

## Carleman Estimates for Parabolic Equations with Nonhomogeneous Boundary Conditions\*\*\*\*

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**Abstract** The authors prove a new Carleman estimate for general linear second order parabolic equation with nonhomogeneous boundary conditions. On the basis of this estimate, improved Carleman estimates for the Stokes system and for a system of parabolic equations with a penalty term are obtained. This system can be viewed as an approximation of the Stokes system.

**Keywords** Controllability, Parabolic equations, Carleman estimates  
**2000 MR Subject Classification** 35K20, 93B05, 93B07

### 1 Introduction

Local Carleman estimates for elliptic and parabolic equations have been known since [1, 7] and among other examples of applications they turned out to be essential for proving unique continuation properties. Global Carleman estimates for parabolic equations with homogeneous boundary conditions have been obtained by several authors in the recent years (see for example [9] for  $L^2(0, T; L^2(\Omega))$  right-hand sides and [13] for  $L^2(0, T; H^{-1}(\Omega))$  right-hand sides). They have been extensively used for obtaining observability inequalities in controllability theory and stability results for some inverse problems.

In the case of elliptic equations with nonhomogeneous boundary conditions and  $H^{-1}(\Omega)$  right-hand sides, sharp Carleman estimates have been obtained in [12] and this result turned out to be essential for obtaining estimates on the pressure in the context of controllability for the Navier-Stokes equations (see [5]).

The main object of the present article is to obtain a similar result of global Carleman estimates for general parabolic equations with nonhomogeneous boundary conditions and right-hand sides in  $L^2(0, T; H^{-1}(\Omega))$ . To this aim, after localization and a change of coordinates, we use a factorization of the operator and successive estimates for first order pseudodifferential operators in order to obtain the Carleman estimate. The article is organized as follows. The main result together with its complete proof is precisely given in Section 2. In Section 3, this result is applied (using also the estimate for elliptic equations) to the Stokes operator. In

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Section 4 it is applied to a compressible Stokes operator where the incompressibility condition is approximated by penalization. In Appendix, we give some useful technical results on calculus for pseudodifferential operators depending on a parameter.

First of all, we need some notations which are introduced below.

Set  $B(\tilde{x}_0, \delta) = \{\tilde{x} \mid \tilde{x} \in \mathbb{R}^m, |\tilde{x} - \tilde{x}_0| \leq \delta\}$ .  $\nu$  is the outward unit normal vector to  $\partial\Omega = \Gamma$ , and  $\frac{\partial}{\partial\nu_A} = \sum_{i,j=1}^n a_{ij}\nu_i \frac{\partial}{\partial x_j}$ .  $Q = (0, T) \times \Omega$ ,  $\Sigma = (0, T) \times \partial\Omega$ ,  $Q_\omega = (0, T) \times \omega$ , where  $\omega$  is a subdomain of  $\Omega$ .  $\mathcal{L}(X, Y)$  is the space of linear continuous operators acting from a Banach space  $X$  into a Banach space  $Y$ , and  $[L, A] = LA - AL$  is the commutator of operators  $L$  and  $A$ . Let

$$\begin{aligned} D &= \left( \frac{1}{i} \frac{\partial}{\partial x_0}, \dots, \frac{1}{i} \frac{\partial}{\partial x_n} \right), & D' &= \left( \frac{1}{i} \frac{\partial}{\partial x_0}, \dots, \frac{1}{i} \frac{\partial}{\partial x_{n-1}} \right), \\ \tilde{D} &= \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \dots, \frac{1}{i} \frac{\partial}{\partial x_n} \right), & D_0 &= \frac{1}{i} \frac{\partial}{\partial x_0}, \\ \nabla &= \left( \frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n} \right), & \nabla' &= \left( \frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_{n-1}} \right), & \nabla_{\tilde{x}} &= \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \\ \xi' &= (\xi_0, \dots, \xi_{n-1}), & \tilde{\xi} &= (\xi_1, \dots, \xi_n), & \xi &= (\xi_0, \dots, \xi_n), \\ \tilde{x} &= (x_1, \dots, x_n), & x &= (x_0, \dots, x_n), & x' &= (x_0, \dots, x_{n-1}). \end{aligned}$$

By  $\hat{u}(\xi')$  we denote the Fourier transform of the function  $u(x')$ :

$$\hat{u}(\xi') = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x') e^{-i\langle x', \xi' \rangle} dx'.$$

We use the following functional spaces:

$$\begin{aligned} H^{1,2}(Q) &= \left\{ y \mid \frac{\partial y}{\partial x_0}, \frac{\partial^2 y}{\partial x_k \partial x_j}, \frac{\partial y}{\partial x_k}, y \in L^2(Q), \forall k, j \in \{1, \dots, n\} \right\}, \\ W(Q) &= \left\{ y \mid y, \frac{\partial y}{\partial x_i} \in L^2(Q), \forall i \in \{1, \dots, n\}, \frac{\partial y}{\partial x_0} \in L^2(0, T; H^{-1}(\Omega)) \right\}, \\ H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n) &= \left\{ y(x_1, \dots, x_n) \mid \|y\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left( 1 + |\xi_1|^{\frac{1}{2}} + \sum_{i=2}^n |\xi_i| \right) |\hat{y}|^2 d\xi < +\infty \right\}. \end{aligned}$$

The space  $H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)$  is the space  $H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)$  equipped with the norm

$$\|y\|_{H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)} = \left( \|y\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}^2 + |s| \|y\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}.$$

## 2 Statement of the Result

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^n$  of class  $C^2$  and let  $\Gamma$  be the boundary of  $\Omega$ . We consider a solution  $y \in W(Q)$  of the following linear second order parabolic equation:

$$\begin{aligned} L(x, D)y &= \frac{\partial y}{\partial x_0} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial y}{\partial x_j} \right) + \sum_{j=1}^n b_j(x) \frac{\partial y}{\partial x_j} + \sum_{i=1}^n \frac{\partial}{\partial x_i} (c_i(x)y) + d(x)y \\ &= f + \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}, \quad \text{in } (0, T) \times \Omega, \end{aligned} \tag{2.1}$$

$$y = g, \quad \text{on } (0, T) \times \Gamma, \tag{2.2}$$

where

$$a_{ij} \in C^2(\overline{Q}), \quad b_j, c_i, d \in L^\infty(Q) \quad \text{for } i, j \in \{1, \dots, n\}, \tag{2.3}$$

$$a_{ij} = a_{ji} \quad \text{for } i, j \in \{1, \dots, n\}, \tag{2.4}$$

and the coefficients  $a_{ij}$  satisfy the standard ellipticity condition

$$\exists \beta > 0, \quad \forall \eta \in \mathbb{R}^n, \quad \forall x \in \overline{Q}, \quad \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \geq \beta|\eta|^2. \tag{2.5}$$

On the other hand, we assume that

$$f \in L^2(Q), \quad f_j \in L^2(Q), \quad \forall j = 1, \dots, n, \tag{2.6}$$

and  $g$  is the boundary value of a function in  $W(Q)$ . For the sake of simplicity, we will assume

$$g \in H^{\frac{1}{4}, \frac{1}{2}}((0, T) \times \Gamma) = H^{\frac{1}{4}, \frac{1}{2}}(\Sigma). \tag{2.7}$$

Our goal is to obtain a sharp global Carleman inequality for solutions of (2.1)–(2.2). In order to formulate our main result, we first have to introduce a suitable weight function.

**Lemma 2.1** *Let  $\omega$  be an arbitrary nonempty open set such that  $\omega \subset \Omega$ . Then there exists a function  $\psi \in C^2(\overline{\Omega})$  such that*

$$\psi = 0, \quad \text{on } \Gamma, \tag{2.8}$$

$$\psi(\tilde{x}) > 0, \quad \forall \tilde{x} \in \Omega, \tag{2.9}$$

$$|\nabla\psi(\tilde{x})| > 0, \quad \forall \tilde{x} \in \overline{\Omega \setminus \omega}. \tag{2.10}$$

**Proof** Let us consider a function  $\theta \in C^2(\mathbb{R}^n)$  such that

$$\Omega = \{\tilde{x} \mid \theta(\tilde{x}) < 0\}, \quad |\nabla\theta(\tilde{x})| \neq 0, \quad \forall \tilde{x} \in \partial\Omega. \tag{2.11}$$

By the theorem on density of Morse functions (see [2]), there exists a sequence of Morse functions  $\{\theta_k(\tilde{x})\}_{k=1}^\infty$  such that

$$\theta_k \rightarrow \theta, \quad \text{in } C^2(\overline{\Omega}), \quad \text{as } k \rightarrow +\infty. \tag{2.12}$$

We construct a Morse function  $\mu \in C^2(\overline{\Omega})$  such that

$$\mu|_{\partial\Omega} = 0, \quad |\nabla\mu(\tilde{x})| > 0, \quad \forall \tilde{x} \in \partial\Omega. \tag{2.13}$$

We denote by  $\mathcal{B} = \{\tilde{x} \in \mathbb{R}^n \mid \nabla\theta(\tilde{x}) = 0\}$  the set of critical points of functions  $\theta$ . Since  $|\nabla\theta||_{\partial\Omega} > 0$ , there exists an open set  $\Theta \subset \mathbb{R}^n$  such that

$$\overline{\Theta} \cap \mathcal{B} = \emptyset, \quad \partial\Omega \subset \Theta. \tag{2.14}$$

Let  $e \in C_0^\infty(\Theta)$ ,  $e|_{\partial\Omega} \equiv 1$ . We set  $\mu_k(\tilde{x}) = \theta_k(\tilde{x}) + e(\theta - \theta_k)(\tilde{x})$ . It is obvious that

$$\mu_k|_{\partial\Omega} = 0. \tag{2.15}$$

By the definition of function  $e(\tilde{x})$ , we have

$$\nabla\mu_k(\tilde{x}) = \nabla\theta_k(\tilde{x}), \quad \forall \tilde{x} \in \overline{\Omega \setminus \Theta}. \tag{2.16}$$

For all  $\tilde{x}$  from the set  $\Theta \cap \Omega$ , there holds

$$\nabla \mu_k(\tilde{x}) = \nabla \theta_k(\tilde{x}) + e(\tilde{x})(\nabla \theta - \nabla \theta_k)(\tilde{x}) + \nabla e(\tilde{x})(\theta - \theta_k)(\tilde{x}). \tag{2.17}$$

By (2.12) and (2.17), for all  $\epsilon > 0$ , there exists  $k_0(\epsilon)$  such that on the set  $\Theta \cap \Omega$ , we have

$$|\nabla \mu_k(\tilde{x})| \geq |\nabla \theta_k(\tilde{x})| - \|e\|_{C^1(\bar{\Omega})} |(\nabla \theta - \nabla \theta_k)(\tilde{x})| - \|e\|_{C^1(\bar{\Omega})} |(\theta - \theta_k)(\tilde{x})| \geq |\nabla \theta_k(\tilde{x})| - \epsilon, \tag{2.18}$$

where  $k > k_0$ .

From (2.12), (2.14), (2.16) and the last inequalities, there exist  $\epsilon > 0$  and  $\widehat{k}$  such that

$$|\nabla \mu_{\widehat{k}}(\tilde{x})| > 0, \quad \text{in } \Theta \cap \Omega. \tag{2.19}$$

Set  $\mu(\tilde{x}) = \mu_{\widehat{k}}(\tilde{x})$ . By (2.15), (2.16) and (2.19), the Morse function  $\mu_{\widehat{k}}(\tilde{x})$  satisfies (2.13).

We denote by  $\mathfrak{M}$  the set of critical points of function  $\mu$ :

$$\mathfrak{M} = \{\widehat{x}_i \in \mathbb{R}^n \mid i = 1, \dots, r\}.$$

Consider the sequence of functions  $\{l_i\}_{i=1}^r \subset C^\infty([0, 1]; \mathbb{R}^n)$ , such that

$$l_i(t) \in \Omega, \quad \forall t \in [0, 1], \quad l_i(t_1) \neq l_i(t_2), \quad \forall t_1, t_2 \in [0, 1], \quad t_1 \neq t_2, \quad i = 1, \dots, r, \tag{2.20}$$

$$l_i(1) = \widehat{x}_i, \quad l_i(0) \in \omega_0, \quad i = 1, \dots, r, \tag{2.21}$$

$$l_i(t_1) \neq l_j(t_2), \quad \forall i \neq j, \quad \forall t_1, t_2 \in [0, 1], \tag{2.22}$$

where  $\omega_0$  is a nonempty open set such that  $\bar{\omega}_0 \subset \omega$ . By (2.20)–(2.22), there exists a sequence of functions  $\{w^{(i)}\}_{i=1}^r \subset C^2(\mathbb{R}^n, \mathbb{R}^n)$  and  $\{e_i\}_{i=1}^r \subset C_0^\infty(\Omega)$  such that

$$\frac{dl_i(t)}{dt} = w^{(i)}(l_i(t)), \quad \forall t \in [0, 1], \quad i = 1, \dots, r, \tag{2.23}$$

$$\text{supp } e_i \subset \Omega, \quad i = 1, \dots, r, \tag{2.24}$$

$$\text{supp } e_i \cap \text{supp } e_j = \emptyset, \quad \forall i \neq j, \tag{2.25}$$

$$e_i(l_i(t)) = 1, \quad \forall t \in [0, 1], \quad i = 1, \dots, r. \tag{2.26}$$

We set

$$V^{(i)}(\tilde{x}) = e_i(\tilde{x})w^{(i)}(\tilde{x}).$$

Consider the system of the ordinary differential equations

$$\frac{d\tilde{x}}{dt} = V^{(i)}(\tilde{x}), \quad \tilde{x}(0) = \tilde{x}_0. \tag{2.27}$$

We denote by  $S_t^{(i)} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  the operator such that  $S_t^{(i)}(\tilde{x}_0) = \tilde{x}(t)$ , where  $\tilde{x}(t)$  is the solution of problem (2.27).

