there is a constant $C > 0$ such that $\|L_x f\|^{w_3,w_4}_{r,1,q} \leq C\|L_x f\|^{w_1,w_2}_{r,1,q}$, we have

$$l^{-1}w''(x) \leq \|L_x f\|^{w_3,w_4}_{r,1,q} \leq C\|L_x f\|^{w_1,w_2}_{r,1,q} \leq Ckw'(x) \tag{2.9}$$

and $w''(x) \leq Cklw'(x)$. This shows that $w' \succ w''$. The second part is seen from [5, Proposition 2.7].

Theorem 2.13 *Let w_1, w_2, w_3 and w_4 be weight functions on G and $1 \leq r \leq \infty$, $1 \leq p, q < \infty$. Then $B^{w_1,w_2^p}_{r,p,q}(G) \subset B^{w_3,w_4^p}_{r,p,q}(G)$, if and only if there exists $c > 0$ such that $\|f\|^{w_3,w_4^p}_{r,p,q} \leq c\|f\|^{w_1,w_2^p}_{r,p,q}$ for all $f \in B^{w_1,w_2^p}_{r,p,q}(G)$.*

Proof The proof is similar to that of Theorem 2.10.

Theorem 2.14 *Let $1 \leq q \leq p < \infty$. Then for any $f \in B^{w_1,w_2^p}_{r,p,q}(G)$, the function $x \to \|L_x f\|^{w_1,w_2^p}_{r,p,q}$ is equivalent to the weight function $w' = w_1 + w_2$, i.e., there exist $c_1(f), c_2(f) > 0$ such that*

$$c_1(f)w'(x) \leq \|L_x f\|^{w_1,w_2^p}_{r,p,q} \leq c_2(f)w'(x). \tag{2.10}$$

Proof Let $f \in B^{w_1,w_2^p}_{r,p,q}(G)$. Since the function $x \to \|L_x f\|_{r,w}$ is equivalent to the weight function w, there exist $k_1(f), k_2(f) > 0$ such that

$$k_1(f)w(x) \leq \|L_x f\|_{r,w} \leq k_2(f)w(x) \tag{2.11}$$

for all $x \in G$ (see [8]). Also by Theorem 2.2 , we write

$$\|L_x f\|_{p,q,w_2^p} \leq w_2(x)\|f\|_{p,q,w_2^p} \tag{2.12}$$

and

$$\|L_x f\|_{p,q,w_2^p} \geq \|L_x f\|^*_{p,q,w_2^p} \geq \|L_x f\|^*_{p,p,w_2^p} = \|L_x f\|_{p,w_2}, \tag{2.13}$$

where $q \leq p$ (see [11]). By (2.11)–(2.13), there are $s_1(f), s_2(f) > 0$ such that

$$s_1(f)w_2(x) \leq \|L_x f\|_{p,w_2} \leq \|L_x f\|_{p,q,w_2^p} \leq w_2(x)\|f\|_{p,q,w_2^p}, \tag{2.14}$$

where $s_2(f) = \|f\|_{p,q,w_2^p}$. If we combine (2.11) with (2.14), then we have

$$s_1(f)w_2(x) + k_1(f)w_1(x) \leq \|L_x f\|^{w_1,w_2^p}_{r,p,q} \leq k_2(f)w_1(x) + s_2(f)w_2(x).$$

Therefore

$$c_1(f)w'(x) \leq \|L_x f\|^{w_1,w_2^p}_{r,p,q} \leq c_2(f)w'(x) \tag{2.15}$$

for all $x \in G$.

Theorem 2.15 *Let w_1, w_2, w_3 and w_4 be weight functions on G and $w' = w_1 + w_2$, $w'' = w_3 + w_4$. If $B^{w_1,w_2^p}_{r,p,q}(G) \subset B^{w_3,w_4^p}_{r,p,q}(G)$, then $w' \succ w''$ for $1 \leq q \leq p < \infty$. Conversely, $B^{w_1,w_2^p}_{r,p,q}(G) \subset B^{w_3,w_4^p}_{r,p,q}(G)$, if $w_1 \succ w_3$ and $w_2 \succ w_4$.*

Proof The proof is similar to that of Theorem 2.12.