By (2.21), (2.23) and (2.26), we have

$$S_1^{(i)}(l_i(0)) = \widehat{x}_i, \quad i = 1, \dots, r.$$

Set

$$\psi(\tilde{x}) = \mu(g_r(\tilde{x})), \quad g_r(\tilde{x}) = S_1^{(1)} \circ S_1^{(2)} \circ \dots \circ S_1^{(r)}(\tilde{x}). \tag{2.28}$$

By (2.24), there exists a domain  $\mathfrak{S} \subset \mathbb{R}^n$  such that  $\partial\Omega \subset \mathfrak{S}$  and

$$S_1^{(i)}(\tilde{x}) = \tilde{x}, \quad \forall \tilde{x} \in \mathfrak{S}, \quad i = 1, \dots, r. \tag{2.29}$$

By (2.29), the mappings  $S_1^{(i)}$  are diffeomorphisms on the domain  $\Omega$ . Therefore  $g_r$  is a diffeomorphism on the domain  $\Omega$ . By (2.29),  $\psi(\tilde{x}) = \mu(\tilde{x}), \forall \tilde{x} \in \mathfrak{S}$ . Hence

$$\psi|_{\partial\Omega} = 0. \tag{2.30}$$

We denote by  $\Psi$  the set of critical points of function  $\psi$ . Since the mapping  $g_r : \Omega \rightarrow \Omega$  is a diffeomorphism, we have

$$\Psi = \{\tilde{x} \in \Omega \mid g_r(\tilde{x}) \in \mathfrak{M}\}. \tag{2.31}$$

By (2.25) and (2.29), we have

$$g_r(l_i(0)) = \widehat{x}_i, \quad i = 1, \dots, r. \tag{2.32}$$

It follows from (2.31) and (2.32) that  $\Psi \subset \omega_0$ . The proof of the lemma is complete.

We say that the function

$$L_2(x, \xi) = i\xi_0 + \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k$$

is the principal symbol of the operator  $L(x, D)$ .

For functions  $f(x, \xi)$  and  $g(x, \xi)$ , we introduce the Poisson bracket

$$\{f, g\} = \sum_{j=0}^n \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \xi_j} \frac{\partial f}{\partial x_j} \right).$$

Henceforth,  $\text{Im}$  and  $\text{Re}$  denote the imaginary and the real parts of a complex number, respectively.

**Definition 2.1** *We say that the function  $\alpha(x)$  is pseudoconvex with respect to the symbol  $L_2(x, \xi)$  if there exists a constant  $\widehat{C} > 0$  such that*

$$\frac{\text{Im} \{ \overline{L_2(x, \xi_0, \tilde{\zeta})}, L_2(x, \xi_0, \tilde{\zeta}) \}}{|s|} > 0, \quad \forall (x, \xi, s) \in \overline{Q \setminus Q_\omega} \times \mathfrak{S},$$

where

$$\begin{aligned} \mathfrak{S} &= \{(x, \xi, s) \mid x \in \overline{Q \setminus Q_\omega}, M(\xi, s) = 1, L_2(x, \xi_0, \tilde{\zeta}) = 0\}, \\ \tilde{\zeta} &= \left( \xi_1 + i|s| \frac{\partial \alpha}{\partial x_1}, \dots, \xi_n + i|s| \frac{\partial \alpha}{\partial x_n} \right), \\ M(\xi, s) &= \left( \xi_0^2 + \sum_{i=1}^n \xi_i^4 + s^4 \right)^{\frac{1}{4}}. \end{aligned}$$

Now, using this function  $\psi$ , we construct two weight functions

$$\varphi(x) = \frac{e^{\lambda\psi(\tilde{x})}}{\ell^\kappa(x_0)}, \quad \alpha(x) = \frac{e^{\lambda\psi(\tilde{x})} - e^{2\lambda\|\psi\|_{C^0(\overline{\Omega})}}}{\ell^\kappa(x_0)}, \tag{2.33}$$

where  $\kappa \geq 2, \lambda \in \mathbb{R}, \lambda \geq 1$  will be chosen later on large enough, and

$$\begin{aligned} \ell &\in C^\infty[0, T], \quad \ell(x_0) > 0, \quad \forall x_0 \in (0, T), \\ \ell(x_0) &= x_0, \quad \forall x_0 \in \left[0, \frac{T}{4}\right], \quad \ell(x_0) = T - x_0, \quad x_0 \in \left[\frac{3T}{4}, T\right]. \end{aligned}$$

**Proposition 2.1** *Let the function  $\alpha$  be given by (2.33). Then there exist  $\widehat{\lambda}$  and  $\widehat{C}$  such that*

$$\text{Im} \{ \overline{L}_2(x, \xi_0, \overline{\zeta}), L_2(x, \xi_0, \tilde{\zeta}) \} \geq \widehat{C} |s| \lambda^4 \frac{e^{\lambda\psi(\bar{x})}}{\ell^\kappa(x_0)} M^2 \left( \xi, s \frac{e^{\lambda\psi(\bar{x})}}{\ell^\kappa(x_0)} \right) \tag{2.34}$$

for all  $(x, \xi, s) \in \mathcal{S}$  and  $\lambda \geq \widehat{\lambda}$ . Here  $\widehat{\lambda}$  is independent of  $s$  and  $\widehat{C}$  is independent of  $s$  and  $\lambda$ .

**Proof** We introduce the following notations:

$$p^{(j)}(x, \xi) = \partial_{\xi_j} p(x, \xi), \quad p^{(j,i)}(x, \xi) = \frac{\partial^2 p(x, \xi)}{\partial \xi_i \partial \xi_j}, \quad p_{(j)}(x, \xi) = \frac{\partial p(x, \xi)}{\partial x_j},$$

$$\tilde{\nabla} \alpha = \left( 0, \frac{\partial \alpha}{\partial x_1}, \dots, \frac{\partial \alpha}{\partial x_n} \right), \quad a(x, \tilde{\xi}, \tilde{\eta}) = \sum_{i,j=1}^n a_{ij} \xi_i \eta_j.$$

After short computations, we obtain

$$\frac{\partial}{\partial \xi_0} L_2(x, \xi_0, \tilde{\zeta}) = i, \quad \frac{\partial}{\partial \xi_0} \overline{L}_2(x, \xi_0, \tilde{\zeta}) = -i,$$

$$\frac{\partial}{\partial x_m} L_2(x, \xi_0, \eta) = L_{2,(m)}(x, \xi_0, \tilde{\zeta}) + i |s| \sum_{k=1}^n L_2^{(k)}(x, \xi_0, \tilde{\zeta}) \frac{\partial^2 \alpha}{\partial x_k \partial x_m},$$

$$\frac{\partial}{\partial x_m} \overline{L}_2(x, \xi_0, \overline{\zeta}) = L_{2,(m)}(x, \xi_0, \overline{\zeta}) - i |s| \sum_{k=1}^n L_2^{(k)}(x, \xi_0, \overline{\zeta}) \frac{\partial^2 \alpha}{\partial x_k \partial x_m}.$$

Then

$$\text{Im} \{ \overline{L}_2(x, \xi_0, \overline{\zeta}), L_2(x, \xi_0, \tilde{\zeta}) \}$$

$$= \text{Im} \left( \sum_{k=0}^n \overline{L}_2^{(k)}(x, \xi_0, \overline{\zeta}) L_{2,(k)}(x, \xi_0, \tilde{\zeta}) - \overline{L}_{2,(k)}(x, \xi_0, \overline{\zeta}) L_2^{(k)}(x, \xi_0, \tilde{\zeta}) \right).$$

Simple computations provide the following formulas:

$$\text{Im} \left( \frac{\partial}{\partial \xi_0} \overline{L}_2(x, \xi_0, \overline{\zeta}) \frac{\partial}{\partial x_0} L_2(x, \xi_0, \tilde{\zeta}) - \frac{\partial}{\partial x_0} \overline{L}_2(x, \xi_0, \overline{\zeta}) \frac{\partial}{\partial \xi_0} L_2(x, \xi_0, \tilde{\zeta}) \right)$$

$$= \text{Im} \left( -i \left( L_{2,(0)}(x, \xi_0, \tilde{\zeta}) + i |s| \sum_{m=1}^n L_2^{(m)}(x, \xi_0, \tilde{\zeta}) \frac{\partial^2 \alpha}{\partial x_m \partial x_0} \right) \right.$$

$$\left. - i \left( L_{2,(0)}(x, \xi_0, \overline{\zeta}) - i |s| \sum_{m=1}^n L_2^{(m)}(x, \xi_0, \overline{\zeta}) \frac{\partial^2 \alpha}{\partial x_m \partial x_0} \right) \right)$$

$$= -2L_{2,(0)}(x, \xi) + 2s^2 \sum_{m=1}^n L_2^{(m)}(x, \tilde{\nabla} \alpha) \frac{\partial^2 \alpha}{\partial x_m \partial x_0} + 2s^2 a_{x_0}(x, \nabla_{\bar{x}} \alpha, \nabla_{\bar{x}} \alpha),$$

where  $a_{x_0}(x, \tilde{\eta}, \tilde{\eta}) = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_0} \eta_i \eta_j$  and

$$\text{Im} \left( \frac{\partial}{\partial \xi_k} \overline{L}_2(x, \xi_0, \overline{\zeta}) \frac{\partial}{\partial x_k} L_2(x, \xi_0, \tilde{\zeta}) - \frac{\partial}{\partial x_k} \overline{L}_2(x, \xi_0, \overline{\zeta}) \frac{\partial}{\partial \xi_k} L_2(x, \xi_0, \tilde{\zeta}) \right)$$

$$= \text{Im} \left( \overline{L}_2^{(k)}(x, \xi_0, \overline{\zeta}) \left( L_{2,(k)}(x, \xi_0, \tilde{\zeta}) + i |s| \sum_{m=1}^n L_2^{(m)}(x, \xi_0, \tilde{\zeta}) \frac{\partial^2 \alpha}{\partial x_k \partial x_m} \right) \right.$$

$$\left. - L_2^{(k)}(x, \xi_0, \tilde{\zeta}) \left( L_{2,(k)}(x, \xi_0, \overline{\zeta}) - i |s| \sum_{m=1}^n L_2^{(m)}(x, \xi_0, \overline{\zeta}) \frac{\partial^2 \alpha}{\partial x_k \partial x_m} \right) \right)$$

$$\begin{aligned}
&= -L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)(L_{2,(k)}(x, \xi) - L_{2,(k)}(x, |s|\tilde{\nabla}\alpha)) + L_2^{(k)}(x, \xi)\text{Im} L_{2,(k)}(x, \xi_0, \tilde{\zeta}) \\
&\quad + |s|L_2^{(k)}(x, \xi) \sum_{m=1}^n L_2^{(m)}(x, \xi) \frac{\partial^2 \alpha}{\partial x_k \partial x_m} + |s|L_2^{(k)}(x, |s|\tilde{\nabla}\alpha) \sum_{m=1}^n L_2^{(m)}(x, |s|\tilde{\nabla}\alpha) \frac{\partial^2 \alpha}{\partial x_k \partial x_m} \\
&\quad - L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)(L_{2,(k)}(x, \xi) - L_{2,(k)}(x, |s|\tilde{\nabla}\alpha)) + L_2^{(k)}(x, \xi)\text{Im} L_{2,(k)}(x, \xi_0, \tilde{\zeta}) \\
&\quad + |s|L_2^{(k)}(x, \xi) \sum_{m=1}^n L_2^{(m)}(x, \xi) \frac{\partial^2 \alpha}{\partial x_k \partial x_m} + |s|L_2^{(k)}(x, |s|\tilde{\nabla}\alpha) \sum_{m=1}^n L_2^{(m)}(x, |s|\tilde{\nabla}\alpha) \frac{\partial^2 \alpha}{\partial x_k \partial x_m}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\text{Im} \{ \bar{L}_2(x, \xi_0, \bar{\zeta}), L_2(x, \xi_0, \tilde{\zeta}) \} \\
&= \frac{1}{2}\text{Im} \left( \sum_{k=0}^n \frac{\partial}{\partial \xi_k} \bar{L}_2(x, \xi_0, \bar{\zeta}) \frac{\partial}{\partial x_k} L_{2,(k)}(x, \xi_0, \tilde{\zeta}) - \frac{\partial}{\partial x_k} \bar{L}_2(x, \xi_0, \bar{\zeta}) \frac{\partial}{\partial \xi_k} L_2(x, \xi_0, \tilde{\zeta}) \right) \\
&= -L_{2,(0)}(x, \xi) + s^2 \sum_{k=1}^n L_2^{(k)}(x, \tilde{\nabla}\alpha) \frac{\partial^2 \alpha}{\partial x_k \partial x_0} + 2s^2 a_{x_0}(x, \nabla_{\bar{x}}\alpha, \nabla_{\bar{x}}\alpha) \\
&\quad - L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)(L_{2,(k)}(x, \xi) - L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)) + L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)\text{Im} L_{2,(k)}(x, \xi_0, \tilde{\zeta}) \\
&\quad + \sum_{m,k=1}^n (|s|L_2^{(k)}(x, \xi)L_2^{(m)}(x, \xi) + |s|L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)L_2^{(m)}(x, |s|\tilde{\nabla}\alpha)) \frac{\partial^2 \alpha}{\partial x_k \partial x_m}.
\end{aligned}$$

Observing that  $\frac{\partial^2 \alpha}{\partial x_k \partial x_m} = (\lambda^2 \frac{\partial \psi}{\partial x_k} \frac{\partial \psi}{\partial x_m} + \lambda \frac{\partial^2 \psi}{\partial x_k \partial x_m}) \frac{e^{\lambda \psi}}{\ell^\kappa}$ , we have

$$\begin{aligned}
\text{I} &= \sum_{m,k=1}^n (|s|L_2^{(k)}(x, \xi)L_2^{(m)}(x, \xi) + |s|L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)L_2^{(m)}(x, |s|\tilde{\nabla}\alpha)) \frac{\partial^2 \alpha}{\partial x_k \partial x_m} \\
&= \lambda^2 |s| \left( a(x, \tilde{\xi}, \nabla_{\bar{x}}\psi)^2 + s^2 \frac{e^{2\lambda\psi(x)}}{\ell^{2\kappa}} a(x, \nabla_{\bar{x}}\psi, \nabla_{\bar{x}}\psi)^2 \right) \frac{e^{\lambda\psi}}{\ell^\kappa} \\
&\quad + \sum_{m,k=1}^n (|s|L_2^{(k)}(x, \xi)L_2^{(m)}(x, \xi) + |s|L_2^{(k)}(x, |s|\tilde{\nabla}\alpha)L_2^{(m)}(x, |s|\tilde{\nabla}\alpha)) \lambda \psi_{x_k x_m} \frac{e^{\lambda\psi}}{\ell^\kappa}.
\end{aligned}$$

Since  $(x, \xi, s) \in \mathcal{S}$ , the following inequality holds:

$$a(x, \tilde{\xi}, \nabla_{\bar{x}}\alpha)^2 = s^2 a(x, \nabla_{\bar{x}}\alpha, \nabla_{\bar{x}}\alpha)^2 \geq \widehat{C} \left| \left( \tilde{\xi}, s \frac{e^{\lambda\psi}}{\ell^\kappa} \right) \right|^2.$$

Taking  $\widehat{\lambda}$  sufficiently large, for all  $\lambda \geq \widehat{\lambda}$ , we have

$$\text{I} \geq \frac{\lambda^4}{2} \widehat{C} |s| \frac{e^{\lambda\psi}}{\ell^\kappa} \left| \left( \tilde{\xi}, s \frac{e^{\lambda\psi}}{\ell^\kappa} \right) \right|^2, \quad \forall (x, \xi, s) \in \mathcal{S}, \quad (2.35)$$

where  $\widehat{C}$  is independent of  $(\lambda, x, \xi, s)$ .

Finally, observing

$$|\xi_0| \leq |a(x, \tilde{\xi}, |s|\nabla_{\bar{x}}\alpha)|, \quad \forall (x, \xi, s) \in \mathcal{S},$$

from (2.35) we find

$$\text{I} \geq \frac{\lambda^4}{2} \widehat{C} \frac{e^{\lambda\psi(\bar{x})}}{\ell^\kappa(x_0)} |s| M^2 \left( \xi, s \frac{e^{\lambda\psi(\bar{x})}}{\ell^\kappa(x_0)} \right), \quad \forall (x, \xi, s) \in \mathcal{S}, \quad (2.36)$$

where  $\widehat{C}$  is independent of  $(\lambda, x, \xi, s)$ . On the other hand,

$$\begin{aligned} & \left| -L_{2,(0)}(x, \xi) + 2s^2 a_{x_0}(x, \nabla_{\bar{x}}\alpha, \nabla_{\bar{x}}\alpha) + s^2 \sum_{k=1}^n L_2^{(k)}(x, \widetilde{\nabla}\alpha) \frac{\partial^2 \alpha}{\partial x_k \partial x_0} \right. \\ & \quad \left. - L_2^{(m)}(x, |s| \widetilde{\nabla}\alpha)(L_{2,(m)}(x, \xi) - L_{2,(m)}(x, |s| \widetilde{\nabla}\alpha)) + L_2^{(m)}(x, |s| \widetilde{\nabla}\alpha) \operatorname{Im} L_{2,(m)}(x, \xi_0, \widetilde{\zeta}) \right| \\ & \leq C |s| \lambda^2 \frac{e^{\lambda\psi}}{\ell^\kappa} M^2 \left( \xi, s \frac{e^{\lambda\psi}}{\ell^\kappa} \right). \end{aligned} \tag{2.37}$$

Inequalities (2.37) and (2.36) imply (2.34).

We formulate our main Carleman estimate for the parabolic equation.

**Theorem 2.1** *Assume that (2.3)–(2.6) hold. Let  $y \in W(Q)$  be a solution of (2.1)–(2.2). Then there exists a constant  $\widehat{\lambda}$ , such that for any  $\lambda > \widehat{\lambda}$  there exist constants  $C > 0$  independent of  $s$  and  $s_0(\lambda)$ , such that*

$$\begin{aligned} & \frac{1}{s} \int_Q \frac{1}{\varphi} \sum_{j=1}^n \left| \frac{\partial y}{\partial x_j} \right|^2 e^{2s\alpha} dx + s \int_Q \varphi |y|^2 e^{2s\alpha} dx \\ & \leq C \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} g e^{s\alpha}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} g e^{s\alpha}\|_{L^2(\Sigma)}^2 + \frac{1}{s^2} \int_Q \frac{|f|^2}{\varphi^2} e^{2s\alpha} dx \right. \\ & \quad \left. + \sum_{j=1}^n \int_Q |f_j|^2 e^{2s\alpha} dx + \int_{Q_w} s \varphi |y|^2 e^{2s\alpha} dx \right), \quad \forall s \geq s_0 > 0. \end{aligned} \tag{2.38}$$

**Remark 2.1** (1) By a density argument, it suffices to prove the result when the solution  $y$  is assumed to be more regular, namely  $y \in H^{1,2}(Q)$  and the right-hand side has a compact support. Actually, there exists a sequence of  $\{f^k, f_1^k, \dots, f_n^k, g^k\} \in (C_0^\infty(Q))^{n+1} \times C_0^\infty(\Sigma)$  converging to  $\{f, f_1, \dots, f_n, g\}$  in  $L^2(Q)^{n+1} \times H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)$ , such that the corresponding solution  $y^k$  to problem (2.1)–(2.2) with right-hand side  $f^k + \sum_{j=1}^n \frac{\partial f_j^k}{\partial x_j}$  and boundary condition  $g^k$  satisfies  $y^k \in H^{1,2}(Q)$  and

$$y^k \rightarrow y, \quad \text{in } L^2(0, T; H^1(\Omega)).$$

Hence, it suffices to prove the estimate (2.38) for solutions  $y \in H^{1,2}(Q)$  and the terms of the right-hand side which have compact supports in  $Q$  and  $\Sigma$ .

(2) Without loss of generality, it is sufficient to consider the case where  $b_j = 0$ ,  $c_i = 0$  and  $d = 0$  as the first and zero order terms in (2.1) can be added to the right-hand side and the corresponding terms in (2.38) can be absorbed by the terms on the left-hand side by choosing  $\widehat{s}$  and  $\widehat{\lambda}$  large enough.

The proof of Theorem 2.1 is divided into several steps and will be the content of the next subsections.

**2.1 Localization in space and time**

For every  $\delta > 0$ , we can consider a covering of  $\overline{Q} = [0, T] \times \overline{\Omega}$  as follows:

$$\overline{Q} \subset \overline{Q}_0 \cup \left( \bigcup_{k=1}^I B(\widehat{x}_k, \delta) \right), \tag{2.39}$$



where  $Q_0 = (0, T) \times \Omega_0$ ,  $\bar{\Omega}_0 \subset \Omega$ ,  $\hat{x}_k \in (0, T) \times \partial\Omega$ .

Let  $(e_k)_{k=0}^I$  be a corresponding partition of unity, i.e.,

$$\begin{aligned} e_0 &\in C_0^\infty(Q_0), \quad e_k \in C_0^\infty(B(\hat{x}_k, \delta)), \quad k = 1, \dots, I, \\ e_k(x) &\geq 0, \quad k = 1, \dots, I, \quad \sum_{k=0}^I e_k(x) = 1, \quad \forall x \in \bar{Q}. \end{aligned}$$

We now define

$$y_k(x) = y(x)e_k(x), \quad k = 0, \dots, I.$$

Then if  $L(x, D)y = \frac{\partial y}{\partial x_0} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial y}{\partial x_j})$  (recall that from the previous remark this corresponds to the general case), we have

$$L(x, D)y_k = [L(x, D), e_k]y + e_k f + \sum_{j=1}^n \frac{\partial(e_k f_j)}{\partial x_j} - \sum_{j=1}^n f_j \frac{\partial e_k}{\partial x_j}, \quad \text{in } Q, \quad (2.40)$$

$$y_k = g e_k, \quad \text{on } (0, T) \times \Gamma \quad (2.41)$$

for each  $k = 0, \dots, I$ , and

$$\text{supp } y_0 \subset Q_0, \quad \text{supp } y_k \subset B(\hat{x}_k, \delta), \quad k = 1, \dots, I. \quad (2.42)$$

Notice that the commutator  $[L, e_k]$  is a first order operator and that  $e_k f_j$  and  $f_j \frac{\partial e_k}{\partial x_j}$  have compact supports in  $B(\hat{x}_k, \delta)$  for all  $k = 1, \dots, I$ .

Assume that the assertion of Theorem 2.1 is true with the additional assumption that

$$\text{supp } y \subset Q_0 \quad \text{or} \quad \text{supp } y \subset B(\hat{x}, \delta), \quad \hat{x} \in (0, T) \times \partial\Omega.$$

Then, as  $y = \sum_{k=0}^I y_k$ , we have (the letter  $C$  denotes various constants independent of  $s$ )

$$\begin{aligned} &\int_Q \left( \frac{1}{s\varphi} \sum_{j=1}^n \left| \frac{\partial y}{\partial x_j} \right|^2 + s\varphi |y|^2 \right) e^{2s\alpha} dx \leq C \sum_{k=0}^I \int_Q \left( \frac{1}{s\varphi} \sum_{j=1}^n \left| \frac{\partial y_k}{\partial x_j} \right|^2 + s\varphi |y_k|^2 \right) e^{2s\alpha} dx \\ &\leq C \sum_{k=0}^I \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} g e_k e^{s\alpha}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} g e_k e^{s\alpha}\|_{L^2(\Sigma)}^2 + \int_Q \frac{|f e_k|^2}{s^2 \varphi^2} e^{2s\alpha} dx \right. \\ &\quad \left. + \sum_{j=1}^n \int_Q |f_j e_k|^2 e^{2s\alpha} dx + \frac{1}{s^2} \sum_{j=1}^n \int_Q \frac{|f_j|^2}{\varphi^2} e^{2s\alpha} dx + \int_Q |y|^2 e^{2s\alpha} dx + \int_{Q_\omega} s\varphi |y_k|^2 e^{2s\alpha} dx \right) \\ &\leq C \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} g e^{s\alpha}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} g e^{s\alpha}\|_{L^2(\Sigma)}^2 + \frac{1}{s^2} \int_Q \frac{|f|^2}{\varphi^2} e^{2s\alpha} dx \right. \\ &\quad \left. + \sum_{j=1}^n \int_Q |f_j|^2 e^{2s\alpha} dx + \int_Q |y|^2 e^{2s\alpha} dx + \int_{Q_\omega} s\varphi |y|^2 e^{2s\alpha} dx \right). \quad (2.43) \end{aligned}$$

Now taking  $\hat{s}$  sufficiently large, we obtain (2.38) for  $s \geq \hat{s}$ . Therefore, it is enough to prove Theorem 2.1 in two cases:

- (i)  $\text{supp } y \subset Q_0$ ,
- (ii)  $\text{supp } y \subset B(\hat{x}, \delta)$ ,  $\hat{x} \in (0, T) \times \partial\Omega$ .

Case (i) immediately follows from [13]. So, we will concentrate on Case (ii).

**2.2 Change of coordinates**

Let us take  $\hat{x} \in (0, T) \times \partial\Omega$ ,  $\delta > 0$  and a solution  $y$  of (2.1)–(2.2) such that  $\text{supp } y \subset B(\hat{x}, \delta)$ . By (2.10), there exists a  $\delta > 0$  sufficiently small, such that for some index  $i_0 \in \{1, \dots, n\}$ ,

$$\frac{\partial\psi}{\partial x_{i_0}}(\tilde{x}) \neq 0, \quad \forall \tilde{x} \text{ such that } (x_0, \tilde{x}) \in B(\hat{x}, \delta).$$

After renumbering, we can assume that  $i_0 = n$  and without loss of generality, we can assume that

$$\frac{\partial\psi}{\partial x_n}(\tilde{x}) \neq 0, \quad \forall \tilde{x} \text{ such that } (x_0, \tilde{x}) \in B(\hat{x}, \delta). \tag{2.44}$$

We now take the new coordinate system

$$\hat{x}_n = \psi(x_1, \dots, x_n), \quad \hat{x}_i = x_i, \quad i = 0, \dots, n - 1. \tag{2.45}$$

Since  $\psi = 0$  on  $\Gamma$ , in the new coordinate system  $((0, T) \times \partial\Omega) \cap B(\hat{x}, \delta)$  corresponds to  $\hat{x}_n = 0$ . Writing

$$\hat{y}(\hat{x}_0, \hat{x}_1, \dots, \hat{x}_n) = y(x_0, x_1, \dots, x_n),$$

from (2.1)–(2.2), we find

$$\begin{aligned} \hat{L}(\hat{x}, D)\hat{y} &= a_0 \frac{\partial\hat{y}}{\partial x_0} - \frac{\partial^2\hat{y}}{\partial \hat{x}_n^2} - \sum_{j=1}^{n-1} \hat{a}_{nj} \frac{\partial^2\hat{y}}{\partial \hat{x}_n \partial \hat{x}_j} - \hat{A}\hat{y} + \hat{B}\hat{y} \\ &= \hat{f} + \sum_{j=1}^n \frac{\partial \hat{f}_j}{\partial \hat{x}_j}, \quad \text{in } [0, T] \times \mathbb{R}^{n-1} \times (0, \gamma), \end{aligned} \tag{2.46}$$

$$\hat{y}(\hat{x}', 0) = \hat{g}(\hat{x}'), \quad \text{on } [0, T] \times \mathbb{R}^{n-1}, \quad \hat{x}' = (\hat{x}_0, \dots, \hat{x}_{n-1}), \tag{2.47}$$

and  $\hat{y}$  vanishes in a neighborhood of the set  $(\partial B'(0, \delta) \times [0, \gamma]) \cup (B'(0, \delta) \times \{\gamma\})$  with  $B'(0, \delta) = \{\hat{x}' \in \mathbb{R}^n \mid |\hat{x}'| \leq \delta\}$  and  $\hat{x}' = (\hat{x}_0, \dots, \hat{x}_{n-1})$ .