3 Compact Embeddings of the Spaces $B^{w_1,w_2}_{r,p,q}(R^d)$

In this section, we will work on R^d with Lebesgue measure $\mathrm{d}x$. We denote by $C_c(R^d)$ the space of complex-valued, continuous functions with compact support.

Theorem 3.1 *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $B^{w_1,w_2}_{r,p,q}(R^d)$. *If* $(f_n)_{n\in\mathbb{N}}$ *converges to zero in* $B^{w_1,w_2}_{r,p,q}(R^d)$, *then* $(f_n)_{n\in\mathbb{N}}$ *also converges to zero in the vague topology (see* [3]*), i.e., for* $n\to\infty$,

$$\int_{R^d} f_n(x)k(x)\mathrm{d}x\to 0$$

for all $k\in C_c(R^d)$.

Proof Let $k\in C_c(R^d)$. We write

$$\Big|\int_{R^d} f_n(x)k(x)\mathrm{d}x\Big|\le \|k\|_{r'}\|f_n\|_r\le \|k\|_{r'}\|f_n\|^{w_1,w_2}_{r,p,q} \tag{3.1}$$

by Hölder's inequality where $\frac{1}{r}+\frac{1}{r'}=1$. Hence the sequence $(f_n)_{n\in\mathbb{N}}$ converges to zero in the vague topology by (3.1).

Theorem 3.2 *Let* w_1, w_2 *and* υ *be Beurling weight functions on* R^d. *If* $w_1\succ w_2$, $w_1\succ \upsilon$ *and* $\frac{\upsilon(x)}{w_1(x)}$ *does not tend to zero in* R^d *for* $x\to\infty$, *then the embedding of the space* $B^{w_1,w_2}_{r,p,q}(R^d)$ *into* $L^r_\upsilon(R^d)$ *is never compact.*

Proof First of all, since $w_1\succ\upsilon$, there is a constant $C>0$ such that $\upsilon(x)\le Cw_1(x)$. This implies that $B^{w_1,w_2}_{r,p,q}(R^d)\subset L^r_\upsilon(R^d)$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in R^d such that $t_n\to\infty$ as $n\to\infty$. Since $\frac{\upsilon(x)}{w_1(x)}$ does not tend to zero in R^d as $x\to\infty$, there exists $\delta>0$ such that $\frac{\upsilon(x)}{w_1(x)}\ge\delta>0$ for $x\to\infty$. To proof that the embedding of the space $B^{w_1,w_2}_{r,p,q}(R^d)$ into $L^r_\upsilon(R^d)$ is never compact, let us take any fixed $f\in B^{w_1,w_2}_{r,p,q}(R^d)$ and define a sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n=w_1^{-1}(t_n)L_{t_n}(f)$. Since $w_1\succ w_2$, there exists $C'>0$ such that $w_2(x)\le C'w_1(x)$ and so the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $B^{w_1,w_2}_{r,p,q}(R^d)$. Indeed, we have

$$\begin{aligned}\|f_n\|^{w_1,w_2}_{r,p,q}&=\|w_1^{-1}(t_n)L_{t_n}(f)\|^{w_1,w_2}_{r,p,q}=w_1^{-1}(t_n)\|L_{t_n}(f)\|^{w_1,w_2}_{r,p,q}\\&\le w_1^{-1}(t_n)\max\{w_1(t_n),w_2(t_n)\}\|f\|^{w_1,w_2}_{r,p,q}\\&\le w_1^{-1}(t_n)\max\{w_1(t_n),C'w_1(t_n)\}\|f\|^{w_1,w_2}_{r,p,q}\\&=\max\{1,C'\}\|f\|^{w_1,w_2}_{r,p,q}.\end{aligned}$$