Now we want to show an inequality analogous to (2.38) corresponding to the weight function

$$\hat{\varphi}(\hat{x}) = e^{\lambda \hat{x}_n}. \tag{2.48}$$

More precisely, we want to show that there exists a constant  $\hat{\lambda}$  such that for any  $\lambda > \hat{\lambda}$ , there exist  $s_0$  and  $\hat{C} > 0$  independent of  $s$  such that for all  $s \geq s_0$ ,

$$\begin{aligned} &\int_{[0, T] \times \mathbb{R}^{n-1} \times [0, \gamma]} \left( \frac{1}{s\hat{\varphi}} \sum_{j=1}^n \left| \frac{\partial\hat{y}}{\partial x_j} \right|^2 + s\hat{\varphi}|\hat{y}|^2 \right) e^{2s\hat{\alpha}} d\hat{x} \\ &\leq \hat{C} \left( s^{-\frac{1}{2}} \|\hat{\varphi}^{-\frac{1}{4}} \hat{g} e^{s\hat{\alpha}}\|_{H^{\frac{1}{4}, \frac{1}{2}}([0, T] \times \mathbb{R}^{n-1})}^2 + s^{-\frac{1}{2}} \|\hat{\varphi}^{-\frac{1}{4} + \frac{1}{\kappa}} \hat{g} e^{s\hat{\alpha}}\|_{L^2([0, T] \times \mathbb{R}^{n-1})}^2 \right) \\ &+ \int_{[0, T] \times \mathbb{R}^{n-1} \times [0, \gamma]} \left( \frac{|\hat{f}|^2}{s^2 \hat{\varphi}^2} + \sum_{j=1}^n |\hat{f}_j|^2 \right) e^{2s\hat{\alpha}} d\hat{x}. \end{aligned} \tag{2.49}$$

In (2.46), the operator  $\hat{A}$  has the form

$$\hat{A}\hat{y} = \sum_{i,j=1}^{n-1} \hat{a}_{ij}(\hat{x}) \frac{\partial^2\hat{y}}{\partial \hat{x}_i \partial \hat{x}_j}, \tag{2.50}$$

and the operator  $\widehat{B}$  is a first order differential operator with  $L^\infty$  coefficients. We have already seen that we can ignore the first order terms. We also omit from now on the notation  $\widehat{\cdot}$ . Then we can write

$$\begin{aligned} L(x, D)y &= a_0 \frac{\partial y}{\partial x_0} - \frac{\partial^2 y}{\partial x_n^2} - \sum_{j=1}^{n-1} a_{nj} \frac{\partial^2 y}{\partial x_n \partial x_j} - Ay \\ &= f + \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}, \quad \text{in } [0, T] \times \mathbb{R}^{n-1} \times (0, \gamma), \end{aligned} \tag{2.51}$$

$$y(x', 0) = g(x'), \quad \text{on } [0, T] \times \mathbb{R}^{n-1}, \tag{2.52}$$

and

$$y \text{ vanishes in the neighborhood of the set } (\partial B'(0, \delta) \times [0, \gamma]) \cup (B'(0, \delta) \times \{\gamma\}). \tag{2.53}$$

Here  $a_0 \in C^1(\overline{Q})$  is a strictly positive function. Notice that  $f, f_j$  also have compact supports in  $B'(0, \delta) \times [0, \gamma)$  and that  $g$  has a compact support in  $B'(0, \delta)$ . Moreover, if we write

$$\tilde{a}(x, \xi^1, \xi^2) = \sum_{i,j=1}^{n-1} a_{ij}(x) \xi_i^1 \xi_j^2 \tag{2.54}$$

for  $x \in \mathbb{R}^{n+1}$  and  $\xi^1, \xi^2 \in \mathbb{R}^{n-1}$ , we have an ellipticity condition corresponding to (2.5), namely, there exists  $\beta > 0$  such that for all  $\tilde{\xi} \in \mathbb{R}^n$ ,

$$\xi_n^2 + \sum_{j=1}^{n-1} a_{nj}(x) \xi_n \xi_j + \tilde{a}(x, \xi^*, \xi^*) \geq \beta |\tilde{\xi}|^2, \quad \forall x \in \Pi_{\delta, \gamma} = B'(0, \delta) \times [0, \gamma], \tag{2.55}$$

where  $\xi^* = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ . This shows that

$$\exists \widehat{\gamma} > 0, \quad \forall \xi^* \in \mathbb{R}^{n-1}, \quad |\xi^*| = 1, \quad 4\tilde{a}(x, \xi^*, \xi^*) - \left( \sum_{j=1}^{n-1} a_{nj}(x) \xi_j \right)^2 \geq \widehat{\gamma}. \tag{2.56}$$

### 2.3 Localization in time

From now on, it is convenient for us to work with the function  $w(x) = e^{|s|\alpha} y(x)$  instead of  $y$ . The function  $w$  verifies the equation

$$L(x, D_0, \widetilde{D} + i|s|\nabla_{\widehat{x}}\alpha)w = F, \quad \text{in } [0, T] \times \mathbb{R}^{n-1} \times (0, \gamma), \tag{2.57}$$

$$w(x', 0) = ge^{|s|\alpha}, \tag{2.58}$$

$$\text{supp } w \subset \Pi_{\delta, \gamma} = B'(0, \delta) \times [0, \gamma], \tag{2.59}$$

where  $F(x) = F_0 + \sum_{k=1}^n \frac{\partial F_k}{\partial x_k}$ ,  $F_0 = e^{|s|\alpha} f - \sum_{j=1}^n |s| \frac{\partial \alpha}{\partial x_j} f_j e^{|s|\alpha} - |s| \frac{\partial \alpha}{\partial x_0} w$ ,  $F_k = f_k e^{|s|\alpha}$ . Since the function  $\alpha$  has singularities at  $x_0 = 0$  and  $x_0 = T$ , it is the same for some coefficients of equation (2.57). We overcome this difficulty by using a localization in time.

Let  $\tilde{\psi} \in C_0^\infty(\frac{1}{2}, 2)$  be a nonnegative function such that  $\sum_{j=-\infty}^\infty \tilde{\psi}(2^{-j}x_0) = 1$  (we may take  $\tilde{\psi}(x_0) = \tilde{\beta}(x_0) - \tilde{\beta}(2x_0)$  where  $\tilde{\beta} \in C_0^\infty(-2, 2)$  and  $\tilde{\beta}(x_0)$  equals 1 for  $|x_0| \leq 1$ ). Denote  $\tilde{\psi}_j(x_0) =$

$\tilde{\psi}(2^{-j}x_0)$ ,  $\mu_j(x_0) = \tilde{\psi}(\frac{2^{-j}}{\ell^\kappa(x_0)})$ ,  $\tilde{\mu}_j(x_0) = \tilde{\psi}'(\frac{2^{-j}}{\ell^\kappa(x_0)})$ ,  $\Psi_j(x_0) = \mu_{j+1}(x_0) + \mu_j(x_0) + \mu_{j-1}(x_0)$  and  $w_j(x) = \mu_j(x_0)w(x)$ ,  $F_{k,j}(x) = \mu_j(x_0)F_k(x)$ ,  $g_j(x) = \mu_j(x_0)g(x)e^{s|\alpha}$ .

The function  $w_j$  satisfies the equation

$$L(x, D_0, \tilde{D} + i|s|\nabla_{\tilde{x}}\alpha)w_j = \frac{\partial\mu_j}{\partial x_0}w + F_{0,j} + \sum_{k=1}^n \frac{\partial F_{k,j}}{\partial x_k}, \quad \text{in } G = \mathbb{R}^n \times (0, \gamma), \tag{2.60}$$

$$w_j(x_0, \dots, x_{n-1}, 0) = g_j, \tag{2.61}$$

$$\text{supp } w_j \subset \Pi_{\delta, \gamma}. \tag{2.62}$$

Suppose that for a function  $w_j$ , the following Carleman estimate is already established:

$$\begin{aligned} & \sum_{k=1}^n \left\| s^{-\frac{1}{2}}\varphi^{-\frac{1}{2}} \frac{\partial w_j}{\partial x_k} \right\|_{L^2(G)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}w_j\|_{L^2(G)} \\ & \leq C \left( \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4} + \frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)} \right. \\ & \quad \left. + \left\| \frac{\frac{\partial\mu_j}{\partial x_0}w + F_{0,j}}{s\varphi} \right\|_{L^2(G)} + \sum_{k=1}^n \|F_{k,j}\|_{L^2(G)} \right), \quad \forall s > s_0 > 1, \end{aligned} \tag{2.63}$$

where  $C$  is independent of  $s$  and  $j$ , and  $s_0$  is independent of  $j$ .

Observe that

$$\text{supp } \mu_j \cap \text{supp } \mu_k \neq \emptyset, \quad \text{supp } \tilde{\mu}_j \cap \text{supp } \tilde{\mu}_k \neq \emptyset, \quad \text{only if } |k - j| \leq 1. \tag{2.64}$$

Let  $\tilde{f}, \tilde{h} \in L^2(G)$  be arbitrary functions and  $\tilde{f}_j = \mu_j\tilde{f}$ ,  $\tilde{h}_j = \tilde{\mu}_j\tilde{h}$ . By (2.64), we have

$$\sum_{j=-\infty}^{\infty} |\tilde{f}_j(x)|^2 \leq C|\tilde{f}(x)|^2, \quad \sum_{j=-\infty}^{\infty} |\tilde{h}_j(x)|^2 \leq C|\tilde{h}(x)|^2, \quad \forall x \in G. \tag{2.65}$$

Therefore

$$\sum_{j=-\infty}^{\infty} \left( \left\| \frac{F_{0,j}}{s\varphi} \right\|_{L^2(G)}^2 + \sum_{k=1}^n \|F_{k,j}\|_{L^2(G)}^2 \right) \leq 3 \left( \left\| \frac{F_0}{s\varphi} \right\|_{L^2(G)}^2 + \sum_{k=1}^n \|F_k\|_{L^2(G)}^2 \right). \tag{2.66}$$

Observing that  $\frac{\partial\mu_j}{\partial x_0} = -\kappa\tilde{\mu}_j \frac{2^{-j}}{\ell^{\kappa+1}}\ell'$ , we have

$$\sum_{j=-\infty}^{\infty} \left\| \frac{\partial\mu_j}{\partial x_0} \frac{w}{s\varphi} \right\|_{L^2(G)}^2 = \sum_{j=-\infty}^{\infty} \left\| \kappa\tilde{\mu}_j \frac{2^{-j}}{\ell^{\kappa+1}}\ell' \frac{w}{s\varphi} \right\|_{L^2(G)}^2 \leq C \sum_{j=-\infty}^{\infty} \left\| \tilde{\mu}_j \frac{1}{\ell} \frac{w}{s\varphi} \right\|_{L^2(G)}^2.$$

Using (2.65) again, we find

$$\sum_{j=-\infty}^{\infty} \left\| \frac{\partial\mu_j}{\partial x_0} \frac{w}{s\varphi} \right\|_{L^2(G)}^2 \leq C \left\| \frac{w}{s\varphi^{1-\frac{1}{\kappa}}} \right\|_{L^2(G)}^2. \tag{2.67}$$

Since the restriction of  $\varphi$  on the hyperplane  $\{x \mid x_n = 0\}$  is independent of  $(x_1, \dots, x_{n-1})$ , we have

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{+\infty} \varphi^{-\frac{1}{2}} \|g_j(x_0, \cdot)\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})}^2 dx_0 &= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{+\infty} \varphi^{-\frac{1}{2}} \mu_j^2 \|g(x_0, \cdot)\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})}^2 dx_0 \\ &\leq C \int_{-\infty}^{+\infty} \varphi^{-\frac{1}{2}} \|g(x_0, \cdot)\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})}^2 dx_0 \\ &\leq C \|\varphi^{-\frac{1}{4}}g\|_{H^{0, \frac{1}{2}}(\mathbb{R}^n)}^2. \end{aligned} \tag{2.68}$$

Next we observe

$$\sum_{j=-\infty}^{\infty} \|\varphi^{-\frac{1}{4}} g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)}^2 = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \|\varphi^{-\frac{1}{4}} g_j(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}(\mathbb{R})}^2 dx_1 \cdots dx_{n-1}.$$

Denote  $\mathcal{G}_j = \text{supp } \mu_j$ ,  $h_j = \varphi^{-\frac{1}{4}} g_j$ ,  $h = \varphi^{-\frac{1}{4}} g$ . According to the definition of the norm in the space  $H^{\frac{1}{4}}$ , we have

$$\begin{aligned} \|h_j(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}(\mathbb{R})}^2 &= C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|(\mu_j h)(y_0, \cdot) - (\mu_j h)(x_0, \cdot)|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 \\ &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mu_j(y_0) - \mu_j(x_0)|^2 |h(y_0, \cdot)|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 \\ &\quad + C \int_{\mathbb{R}} \int_{\mathcal{G}_j} \frac{|h(y_0, \cdot) - h(x_0, \cdot)|^2 |\mu_j(x_0)|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 \\ &= C(I_1(j) + I_2(j)). \end{aligned} \quad (2.69)$$

We estimate the terms  $I_1$  and  $I_2$  separately. By Young's inequality, we have

$$\begin{aligned} I_1 &= \int_{\mathcal{G}_j} \int_{[-\delta, \delta]} |\mu_j'(\zeta)|^2 |h(y_0, x_1, \dots, x_{n-1})|^2 |y_0 - x_0|^{\frac{1}{2}} dx_0 dy_0 \\ &\leq \int_{\mathcal{G}_j} \int_{[-\delta, \delta]} |h(y_0, x_1, \dots, x_{n-1})|^2 |y_0 - x_0|^{\frac{1}{2}} dx_0 dy_0 \|\mu_j\|_{C^1(\mathbb{R})}^2 \\ &\leq C \|\mu_j\|_{C^1(\mathbb{R})}^2 \|h(\cdot, x_1, \dots, x_{n-1})\|_{L^2(\mathcal{G}_j)}^2. \end{aligned}$$

Here  $\zeta$  is some number between  $x_0$  and  $y_0$ . Obviously,

$$\|\mu_j\|_{C^1(\mathbb{R})} \leq C(1 + \|\varphi^{\frac{1}{\kappa}}(\cdot, x_1, \dots, x_{n-1}, 0)\|_{C^0(\mathcal{G}_j)}).$$

Next we notice

$$\frac{2^{-j}}{\ell^\kappa(x_0)} \in \left[\frac{1}{2}, 2\right], \quad \forall x_0 \in \mathcal{G}_j.$$

Since  $\varphi(\cdot, x_1, \dots, x_{n-1}, 0) = \frac{1}{\ell^\kappa(x_0)}$ , this implies

$$\varphi(\cdot, x_1, \dots, x_{n-1}, 0) \in [2^{j-1}, 2^j], \quad \forall x_0 \in \mathcal{G}_j$$

and

$$I_1 \leq C \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}}(\cdot, x_1, \dots, x_{n-1}, 0) g(\cdot, x_1, \dots, x_{n-1})\|_{L^2(\mathcal{G}_j)}^2, \quad (2.70)$$

$$\begin{aligned} I_2 &\leq \|\mu_j\|_{C^0(\mathbb{R})}^2 \int_{\mathbb{R}} \int_{\mathcal{G}_j} \frac{|h(y_0, x_1, \dots, x_{n-1}) - h(x_0, x_1, \dots, x_{n-1})|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 \\ &\leq \|\mu_j\|_{C^0(\mathbb{R})}^2 \left( \|h(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}\left(\bigcup_{l=j-2}^{j+2} \mathcal{G}_l\right)}^2 + \int_{\mathbb{R} \setminus \bigcup_{l=j-1}^{j+1} \mathcal{G}_l} \int_{\mathcal{G}_j} \frac{|h(x_0, x_1, \dots, x_{n-1})|^2}{|y_0 - x_0|^{\frac{3}{2}}} dx_0 dy_0 \right) \\ &\leq C \left( \|h(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}\left(\bigcup_{l=j-2}^{j+2} \mathcal{G}_l\right)}^2 + \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} g(\cdot, x_1, \dots, x_{n-1})\|_{L^2(\mathcal{G}_j)}^2 \right). \end{aligned} \quad (2.71)$$

Here we used the fact that

$$\begin{aligned} \int_{\mathbb{R} \setminus \bigcup_{l=j-1}^{j+1} \mathcal{G}_l} \frac{1}{|y_0 - x_0|^{\frac{3}{2}}} dy_0 &\leq C \int_{\{|y_0|^\kappa \geq 2^{-j-2}\}} \frac{1}{|y_0|^{\frac{3}{2}}} dy_0 \\ &\leq C \|\varphi^{-\frac{1}{4}}(\cdot, x_1, \dots, x_{n-1}, 0)\|_{C^0(\mathcal{G}_j)}^2, \quad \forall x_0 \in \mathcal{G}_j. \end{aligned}$$

By (2.64),

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} \|h(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}(\mathcal{G}_j)}^2 \\ & \leq \sum_{j=-\infty}^{\infty} \|h(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}(\mathcal{G}_{2j})}^2 + \sum_{j=-\infty}^{\infty} \|h(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}(\mathcal{G}_{2j+1})}^2 \\ & \leq 2\|h(\cdot, x_1, \dots, x_{n-1})\|_{H^{\frac{1}{4}}(\mathbb{R})}^2. \end{aligned} \tag{2.72}$$

By (2.70)–(2.72),

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}^2 &= \sum_{j=-\infty}^{\infty} \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g_j\|_{H^{0, \frac{1}{2}}(\mathbb{R}^n)}^2 + \sum_{j=-\infty}^{\infty} \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)}^2 \\ &\leq C\left(\|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g\|_{H^{0, \frac{1}{2}}(\mathbb{R}^n)}^2 + s^{-\frac{1}{2}} \sum_{j=-\infty}^{\infty} (\mathbf{I}_1(j) + \mathbf{I}_2(j))\right) \\ &\leq C(\|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}^2 + \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4} + \frac{1}{\kappa}}g\|_{L^2(\mathbb{R}^n)}^2). \end{aligned} \tag{2.73}$$

By (2.63), for any  $s \geq s_0$ , we have

$$\begin{aligned} & \sum_{k=1}^n \left\| s^{-\frac{1}{2}}\varphi^{-\frac{1}{2}} \frac{\partial w}{\partial x_k} \right\|_{L^2(G)}^2 + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}w\|_{L^2(G)}^2 \\ & \leq \sum_{j=-\infty}^{\infty} \left( \sum_{k=1}^n \left\| s^{-\frac{1}{2}}\varphi^{-\frac{1}{2}} \frac{\partial w_j}{\partial x_k} \right\|_{L^2(G)}^2 + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}w_j\|_{L^2(G)}^2 \right) \\ & \leq C \sum_{j=-\infty}^{\infty} \left( \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}^2 + \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4} + \frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}^2 \right. \\ & \quad \left. + \left\| \frac{\frac{\partial \mu_j}{\partial x_0} w + F_{0,j}}{s\varphi} \right\|_{L^2(G)}^2 + \sum_{k=1}^n \|F_{k,j}\|_{L^2(G)}^2 \right). \end{aligned} \tag{2.74}$$

Finally, we estimate the right-hand side of (2.74) by using (2.66), (2.67) and (2.73),

$$\begin{aligned} & \sum_{k=1}^n \left\| s^{-\frac{1}{2}}\varphi^{-\frac{1}{2}} \frac{\partial w}{\partial x_k} \right\|_{L^2(G)}^2 + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}w\|_{L^2(G)}^2 \\ & \leq C\left(\|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4}}g\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}^2 + \|s^{-\frac{1}{4}}\varphi^{-\frac{1}{4} + \frac{1}{\kappa}}g\|_{L^2(\mathbb{R}^n)}^2 \right. \\ & \quad \left. + \left\| \frac{w}{s\varphi^{1-\frac{1}{\kappa}}} \right\|_{L^2(G)}^2 + \left\| \frac{F_0}{s\varphi} \right\|_{L^2(G)}^2 + \sum_{k=1}^n \|F_k\|_{L^2(G)}^2 \right), \quad \forall s > s_0. \end{aligned} \tag{2.75}$$

Using the definition of the functions  $F_k$  and increasing the parameter  $s_0$  if necessary, we obtain (2.49) from (2.75). Thus, in order to prove (2.49), it suffices to prove (2.63). We concentrate on proving this estimate below.

### 2.4 Auxiliary problem

In the previous subsection, we showed that in order to prove the Carleman estimate (2.38), it suffices to establish a countable number of Carleman estimates for slightly simpler problems.

We put all these problems in the following general framework. Consider the following partial differential equation:

$$L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)w = \tilde{f}, \quad \text{in } G, \quad (2.76)$$

$$w(x', 0) = \tilde{g}, \quad (2.77)$$

$$\text{supp } w \subset \Pi_{\delta, \gamma}. \quad (2.78)$$

Here  $\tilde{f} \in H^{-\frac{1}{2}, -1, \tau}(G)$ ,  $\text{supp } \tilde{f} \subset \Pi_{\delta, \gamma}$ , where the space  $H_0^{\frac{1}{2}, 1, \tau}(G)$  can be defined similarly to  $H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)$  and

$$H^{-\frac{1}{2}, -1, \tau}(G) = (H_0^{\frac{1}{2}, 1, \tau}(G))^*, \quad \|\tilde{f}\|_{H^{-\frac{1}{2}, -1, \tau}(G)} = \sup_{w \in H_0^{\frac{1}{2}, 1, \tau}(G)} \frac{|\langle \tilde{f}, w \rangle|}{\|w\|_{H_0^{\frac{1}{2}, 1, \tau}(G)}}.$$

We take a function  $\beta$  such that

$$\beta \in \mathcal{U}, \quad (2.79)$$

where the set  $\mathcal{U}$  is constructed in the following way. First, we extend a function  $\psi(x_1, \dots, x_n)$  on the set  $\mathbb{R}^{n-1} \times [0, \gamma]$  up to a  $C^2$  function in such a way that  $\psi$  is constant outside a ball with a sufficiently large radius and  $\psi(x) < 2\|\psi\|_{C^0(\overline{\Omega})}$  on  $\mathbb{R}^{n-1} \times [0, \gamma]$ . (Here  $\|\psi\|_{C^0(\overline{\Omega})}$  is the norm of function  $\psi$  in the original coordinates.) We fix a sequence  $\{x_{0,j}\}$  such that  $x_{0,j} \in \text{supp } \mu_j$ . The set  $\mathcal{U}$  consists of the functions of the form

$$(e^{\lambda\psi} - e^{2\lambda\|\psi\|_{C^0(\overline{\Omega})}}) \frac{\ell^\kappa(x_{0,j})}{\tilde{\ell}_j^\kappa(x_0)}.$$

The sequence of functions  $\{\tilde{\ell}_j\}$  is constructed in the following way. We fix sufficiently large  $\widehat{j}$  such that for all  $j \geq \widehat{j}$ ,  $\text{supp } \mu_j \subset [0, \frac{T}{8}] \cup [\frac{7T}{8}, T]$ . There exist  $0 < T_0(j) < T_1(j) < T$  such that  $\text{supp } \mu_j \subset [T_0(j), T_1(j)]$ . Hence, for  $j \leq \widehat{j}$ , we define  $\tilde{\ell}_j$  to be a smooth, strictly positive function on  $[0, T]$ , which coincides with  $\ell$  on the segment  $[T_0(j), T_1(j)]$  and is equal to some constants on  $[T, +\infty)$  and  $(-\infty, 0]$ . If  $j \geq \widehat{j}$ , then  $\text{supp } \mu_j \subset [2^{-\frac{j+1}{\kappa}}, 2^{-\frac{j}{\kappa}}] \cup [T - 2^{-\frac{j+1}{\kappa}}, T - 2^{-\frac{j}{\kappa}}]$ . Set  $\tilde{\ell} = 2^{-\frac{j+2}{\kappa}}$  on the segment  $[0, 2^{-\frac{j+2}{\kappa}}]$  and on the segment  $[2^{-\frac{j+2}{\kappa}}, 2^{-\frac{j+1}{\kappa}}]$ . We extend  $\tilde{\ell}$  as a linear function in such a way that the resulting function is continuous. Similarly, on the segment  $[T - 2^{-\frac{j-1}{\kappa}}, T]$  we set  $\tilde{\ell}(x_0) = T - 2^{-\frac{j-1}{\kappa}}$ , and on the segment  $[T - 2^{-\frac{j}{\kappa}}, T - 2^{-\frac{j-1}{\kappa}}]$  we let  $\tilde{\ell}$  be a linear function such that the resulting function is continuous. Finally, on the segment  $[2^{-\frac{j}{\kappa}}, T - 2^{-\frac{j+1}{\kappa}}]$  the function  $\tilde{\ell}$  coincides with  $\ell$ . It is not difficult to establish the following properties for functions of the set  $\mathcal{U}$ .

**Proposition 2.2** (1) *There exists a positive constant  $C$  such that, for all  $\beta \in \mathcal{U}$ ,*

$$\frac{\partial \beta}{\partial x_n}(x) \geq C > 0, \quad \forall x \in \Pi_{\delta, \gamma}, \quad \frac{\partial \beta}{\partial x_i} = 0, \quad \forall i \in \{1, \dots, n-1\}, \quad \forall x \in \{x_n = 0\}. \quad (2.80)$$

(2) *There exists a positive constant  $\widehat{C}$  such that*

$$\text{Im} \{ \overline{L_2}(x, \xi_0, \tilde{\xi} - i|\tau|\nabla_{\tilde{x}}\beta), L_2(x, \xi_0, \tilde{\xi} + i|\tau|\nabla_{\tilde{x}}\beta) \} \geq \widehat{C}|\tau|M^2(\xi, \tau) \quad (2.81)$$

for all  $(x, \xi, \tau) \in \{(x, \xi, \tau) \mid x \in \Pi_{\delta_0, \gamma}, L_2(x, \xi_0, \tilde{\xi} + i|\tau|\nabla_{\tilde{x}}\beta) = 0\}$ , where  $\delta_0 > \delta$  is some constant independent of  $\beta$ .

Let us show that problem (2.60)–(2.62) can be reduced to problem (2.76)–(2.78). Set  $w = w_j$ ,  $\tilde{f} = \frac{\partial \mu_j}{\partial x_0} w + F_{0,j} + \sum_{k=1}^n \frac{\partial F_{k,j}}{\partial x_k}$ ,  $\tilde{g} = g_j$ , and

$$\tau = \frac{s}{\ell^\kappa(x_{0,j})}, \quad \beta(x) = \alpha(x)\ell^\kappa(x_{0,j}), \quad \forall x_0 \in \text{supp } \mu_j, \quad x_{0,j} \in \text{supp } \mu_j. \tag{2.82}$$

Since

$$\beta(x) = \alpha(x)\ell^\kappa(x_{0,j}), \quad \forall x_0 \in \text{supp } \mu_j,$$

we have

$$\begin{aligned} L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)w &= L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)w_j = L(x, D_0, \tilde{D} + i|\tau|(\nabla_{\tilde{x}}\alpha)\ell^\kappa(x_{0,j}))w_j \\ &= L(x, D_0, \tilde{D} + i|s|\nabla_{\tilde{x}}\alpha)w_j = \frac{\partial \mu_j}{\partial x_0} w + F_{0,j} + \sum_{k=1}^n \frac{\partial F_{k,j}}{\partial x_k}. \end{aligned}$$

Next, we claim that the solutions of system (2.76)–(2.78) satisfy the following Carleman estimate: there exist constants  $C$  and  $\tau_0$ , both independent of  $\beta \in \mathcal{U}$ , such that

$$\begin{aligned} &\sum_{j=1}^n \left\| \tau^{-\frac{1}{2}} \frac{\partial w}{\partial x_j} \right\|_{L^2(G)} + \|\tau^{\frac{1}{2}} w\|_{L^2(G)} \\ &\leq C(\tau^{-\frac{1}{4}} \|\tilde{g}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + \tau^{-\frac{1}{4} + \frac{1}{\kappa}} \|\tilde{g}\|_{L^2(\mathbb{R}^n)} + \|\tilde{f}\|_{H^{-\frac{1}{2}, -1, \tau}(G)}), \quad \forall \tau \geq \tau_0. \end{aligned} \tag{2.83}$$

Suppose for the moment that this estimate is proved already. Let us show that it implies (2.63). We know already that functions  $(w_j, g_j, \frac{\partial \mu_j}{\partial x_0} w + F_{0,j} + \sum_{k=1}^n \frac{\partial F_{k,j}}{\partial x_k})$  satisfy (2.76)–(2.78) with  $\tau$  and  $\beta$  defined in (2.82). Making the change of unknown in (2.83), we arrive at the inequality

$$\begin{aligned} &\sum_{k=1}^n \left\| \left( \frac{s}{\ell^\kappa(x_{0,j})} \right)^{-\frac{1}{2}} \frac{\partial w_j}{\partial x_k} \right\|_{L^2(G)} + \left\| \left( \frac{|s|}{\ell^\kappa(x_{0,j})} \right)^{\frac{1}{2}} w_j \right\|_{L^2(G)} \\ &\leq C \left( \left( \frac{|s|}{\ell^\kappa(x_{0,j})} \right)^{-\frac{1}{4}} \|g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + \left( \frac{|s|}{\ell^\kappa(x_{0,j})} \right)^{-\frac{1}{4} + \frac{1}{\kappa}} \|g_j\|_{L^2(\mathbb{R}^n)} \right. \\ &\quad \left. + \left\| \partial_{x_0} \mu_j w + F_{0,j} + \sum_{k=1}^n \frac{\partial F_{k,j}}{\partial x_k} \right\|_{H^{-\frac{1}{2}, -1, (\frac{s}{\ell^\kappa(x_{0,j})})}(G)} \right), \end{aligned} \tag{2.84}$$

which holds for all  $\tau \geq \tau_0$ .