Now, we will prove that there would not exist a subsequence of $(f_n)_{n\in\mathbb{N}}$ which is convergent in $L^r_\upsilon(R^d)$. The sequence in the above certainly converges in the vague topology. Indeed, for all $k\in C_c(R^d)$, we get

$$\begin{aligned}\Big|\int_{R^d} f_n(x)k(x)\mathrm{d}x\Big|&=\Big|\int_{R^d} w_1^{-1}(t_n)L_{t_n}f(x)k(x)\mathrm{d}x\Big|\\&=\frac{1}{w_1(t_n)}\Big|\int_{R^d} L_{t_n}f(x)k(x)\mathrm{d}x\Big|\\&\le\frac{1}{w_1(t_n)}\|k\|_{r'}\|f_n\|_r\le\frac{1}{w_1(t_n)}\|k\|_{r'}\|f_n\|^{w_1,w_2}_{r,p,q}.\end{aligned} \tag{3.2}$$

Since the right-hand side of (3.2) tends to zero as $n\to\infty$, we have

$$\int_{R^d} f_n(x)k(x)\mathrm{d}x\to 0.$$

Finally by Theorem 3.1, the only possible limit of (f_n) in $L^r_\upsilon(R^d)$ is zero. It is known that the function $x \to \|L_x f\|_{r,\upsilon}$ is equivalent to the weight function υ, i.e., there exist $c_1(f), c_2(f) > 0$ depending on f such that

$$c_1\upsilon(x) \le \|L_x f\|_{r,\upsilon} \le c_2\upsilon(x)$$

for all $x \in G$. Therefore

$$\|f_n\|_{r,\upsilon} = w_1^{-1}(t_n)\|L_{t_n}(f)\|_{r,\upsilon} \ge c_1 w_1^{-1}(t_n)\upsilon(x). \tag{3.3}$$

Since $\frac{\upsilon(t_n)}{w_1(t_n)} \ge \delta > 0$ for all t_n, by using (3.3) we write

$$\|f_n\|_{r,\upsilon} \ge c_1 w_1^{-1}(t_n)\upsilon(x) \ge c_1\delta.$$

This means that it is not possible to find a norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^r_\upsilon(R^d)$.

Theorem 3.3 *For $1 \le q \le p < \infty$, let w_1, w_2 and w_3 be Beurling weight functions on R^d. If*

(1) *$w_1 \succ w_2 \succ w_3$ and $\frac{w_3(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(2) *$w_1 \prec w_2$, $w_3 \prec w_2$ and $\frac{w_3(x)}{w_2(x)}$ does not tend to zero in R^d as $x \to \infty$,*

then the embedding of the space $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $L(p,q,w_3^p \mathrm{d}\mu)(R^d)$ is never compact.

Proof (1) First of all, since $w_1 \succ w_2 \succ w_3$, there are constants $C_1, C_2 > 0$ such that $w_3(x) \le C_2 w_2(x)$ and $w_2(x) \le C_1 w_1(x)$. This implies that $B^{w_1,w_2^p}_{r,p,q}(R^d) \subset L(p,q,w_3^p\mathrm{d}\mu)(R^d)$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$. Since $\frac{w_3(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$, there exists $\delta > 0$ such that $\frac{w_3(x)}{w_1(x)} \ge \delta > 0$ for $x \to \infty$. To proof that the embedding of the space $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $L(p,q,w_3^p\mathrm{d}\mu)(R^d)$ is never compact, let us take any fixed $f \in B^{w_1,w_2^p}_{r,p,q}(R^d)$ and define a sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n = w_1^{-1}(t_n)L_{t_n}(f)$. This sequence is bounded in $B^{w_1,w_2^p}_{r,p,q}(R^d)$. Indeed, we write

$$\begin{aligned}
\|f_n\|^{w_1,w_2^p}_{r,p,q} &= \|w_1^{-1}(t_n)L_{t_n}(f)\|^{w_1,w_2^p}_{r,p,q} = w_1^{-1}(t_n)\|L_{t_n}(f)\|^{w_1,w_2^p}_{r,p,q} \\
&\le w_1^{-1}(t_n)\max\{w_1(t_n), w_2(t_n)\}\|f\|^{w_1,w_2^p}_{r,p,q} \\
&\le w_1^{-1}(t_n)\max\{w_1(t_n), C_1 w_1(t_n)\}\|f\|^{w_1,w_2^p}_{r,p,q} \\
&= \max\{1, C_1\}\|f\|^{w_1,w_2^p}_{r,p,q}.
\end{aligned}$$

Now, we will prove that there would not exist a subsequence of $(f_n)_{n\in\mathbb{N}}$ which is convergent in $L(p,q,w_3^p\mathrm{d}\mu)(R^d)$. The sequence in the above certainly converges in the vague topology. Indeed, for all $k \in C_c(R^d)$, we get

$$\begin{aligned}
\Big|\int_{R^d} f_n(x)k(x)\mathrm{d}x\Big| &= \Big|\int_{R^d} w_1^{-1}(t_n)L_{t_n}f(x)k(x)\mathrm{d}x\Big| \\
&= \frac{1}{w_1(t_n)}\Big|\int_{R^d} L_{t_n}f(x)k(x)\mathrm{d}x\Big| \\
&\le \frac{1}{w_1(t_n)}\|k\|_{r'}\|f_n\|_r \le \frac{1}{w_1(t_n)}\|k\|_{r'}\|f_n\|^{w_1,w_2^p}_{r,p,q}.
\end{aligned} \tag{3.4}$$

Since the right-hand side of (3.4) tends to zero as $n \to \infty$, we have

$$\int_{R^d} f_n(x)k(x)\mathrm{d}x \to 0.$$

Finally by Theorem 3.1, the only possible limit of (f_n) in $L(p,q,w_3^p \mathrm{d}\mu)(R^d)$ is zero. By Theorem 2.14, it is known that the function $x \to \|L_x f\|_{p,q,w_3^p}$ is equivalent to the weight function w_3, i.e., there exist $c_1(f), c_2(f) > 0$ such that

$$c_1 w_3(x) \leq \|L_x f\|_{p,q,w_3^p} \leq c_2 w_3(x)$$

for all $x \in G$. Therefore

$$\|f_n\|_{p,q,w_3^p} = w_1^{-1}(t_n)\|L_{t_n}(f)\|_{p,q,w_3^p} \geq c_1 w_1^{-1}(t_n) w_3(x). \tag{3.5}$$

Since $\frac{w_3(x)}{w_1(x)} \geq \delta > 0$ for all t_n, by using (3.5) we write

$$\|f_n\|_{p,q,w_3^p} \geq c_1 w_1^{-1}(t_n) w_3(x) \geq c_1 \delta.$$

This means that it is not possible to find a norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^r_\upsilon(R^d)$.

(2) This part is similar to part (1). Briefly, the sequence will be formed as $f_n = w_2^{-1}(t_n) \cdot L_{t_n}(f)$ and the rest.

Now, we will introduce a proposition whose proof is easy.

Theorem 3.4 *Let $w_1 \approx w_2, w_3$ and w_4 be Beurling weight functions on R^d and $1 \leq q \leq p < \infty$. Then the embedding $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $B^{w_3,w_4^p}_{r,p,q}(R^d)$ is continuous if and only if $w_3 \prec w_1$, $w_4 \prec w_2$.*

Theorem 3.5 *Let w_1, w_2, w_3 and w_4 be Beurling weight functions on R^d and $1 \leq q \leq p < \infty$. If*

(1) *$w_4 \prec w_2 \prec w_1$, $w_3 \prec w_1$ and $\frac{w_3(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(2) *$w_3 \prec w_1 \prec w_2$, $w_4 \prec w_2$ and $\frac{w_3(x)}{w_2(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(3) *$w_4 \prec w_2$, $w_3 \prec w_1 \prec w_2$ and $\frac{w_4(x)}{w_2(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(4) *$w_4 \prec w_2 \prec w_1$, $w_3 \prec w_1$ and $\frac{w_4(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$,*

then the embedding of the space $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $B^{w_3,w_4^p}_{r,p,q}(R^d)$ is never compact.