Note that there exist two constants  $C_1 > 0$  and  $C_2 > 0$  independent of  $s, j$ , such that

$$C_1 \frac{1}{\varphi(x_0, \dots, x_{n-1}, 0)} \leq \ell^k(x_{0,j}) \leq C_2 \frac{1}{\varphi(x_0, \dots, x_{n-1}, 0)}, \quad \forall x_0 \in \text{supp } \mu_j. \tag{2.85}$$

Then the previous inequality can be written in the form

$$\begin{aligned} &\sum_{k=1}^n \left\| |s|^{-\frac{1}{2}} \varphi^{-\frac{1}{2}} \frac{\partial w_j}{\partial x_k} \right\|_{L^2(G)} + \left\| |s|^{\frac{1}{2}} \varphi^{\frac{1}{2}} w_j \right\|_{L^2(G)} \\ &\leq C \left( \left( \frac{|s|}{\ell^\kappa(x_{0,j})} \right)^{-\frac{1}{4}} \|g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + \left( |s|\varphi \right)^{-\frac{1}{4} + \frac{1}{\kappa}} \|g_j\|_{L^2(\mathbb{R}^n)} \right. \\ &\quad \left. + \left\| \frac{\frac{\partial \mu_j}{\partial x_0} w + F_{0,j}}{s\varphi} \right\|_{L^2(G)} + \sum_{k=1}^n \|F_{k,j}\|_{L^2(G)} \right), \quad \forall s \geq s_0. \end{aligned} \tag{2.86}$$



What remains to be shown is the existence of a constant  $C > 0$  such that

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac{1}{4}} \|g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} \leq C(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + \|(|s|\varphi)^{-\frac{1}{4} + \frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}).$$

Since (2.85) immediately implies the inequality

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac{1}{4}} \|g_j\|_{H^{0, \frac{1}{2}}(\mathbb{R}^n)} \leq C\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{0, \frac{1}{2}}(\mathbb{R}^n)},$$

it suffices to prove

$$\left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac{1}{4}} \|g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)} \leq C(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)} + \|(|s|\varphi)^{-\frac{1}{4} + \frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}). \quad (2.87)$$

Then, elementary computations provide that for any  $x_0 \in \mathcal{G}_j$ ,

$$\begin{aligned} & |h(x_0)\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0) - h(y_0)\varphi^{-\frac{1}{4}}(y_0, \dots, y_{n-1}, 0)|^2 \\ &= |\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0)(h(x_0) - h(y_0)) + h(y_0)(\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0) - \varphi^{-\frac{1}{4}}(y_0, \dots, y_{n-1}, 0))|^2 \\ &\geq |\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0)|^2 |h(x_0) - h(y_0)|^2 \\ &\quad - 4|h(y_0)|^2 |\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0) - \varphi^{-\frac{1}{4}}(y_0, \dots, y_{n-1}, 0)|^2 \\ &\geq C\left(\frac{1}{\ell^k(x_{0,j})}\right)^{-\frac{1}{4}} |h(x_0) - h(y_0)|^2 \\ &\quad - 4|h(y_0)|^2 |\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0) - \varphi^{-\frac{1}{4}}(y_0, \dots, y_{n-1}, 0)|^2. \end{aligned} \quad (2.88)$$

Using (2.88) and the definition of  $H^{\frac{1}{4}, 0}$ -norm, we have

$$\begin{aligned} & \left(\frac{|s|}{\ell^k(x_{0,j})}\right)^{-\frac{1}{4}} \|g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)} \leq C\left(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)} \right. \\ & \quad \left. + \sqrt{\int_{\mathbb{R}^{n-1}} \int_{\mathcal{G}_j} \int_{\mathcal{G}_j} \frac{|\varphi^{-\frac{1}{4}}(x_0, \dots, x_{n-1}, 0) - \varphi^{-\frac{1}{4}}(y_0, \dots, y_{n-1}, 0)|^2}{|s|^{\frac{1}{4}}|h(y_0)|^{-2}|x_0 - y_0|^{\frac{3}{2}}} dx_0 dy_0 dx_1 \cdots dx_{n-1}}\right) \\ & \leq C\left(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)} \right. \\ & \quad \left. + \sqrt{\int_{\mathbb{R}^{n-1}} \int_{\mathcal{G}_j} \int_{\mathcal{G}_j} |s|^{-\frac{1}{4}}|h(y_0)|^2(\ell^{\frac{\kappa}{4}})'(\zeta)|^2|x_0 - y_0|^{\frac{1}{2}} dx_0 dy_0 dx_1 \cdots dx_{n-1}}\right) \\ & \leq C(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4}, 0}(\mathbb{R}^n)} + \| |s|^{-\frac{1}{4}}\varphi^{-\frac{1}{4} + \frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}). \end{aligned}$$

Thus estimate (2.87) is established.

In (2.76), without loss of generality, one can assume  $a_{nn} \equiv 1$ . The principal symbol of operator  $L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)$  can be written in the form

$$L_2(x, \xi, \tau) = ia_0(x)\xi_0 + \sum_{k,j=1}^n a_{kj}(x)\xi_k\xi_j,$$

where  $\zeta = \xi + i|\tau|\nabla\beta$ . Consider the equation  $L_2(x, \xi, \tau) = 0$ . The two roots in  $\xi_n$  of this equation are

$$-i|\tau|\frac{\partial\beta(x)}{\partial x_n} + \lambda^\pm(x, \xi', \tau),$$

where

$$\lambda^\pm(x, \xi', \tau) = - \sum_{j=1}^{n-1} a_{nj}(x)\zeta_j \pm \sqrt{-(a(x, \zeta', \zeta') + ia_0(x)\xi_0) + \left(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\right)^2},$$

and  $\zeta' = \xi' + i|\tau|\nabla'\beta$ .

Let  $\mathcal{M} = \{(\xi', \tau) \mid \xi_0^2 + \tau^4 + \sum_{i=1}^{n-1} \xi_i^4 = 1\}$ . If

$$(x, \xi', \tau) \in \Phi = \left\{ (x, \xi', \tau) \in \Pi_{\delta, \gamma} \times \mathcal{M} \mid -(a(x, \zeta', \zeta') + ia_0(x)\xi_0) + \left(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\right)^2 \in \mathbb{R}_+^1 \right\},$$

we assume that

$$\text{Im} \sqrt{-(a(x, \zeta', \zeta') + ia_0(x)\xi_0) + \left(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\right)^2}$$

is positive. Therefore, outside the set  $\Phi$ , the functions  $\lambda^\pm$  are smooth. In order to regularize this expression for  $\lambda^\pm(x, \xi', \tau)$  near  $(\xi', \tau) = 0$ , we consider  $v \in C^\infty(\mathbb{R}^+)$  such that

$$\begin{aligned} v(t) &= 0 \quad \text{for } t \in \left[0, \frac{1}{2}\right], \\ v(t) &= 1 \quad \text{for } t > 1, \\ 0 \leq v(t) &\leq 1, \quad \forall t \in \mathbb{R}^+, \end{aligned}$$

and determine  $\tilde{\lambda}^\pm(x, \xi', \tau)$  as

$$\begin{aligned} \tilde{\lambda}^\pm(x, \xi', \tau) &= - \sum_{j=1}^{n-1} a_{nj}(x)\zeta_j \pm v(M(\xi', \tau)) \sqrt{-(a(x, \zeta', \zeta') + ia_0(x)\xi_0) + \left(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\right)^2}, \\ M(\xi', \tau) &= \left(\xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + \tau^4\right)^{\frac{1}{4}}. \end{aligned}$$

We set

$$r_\pm(x, \xi', \tau) = |\tau| \frac{\partial \beta}{\partial x_n}(x) + i\tilde{\lambda}^\pm(x, \xi', \tau), \tag{2.89}$$

and introduce the following sets

$$\begin{aligned} \Upsilon &= \{(\xi', \tau) \mid \exists x \in \Pi_{\delta, \gamma} \text{ such that } (x, \xi', \tau) \in \Phi\}, \\ \Upsilon_\epsilon &= \{(\xi', \tau) \mid \text{dist}((\xi', \tau), \Upsilon) \leq \epsilon\}, \quad \Upsilon_\epsilon^1 = \mathcal{M} \setminus \Upsilon_{\frac{\epsilon}{2}}. \end{aligned}$$

We claim that one can take a parameter  $\gamma > 0$  small enough such that there exists a positive  $\epsilon(\gamma)$ , which can be taken arbitrarily small, and a pair of functions  $\{\chi_0, \chi_1\}$  independent of  $\beta \in \mathcal{U}$  such that

$$\chi_0, \chi_1 \in C^\infty(\mathcal{M}), \tag{2.90}$$

$$\text{supp } \chi_0 \subset \Upsilon_\epsilon, \quad \text{supp } \chi_1 \subset \Upsilon_\epsilon^1, \tag{2.91}$$

$$\chi_0 \geq 0, \quad \chi_1 \geq 0, \quad \chi_0 + \chi_1 \geq 1, \quad \text{on } \mathcal{M}, \tag{2.92}$$

$$\text{dist}(\Upsilon, (0, \dots, 0, 1)) \rightarrow +0, \quad \text{as } \gamma \rightarrow +0. \tag{2.93}$$

Really, we observe that  $\frac{\partial \beta}{\partial x_i}(x', 0) = 0$  for  $i \in \{1, \dots, n-1\}$  and

$$\min_{x \in \{x_n=0, |x'| < \delta\}} \min\{\operatorname{Re} Z, -|\operatorname{Im} Z|\} < 0,$$

where  $Z = -4(a(x, \zeta', \zeta') + ia_0(x)\xi_0) + \left(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\right)^2$ . Therefore, if a point  $(\xi', \tau)$  belongs to

$\Upsilon$  and  $M(\xi', 0) > 0$ , the quantity  $\sum_{j=1}^{n-1} |\tau \frac{\partial \beta}{\partial x_j}|$  can not be closed to zero. Indeed, if  $\tau = 0$ , then  $\zeta'$  is a real vector. Since  $a_0$  is a strictly positive function,  $\xi_0 = 0$ . However, in this case, the inequality (2.56) implies that  $Z$  is a negative number. This contradicts the fact  $(\xi, \tau) \in \Upsilon$ . On the other hand,

$$\max_{x \in \Pi_{\delta, \gamma}} \sum_{j=1}^{n-1} \left| \frac{\partial \beta}{\partial x_j} \right| \rightarrow +0, \quad \text{as } \gamma \rightarrow +0.$$

Therefore, (2.93) holds true, and for sufficiently small positive  $\epsilon$  the choice of the functions  $\chi_i$  is possible.

Next, we extend  $\chi_\mu$  to the set  $\{(\xi', \tau) \mid M(\xi', \tau) > 1\}$  by the formula

$$\chi_\mu(\xi', \tau) = \chi_\mu\left(\frac{\xi_0}{M^2(\xi', \tau)}, \frac{\xi_1}{M(\xi', \tau)}, \dots, \frac{\xi_n}{M(\xi', \tau)}, \frac{\tau}{M(\xi', \tau)}\right).$$

Then we extend functions  $\chi_\mu$  up to  $C^\infty$  function on  $\{(\xi', \tau) \mid M(\xi', \tau) < 1\}$ . Let  $\chi_\mu(D', \tau)$  be the pseudodifferential operator with symbol  $\chi_\mu(\xi', \tau)$ .

Applying the operator  $\chi_\mu(D', \tau)$  to the both sides of equation (2.76), we find

$$L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)w_\mu = \chi_\mu \tilde{f} - [\chi_\mu, L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)]w = \tilde{f}_\mu, \quad \text{in } G, \quad (2.94)$$

$$w_\mu(x', 0) = (\chi_\mu \tilde{g})(x'), \quad x' \in \mathbb{R}^n, \quad (2.95)$$

where  $w_\mu = \chi_\mu(D', \tau)w$ .

By Lemma A.4, we observe

$$\|[\chi_\mu, L(x, D_0, \tilde{D} + i|\tau|\nabla_{\tilde{x}}\beta)]w\|_{H^{-\frac{1}{2}, -1, \tau}(G)} \leq \frac{C\|w\|_{H^{\frac{1}{2}, 1, \tau}(G)}}{(1 + |\tau|)^{1 - \frac{1}{\kappa}}}, \quad \mu \in \{0, 1\}. \quad (2.96)$$

## 2.5 Proof of the main estimate

First, we obtain an a priori estimate for the function  $w_0 = \chi_0(D', \tau)w$ . We claim that there exists a constant  $C > 0$  such that

$$\|w_0\|_{H^{\frac{1}{2}, 1, \tau}(G)} \leq C\left(\|\tilde{f}\|_{H^{-\frac{1}{2}, -1, \tau}(G)} + \frac{\|w\|_{H^{\frac{1}{2}, 1, \tau}(G)}}{(1 + |\tau|)^{1 - \frac{1}{\kappa}}}\right). \quad (2.97)$$

Really, using the notations  $W = (W_1, W_2)$  and  $F = (0, i\tilde{f}_0)$ , where  $W_1 = \tilde{\Lambda}(D', \tau)w_0$ ,  $W_2 = \frac{\partial w_0}{\partial x_n} - |\tau|\frac{\partial \beta}{\partial x_n}w_0$  and  $\tilde{\Lambda}(D', \tau)w_0 = \int_{\mathbb{R}^n} (1 + M(\xi', \tau))\hat{w}_0 e^{i(\xi', x')} d\xi'$ , we rewrite system (2.94)–(2.95) in the form

$$\frac{\partial W}{\partial x_n} = K(x, D', \tau)W + F, \quad \text{in } G, \quad (2.98)$$

$$W(x', \gamma) = \frac{\partial}{\partial x_n} W(x', \gamma) = 0. \quad (2.99)$$

Here we set

$$K(x, D', \tau) = |\tau| \frac{\partial \beta}{\partial x_n} I + \begin{pmatrix} 0 & \tilde{\Lambda}(D', \tau) + \left[ \tilde{\Lambda}, |\tau| \frac{\partial \beta}{\partial x_n} \right] \\ K_{12}(x, D', \tau) & K_{22}(x, D', \tau) \end{pmatrix},$$

where

$$K_{12}(x, D', \tau) = \sum_{j,k=1}^{n-1} a_{kj}(x) \left( D_j + i|\tau| \frac{\partial \beta}{\partial x_j} \right) \left( D_k + i|\tau| \frac{\partial \beta}{\partial x_k} \right) \tilde{\Lambda}^{-1}(D', \tau) + ia_0 D_0 \tilde{\Lambda}^{-1}(D', \tau),$$

$$K_{22}(x, D', \tau) = -i \sum_{j=1}^{n-1} a_{jn}(x) \left( D_j + i|\tau| \frac{\partial \beta}{\partial x_j} \right).$$

The eigenvalues of the matrix  $K(x, \xi', \tau)$  are  $r_{\pm}(x, \xi', \tau)$ . Therefore, by (2.93), there exists a positive constant  $C$  independent of  $\beta \in \mathcal{U}$  such that

$$\text{Spec } K(x, \xi', \tau) \subset \left\{ z \in \mathbb{C} : \text{Re } z \geq C \left( \xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + \tau^4 \right)^{\frac{1}{4}} \right\} \tag{2.100}$$

for all  $(x, \xi', \tau) \in \Pi_{\delta, \gamma} \times (\Upsilon_{\epsilon} \cap \{(\xi', s) \mid M(\xi', \tau) \geq 1\})$ . Here  $\text{Spec } K$  means the spectrum of  $K$ . We extend the symbol  $K(x, \xi', \tau)$  from  $\Pi_{\delta, \gamma} \times \Upsilon_{\epsilon}$  on  $G \times \mathbb{R}^{n+1}$  in such a way that the new symbol  $\tilde{K}(x, \xi', \tau) \in C_{\text{cl}}^1 S^{\frac{1}{2}, 1, \tau}(G)$  and inequality (2.100) holds true on  $G \times \mathbb{R}^{n+1}$  with the constant  $\frac{C}{2}$ . Here and henceforth, we understand that  $C_{\text{cl}}^k S^{\frac{\kappa}{2}, \kappa, s}(\mathcal{O})$  is specified in Definition A.1 in Appendix. Moreover, the symbol  $\tilde{K}$  is independent of  $x'$  if  $|x'|$  is sufficiently large. The function  $W$  verifies

$$\frac{\partial W}{\partial x_n} = \tilde{K}(x, D', \tau)W + \tilde{F}, \quad \text{in } G, \tag{2.101}$$

$$W(x', \gamma) = \frac{\partial}{\partial x_n} W(x', \gamma) = 0. \tag{2.102}$$

Applying Lemma A.7 and using (2.96), we obtain (2.97).

Next, we obtain an estimate for the function  $w_1 = \chi_1(D', \tau)w$ . The symbols  $r_{\pm}(x, \xi', \tau)$  are smooth on  $\overline{\Pi_{\delta, \gamma}} \times \Upsilon_{\epsilon}^1$ . Our goal is to extend symbols  $r_{\pm}$  on the set  $G \times \mathbb{R}^{n+1}$ . First, for some positive constant  $C$ , we observe

$$\text{Re } r_{-}(x, \xi', \tau) \geq CM(\xi', \tau), \quad \forall (x, \xi', \tau) \in \Pi_{\delta, \gamma} \times \Upsilon_{\epsilon}^1.$$

Therefore, we extend the symbol  $r_{-}$  to the set  $G \times \mathbb{R}^{n+1}$  in such a way that  $r_{-} \in C_{\text{cl}}^1 S^{\frac{1}{2}, 1}(G)$ , the previous inequality holds true with a constant  $\frac{C}{2}$  on  $G \times \mathbb{R}^{n+1}$ , and  $r_{-}$  is independent of  $x$  for all  $|x'| \geq \hat{K}$ ,

$$\text{Re } r_{-}(x, \xi', \tau) > \frac{C}{2} M(\xi', \tau), \quad \forall (x, \xi', \tau) \in G \times \mathbb{R}^{n+1}. \tag{2.103}$$

Next we extend the symbol  $r_{+}$ . Note that

$$-\text{Re } r_{+}(x, \xi', \tau) \geq CM(\xi', \tau), \quad \forall (x, \xi', \tau) \in \{(x, \xi', \tau) \mid x \in \Pi_{\delta, \gamma}, (\xi', \tau) \in \partial \Upsilon_{\epsilon}^1\}.$$

Therefore, we extend  $r_{+}$  on  $\Pi_{\delta', \gamma} \times (\mathbb{R}^{n+1} \setminus \Upsilon_{\epsilon}^1)$  in such a way that

$$-\text{Re } r_{+}(x, \xi', \tau) \geq CM(\xi', \tau), \quad \forall (x, \xi', \tau) \in \{(x, \xi', \tau) \mid x \in \Pi_{\delta', \gamma}, (\xi', \tau) \in \mathbb{R}^{n+1} \setminus \Upsilon_{\epsilon}^1\}. \tag{2.104}$$

This is possible if the difference  $\delta' - \delta > 0$  is small. Then in the definition of the symbol of operator  $\lambda^+$ , we substitute the function  $\beta$  by  $\beta\chi_{-1}$ , where  $\chi_{-1} \in C_0^\infty(B(0, \delta'))$  and  $\chi_{-1}|_{B(0, \delta)} = 1$ . Finally, we extend  $r_+$  from  $\Pi_{\delta', \gamma} \times \mathbb{R}^{n+1}$  on  $G \times \mathbb{R}^{n+1}$  up to a symbol of a class  $C_{cl}^1 S^{\frac{1}{2}, 1, \tau}(G)$  in such a way that the symbol  $r_+$  is independent of  $x$  for all  $x'$  such that  $|x'| > \widehat{C}$  and

$$-\operatorname{Re} r_+(x, \xi', \tau) \geq CM(\xi', \tau), \quad \forall (x, \xi', \tau) \in \{(x, \xi', \tau) \mid |x'| > \widehat{C}, (\xi', \tau) \in \mathbb{R}^{n+1}\}.$$

We denote by  $R_\pm(x, D', \tau)$  the pseudodifferential operator with symbol  $r_\pm(x, \xi', \tau)$ , namely,

$$R_\pm(x, D', \tau)u(x) = \int_{\mathbb{R}^n} r_\pm(x, \xi', \tau)\widehat{u}(\xi', x_n)e^{i(x', \xi')}d\xi'. \quad (2.105)$$

Since  $r_\pm(x, \xi', \tau) \in C_{cl}^1 S^{\frac{1}{2}, 1, s}(G)$ , by Lemma A.1, we have  $R_\pm \in \mathcal{L}(H^{\frac{1}{2}, 1, \tau}(G); L^2(G))$ . Using these pseudodifferential operators  $R_\pm(x, D', \tau)$ , we construct two operators

$$L_-(x, D, \tau) = \frac{\partial}{\partial x_n} - R_-(x, D', \tau), \quad L_+(x, D, \tau) = \frac{\partial}{\partial x_n} - R_+(x, D', \tau).$$

Symbols of the operators  $L_-(x, D, \tau)$  and  $L_+(x, D, \tau)$  are

$$L_-(x, \xi, \tau) = i\xi_n - r_-(x, \xi', \tau), \quad L_+(x, \xi, \tau) = i\xi_n - r_+(x, \xi', \tau). \quad (2.106)$$

If  $\operatorname{supp} \widehat{w}(\xi', x_n, \tau) \subset \Upsilon_\epsilon^1$  for any  $x_n \in [0, \gamma]$ , the operator  $L(x, D, \tau)$  can be represented as

$$\begin{aligned} & L(x, D_0, \widetilde{D} + i|\tau|\nabla_{\widetilde{x}}\beta)w \\ &= \left(\frac{\partial}{\partial x_n} - R_-(x, D', \tau)\right)\left(\frac{\partial}{\partial x_n} - R_+(x, D', \tau)\right)w + K(x_n)w, \quad x_n \in [0, \gamma], \end{aligned} \quad (2.107)$$

where

$$Kw \in L^\infty(0, \gamma; \mathcal{L}(H^{\frac{1}{2}, 1, \tau}(\mathbb{R}^n), L^2(\mathbb{R}^n))), \quad \|K(x_n)\|_{\mathcal{L}(H^{\frac{1}{2}, 1, \tau}, L^2)} \leq C(1 + |\tau|)^{\frac{1}{\kappa}} \quad (2.108)$$

for all  $x_n \in [0, \gamma]$ . Indeed, note that

$$\begin{aligned} & \left(\frac{\partial}{\partial x_n} - R_-(x, D', \tau)\right)\left(\frac{\partial}{\partial x_n} - R_+(x, D', \tau)\right) \\ &= \frac{\partial^2}{\partial x_n^2} + R_-(x, D', \tau)R_+(x, D', \tau) \\ & \quad - (R_+(x, D', \tau) + R_-(x, D', \tau))\frac{\partial}{\partial x_n} - R_{+, (x_n)}(x, D', \tau). \end{aligned} \quad (2.109)$$

According to Lemma A.3, the operator  $R_-R_+ = R + K_1$ , where  $R$  is the pseudodifferential operator of the form (2.105) with symbol  $r_+(x, \xi', \tau)r_-(x, \xi', \tau)$ , and the operator  $K_1(x_n) \in \mathcal{L}(H^{\frac{1}{2}, 1, \tau}(G), L^2(G))$  is such that  $\sup_{x_n \in [0, \gamma]} \|K_1(x_n)\| \leq C(\pi_{C^1(r_+)}, \pi_{C^1(r_-)}) \leq C(1 + |\tau|)^{\frac{1}{\kappa}}$ . The

operator  $R_{+, (x_n)}$  is the pseudodifferential operator with symbol  $\frac{\partial}{\partial x_n}r_+(x, \xi', \tau) \in C_{cl}^0 S^{\frac{1}{2}, 1, \tau}(\mathbb{R}^n)$  for any  $x_n$  in  $[0, \gamma]$ . By Lemma A.1, it belongs to  $L^\infty(0, \gamma; \mathcal{L}(H^{\frac{1}{2}, 1, \tau}(\mathbb{R}^n), L^2(\mathbb{R}^n)))$ . Next, observe  $(R_+(x, D', \tau) + R_-(x, D', \tau))\frac{\partial}{\partial x_n} = \sum_{j=1}^{n-1} a_{nj}(D_j + i|\tau|\frac{\partial\beta}{\partial x_j})\frac{\partial}{\partial x_n}$ . By Lemma A.3, we have

$$R_-(x, D', \tau)R_+(x, D', \tau) = a(x, D' + i|\tau|\nabla'\beta, D' + i|\tau|\nabla'\beta) - iD_0 + K_2(x_n),$$

where the operator  $K_2(x_n) \in \mathcal{L}(H^{\frac{1}{2},1,\tau}(G), L^2(G))$  is such that  $\sup_{x_n \in [0, \gamma]} \|K_2(x_n)\| \leq C(\pi_{C^1(r_+)}, \pi_{C^1(r_-)}) \leq C(1 + |\tau|)^{\frac{1}{\kappa}}$ . Here and henceforth, we understand that the seminorm  $\pi_{C^k(a)}$  is defined after Definition A.1 in Appendix. This proves (2.107) and (2.108).