Proof (1) Let us assume that $w_4 \prec w_2 \prec w_1$, $w_3 \prec w_1$. Then there are constants $C_1, C_2 > 0$ such that $w_4(x) \leq C_1 w_2(x)$ and $w_3(x) \leq C_2 w_1(x)$. By Theorem 2.15, this implies that $B^{w_1,w_2^p}_{r,p,q}(R^d) \subset B^{w_3,w_4^p}_{r,p,q}(R^d)$ and the unit function i from $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $B^{w_3,w_4^p}_{r,p,q}(R^d)$ is continuous. Now assume that $\frac{w_3(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$ and $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $B^{w_1,w_2^p}_{r,p,q}(R^d)$. If any subsequence of $(f_n)_{n\in\mathbb{N}}$ is convergent in $B^{w_3,w_4^p}_{r,p,q}(R^d)$, then this subsequence is also convergent in $L^r_{w_3}(R^d)$. However, this is not possible by Theorem 3.2, since the embedding of the space $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $L^r_{w_3}(R^d)$ is never compact.

(2) This part is similar to part (1).

(3) Let us assume that $w_4 \prec w_2$, $w_3 \prec w_1 \prec w_2$. Then there are constants $C_1, C_2 > 0$ such that $w_4(x) \leq C_1 w_2(x)$ and $w_3(x) \leq C_2 w_1(x)$. By Theorem 2.15, this implies that $B^{w_1,w_2^p}_{r,p,q}(R^d) \subset B^{w_3,w_4^p}_{r,p,q}(R^d)$ and the unit function i from $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $B^{w_3,w_4^p}_{r,p,q}(R^d)$ is continuous. Now assume that $\frac{w_4(x)}{w_2(x)}$ does not tend to zero in R^d as $x \to \infty$ and $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $B^{w_1,w_2^p}_{r,p,q}(R^d)$. If any subsequence of $(f_n)_{n\in\mathbb{N}}$ is convergent in $B^{w_3,w_4^p}_{r,p,q}(R^d)$, then this subsequence is also convergent in $L(p,q,w_4^p \mathrm{d}\mu)(R^d)$. However this is not possible by Theorem 3.3, since the embedding of the space $B^{w_1,w_2^p}_{r,p,q}(R^d)$ into $L(p,q,w_4^p \mathrm{d}\mu)(R^d)$ is never compact.

(4) This part is similar to part (3).

Theorem 3.6 *For $0 < q \leq 1$, let w_1, w_2 and w_3 be Beurling weight functions on R^d. If $w_1 \succ w_2 \succ w_3$ and $\frac{w_3(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$, then the embedding of the space $B^{w_1,w_2}_{r,1,q}(R^d)$ into $L(1, q, w_3 \mathrm{d}\mu)(R^d)$ is never compact.*

Theorem 3.7 *Let w_1, w_2, w_3 and w_4 be Beurling weight functions on R^d and $0 < q \leq 1$. If*

(1) *$w_4 \prec w_2 \prec w_1$, $w_3 \prec w_1$ and $\frac{w_3(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(2) *$w_3 \prec w_1 \prec w_2$, $w_4 \prec w_2$ and $\frac{w_3(x)}{w_2(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(3) *$w_4 \prec w_2$, $w_3 \prec w_1 \prec w_2$ and $\frac{w_4(x)}{w_2(x)}$ does not tend to zero in R^d as $x \to \infty$, or*

(4) *$w_4 \prec w_2 \prec w_1$, $w_3 \prec w_1$ and $\frac{w_4(x)}{w_1(x)}$ does not tend to zero in R^d as $x \to \infty$,*

then the embedding of the space $B^{w_1,w_2}_{r,1,q}(R^d)$ into $B^{w_3,w_4}_{r,1,q}(R^d)$ is never compact.

The proofs of Theorem 3.6 and 3.7 can be derived from Theorems 3.3–3.5.

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