Denote the function  $L_+(x, D, \tau)w_1$  as  $z$ ,

$$L_+(x, D, \tau)w_1 = z, \quad \text{in } G. \quad (2.110)$$

Consider the initial value problem

$$L_-(x, D, \tau)z = \tilde{f}_1 - Kw_1, \quad \text{in } G, \quad z(\cdot, \gamma) = 0. \quad (2.111)$$

By (2.103) and Lemma A.7, there exists a constant  $C$  independent of  $\beta$  such that

$$\|z\|_{L^2(G)} \leq C(\|\tilde{f}_1\|_{H^{-\frac{1}{2},-1,\tau}(G)} + \|Kw_1\|_{H^{-\frac{1}{2},-1,\tau}(G)}). \quad (2.112)$$

Now we concentrate on obtaining an a priori estimate for equation (2.110). We introduce the operators

$$Q(x, D', \tau) = \frac{1}{2}(L_+(x, D, \tau) + L_+(x, D, \tau)^*), \quad (2.113)$$

$$\begin{aligned} P(x, D, \tau) &= \frac{1}{2}(L_+(x, D, \tau) - L_+(x, D, \tau)^*) \\ &= \frac{\partial}{\partial x_n} - \frac{1}{2}(R_+(x, D', \tau) - R_+(x, D', \tau)^*). \end{aligned} \quad (2.114)$$

Therefore

$$Q(x, D', \tau) = Q(x, D', \tau)^*, \quad P(x, D, \tau)^* = -P(x, D', \tau).$$

Then equation (2.110) can be written in the form

$$Q(x, D', \tau)w_1 + P(x, D, \tau)w_1 = z, \quad \text{in } G.$$

Taking the  $L^2$ -norms of the left- and right-hand sides of this equation, we have

$$\|Qw_1\|_{L^2(G)}^2 + \|Pw_1\|_{L^2(G)}^2 + \operatorname{Re}(Qw_1, Pw_1)_{L^2(G)} = \|z\|_{L^2(G)}^2.$$

Observe that

$$\begin{aligned} \operatorname{Re}(Qw_1, Pw_1)_{L^2(G)} &= (Qw_1, Pw_1)_{L^2(G)} + (Pw_1, Qw_1)_{L^2(G)} \\ &= ([Q, P]w_1, w_1)_{L^2(G)} - (Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)}. \end{aligned} \quad (2.115)$$

Therefore

$$\begin{aligned} \|z\|_{L^2(G)}^2 &= \|Qw_1\|_{L^2(G)}^2 + \|Pw_1\|_{L^2(G)}^2 + ([Q, P]w_1, w_1)_{L^2(G)} \\ &\quad - (Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)}. \end{aligned} \quad (2.116)$$

By (2.81), there exists a positive constant  $\widehat{C}$  such that

$$\operatorname{Re}\{Q, P\}(x, \xi', \tau) > \widehat{C}M(\xi', \tau) \quad (2.117)$$

for all  $(x, \xi, \tau) \in \{(x, \xi, \tau) \mid x \in \Pi_{\delta', \gamma}, Q(x, \xi', \tau) = 0, |\tau| > 1\}$ , where  $\delta' > \delta$ .

**Proposition 2.3** *Suppose that (2.117) holds true. Let  $w \in H^{\frac{1}{2},1}(G)$  and  $\text{supp } w \subset \Pi_{\beta',\gamma}$ . Then there exist positive constants  $C_0$  and  $C_1$ , such that*

$$\|Qw\|_{L^2(G)}^2 + \|Pw\|_{L^2(G)}^2 + \text{Re}([Q, P]w, w)_{L^2(G)} \geq C_0 \|w\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)}^2 - C_1 \|w\|_{L^2(G)}^2.$$

**Proof** The pseudodifferential operators  $P$  and  $Q$  have the symbols with  $C^1$ -smoothness in variable  $x$ . We approximate these operators by pseudodifferential operators with smooth symbols. The approximations are constructed in the following way:

$$Q_\tau = \frac{1}{2}(L_{+, \tau}(x, D, \tau) + L_{+, \tau}(x, D, \tau)^*), \quad P_\tau = \frac{1}{2}(L_{+, \tau}(x, D, \tau) - L_{+, \tau}(x, D, \tau)^*),$$

where  $L_{+, \tau} = \frac{\partial}{\partial x_n} - R_{+, \tau}(x, D', \tau)$ . The symbol of the operator  $R_{+, \tau}$  is given by the formula

$$r_{+, \tau}(x, \xi', \tau) = |\tau| \frac{\partial \beta_\tau}{\partial x_n} + i\lambda_\tau^+(x, \xi', \tau),$$

where

$$\lambda_\tau^+(x, \xi', \tau) = \sum_{j=1}^{n-1} a_{\tau, nj}(x) \tilde{\zeta}_j \pm v(M(\xi', \tau)) \sqrt{-(a_\tau(x, \tilde{\zeta}', \tilde{\zeta}')) + ia_{\tau, 0}(x) \xi_0 + \left( \sum_{j=1}^{n-1} a_{\tau, nj}(x) \tilde{\zeta}_j \right)^2}.$$

Here

$$\begin{aligned} a_{\tau, 0} &= a_0 * \eta_{\tau^{\frac{1}{2} + \epsilon}}, \quad a_{\tau, ij} = a_{ij} * \eta_{\tau^{\frac{1}{2} + \epsilon}}, \quad \beta_\tau = \beta * \eta_{\tau^{\frac{1}{2} + \epsilon}}, \quad \tilde{\zeta}_j = \xi_j + i|\tau| \frac{\partial \beta_\tau}{\partial x_j}, \\ \tilde{\zeta}' &= (\tilde{\zeta}_0, \dots, \tilde{\zeta}_{n-1}), \end{aligned} \quad (2.118)$$

where the function  $\eta_\varepsilon$  is the standard mollifier (see e.g. [3, p. 629]), and  $\epsilon$  is a positive parameter.

Using the properties of mollifiers, we obtain

$$\begin{aligned} &\|Q - Q_\tau\|_{\mathcal{L}(H^{\frac{1}{2}, 1, \tau}(G); L^2(G))} + \|P - P_\tau\|_{\mathcal{L}(H^{\frac{1}{2}, 1, \tau}(G); L^2(G))} \\ &\leq C(\pi_{C^0(Q-O_\tau)} + \pi_{C^0(P-P_\tau)}) \leq \frac{C_\epsilon}{(1 + |\tau|)^{\frac{1}{2} + \epsilon}}, \end{aligned} \quad (2.119)$$

$$(1 + |\tau|) \|\tilde{\Lambda}^{-\frac{1}{2}} Q_\tau w\|_{L^2(G)}^2 \leq \frac{\hat{C}}{2} \|Q_\tau w\|_{L^2(G)}^2, \quad (2.120)$$

where

$$\tilde{\Lambda}^{-\frac{1}{2}} w = \int_{\mathbb{R}^n} \left(1 + \xi_0^2 + \tau^4 + \sum_{i=1}^{n-1} \xi_i^4\right)^{-\frac{1}{4}} \hat{w} e^{i\langle x', \xi' \rangle} d\xi'.$$

By (2.117) and (2.118), there exists a  $\tau_0 > 0$  such that

$$\frac{\tau_0 |Q_\tau(x, \xi', \tau)|^2}{M(\xi', \tau)} + \text{Re}\{Q_\tau, P_\tau\}(x, \xi, \tau) > CM(\xi', \tau). \quad (2.121)$$

Some short computations provide

$$\begin{aligned} &\tau_0 \|Q_\tau \tilde{\Lambda}^{-\frac{1}{2}} w\|_{L^2(G)}^2 + ([Q_\tau, P_\tau]w, w)_{L^2(G)} \\ &= (\tau_0 (Q_\tau \tilde{\Lambda}^{-\frac{1}{2}})^* Q_\tau \tilde{\Lambda}^{-\frac{1}{2}} w, w)_{L^2(G)} + ([Q_\tau, P_\tau]w, w)_{L^2(G)} \\ &= \text{Re}((\tau_0 (Q_\tau \tilde{\Lambda}^{-\frac{1}{2}})^* Q_\tau \tilde{\Lambda}^{-\frac{1}{2}} + [Q_\tau, P_\tau])w, w)_{L^2(G)}. \end{aligned}$$

By (2.121) and applying Gårding’s inequality, we have

$$\begin{aligned} & \tau_0 \|Q_\tau \tilde{\Lambda}^{-\frac{1}{2}} w\|_{L^2(G)}^2 + \operatorname{Re}([Q_\tau, P_\tau]w, w)_{L^2(G)} \\ & \geq C \int_0^\gamma \|w(\cdot, x_n)\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(\mathbb{R}^n)}^2 dx_n - C_1 \|w\|_{L^2(G)}^2. \end{aligned} \tag{2.122}$$

Next, we observe

$$\begin{aligned} \|Q_\tau \tilde{\Lambda}^{-\frac{1}{2}} w\|_{L^2(G)} &= \|\tilde{\Lambda}^{-\frac{1}{2}} Q_\tau w + [Q_\tau, \tilde{\Lambda}^{-\frac{1}{2}}]w\|_{L^2(G)} \\ &\leq C_0 (\|\tilde{\Lambda}^{-\frac{1}{2}} Q_\tau w\|_{L^2(G)} + \|w\|_{L^2(G)}) \\ &\leq C \left( \frac{\|Q_\tau w\|_{L^2(G)}}{1 + |\tau|} + \|w\|_{L^2(G)} \right), \end{aligned} \tag{2.123}$$

where in the last inequality we used the estimate (2.120). Combining (2.123) and (2.122), we obtain

$$\|Q_\tau w\|_{L^2(G)}^2 + \operatorname{Re}([Q_\tau, P_\tau]w, w)_{L^2(G)} \geq C \int_0^\gamma \|w(\cdot, x_n)\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(\mathbb{R}^n)}^2 dx_n - C_1 \|w\|_{L^2(G)}^2. \tag{2.124}$$

Since there exists a constant  $\tilde{C}$  independent of  $\tau$  such that

$$|Q_\tau(x, \xi', \tau)|^2 + \tilde{C}|\tau|^2 \geq M^2(\xi', \tau),$$

Gårding’s inequality yields

$$\|Q_\tau w\|_{L^2(G)}^2 + \tau^2 \|w\|_{L^2(G)}^2 \geq C \|w\|_{H^{\frac{1}{2}, 1, \tau}(G)}^2.$$

This inequality and (2.124) imply

$$\|Q_\tau w\|_{L^2(G)}^2 + \operatorname{Re}([Q_\tau, P_\tau]w, w)_{L^2(G)} \geq C \int_0^\gamma \frac{\|w(\cdot, x_n)\|_{H^{\frac{1}{2}, 1, \tau}(\mathbb{R}^n)}^2}{1 + |\tau|} dx_n - C_1 \|w\|_{L^2(G)}^2. \tag{2.125}$$

On the other hand,

$$\left\| \frac{\partial w}{\partial x_n} \right\|_{L^2(G)} \leq \|P_\tau w\|_{L^2(G)} + C \|w\|_{H^{\frac{1}{2}, 1, \tau}(G)}.$$

Hence (2.125) can be transformed into the following estimate:

$$\|P_\tau w\|_{L^2(G)}^2 + \|Q_\tau w\|_{L^2(G)}^2 + \operatorname{Re}([Q_\tau, P_\tau]w, w)_{L^2(G)} \geq C \|w\|_{H^{\frac{1}{2}, 1, \tau}(G)}^2 - C_1 \|w\|_{L^2(G)}^2. \tag{2.126}$$

By (2.119), we can put in the left-hand side of (2.126) the  $L^2$ -norms of the functions  $Pw$  and  $Qw$  instead of  $P_\tau w$  and  $Q_\tau w$  respectively. The proof of the proposition is finished.

By (2.116) and Proposition 2.3, there exist two positive constants  $C$  and  $C_1$  independent of  $\beta, \tau$  such that

$$\begin{aligned} \|z\|_{L^2(G)}^2 &\geq \frac{1}{2} \|Qw_1\|_{L^2(G)}^2 + \frac{1}{2} \|Pw_1\|_{L^2(G)}^2 + C \|w_1\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)}^2 - C_1 \|w_1\|_{L^2(G)}^2 \\ &\quad - \operatorname{Re}(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.127}$$

Depending on the sign of the fifth term on the right-hand side of (2.127), we consider two cases.



**Case 1** Assume that  $\operatorname{Re}(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)} \leq 0$ . Then (2.127) implies

$$\|w_1\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)} \leq C(\|\tilde{f}_1\|_{H^{-\frac{1}{2}, -1, \tau}(G)} + \|w_1\|_{L^2(G)}). \quad (2.128)$$

**Case 2** Assume that  $\operatorname{Re}(Q(x', 0, D', \tau)w_1(\cdot, 0), w_1(\cdot, 0))_{L^2(\mathbb{R}^n)} \geq 0$ . By (2.80), (2.89) and (2.106), there exists a constant  $C > 0$  independent of  $\tau, \beta$  such that

$$\sqrt{1 + |\tau|} \|w_1(\cdot, 0)\|_{L^2(\mathbb{R}^n)} \leq C \|w_1(\cdot, 0)\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}. \quad (2.129)$$

Let us consider the following (adjoint) problem:

$$L_+(x, D, \tau)^* p = \left( -\frac{\partial}{\partial x_n} - R_+(x, D', \tau)^* \right) p = (1 + |\tau|)w_1 + v, \quad \text{in } G. \quad (2.130)$$

Set

$$m_\tau(x) = 1, \quad x \in \Pi_{\delta_0, \gamma}, \quad m_1(x) = (1 + |\tau|)^{-1}, \quad x \in \mathbb{R}^{n+1} \setminus \Pi_{\delta_0, \gamma},$$

where  $\delta_0 \in (\delta, \delta')$ . We have

**Lemma 2.2** *There exist a constant  $C > 0$  independent of  $\tau$  and a pair  $(p, v)$  satisfying (2.130) such that*

$$\sqrt{1 + |\tau|} \int_G (|p|^2 + m_\tau^2 |v|^2) dx + \int_{\mathbb{R}^n} m_\tau^2 |p(x', 0)|^2 dx' \leq C(1 + |\tau|)^{\frac{3}{2}} \int_G |w_1|^2 dx. \quad (2.131)$$

**Proof** For  $\epsilon > 0$ , let us consider the functional

$$J_\epsilon(p, v) = \frac{1}{2} \|p\|_{L^2(G)}^2 + \frac{1}{2} \|m_\tau v\|_{L^2(G)}^2 + \frac{1}{2\epsilon} \left\| \frac{\partial p}{\partial x_n} + R_+(x, D', \tau)^* p + (1 + |\tau|)w_1 + v \right\|_{L^2(G)}^2. \quad (2.132)$$

Note that there exists a pair  $(p, v)$  such that  $J_\epsilon(p, v)$  is finite, for example  $(p, v) = 0$ . We consider the minimization problem

$$\min_{(p, v) \in U} J_\epsilon(p, v),$$

where

$$U = \left\{ (p, v) \in L^2(G) \times L^2(G) \mid \frac{\partial p}{\partial x_n} + R_+(x, D', \tau)^* p + (1 + |\tau|)w_1 + v \in L^2(G) \right\}.$$

We will follow a typical argument for the minimization problem (e.g., [14]). There exists a minimizing sequence  $\{(p_k, v_k)\}_{k=1}^\infty$  such that  $(p_k, v_k) \in U$  and

$$J_\epsilon(p_k, v_k) \rightarrow \inf_{(p, v) \in U} J_\epsilon(p, v).$$

Then  $\|(p_k, v_k)\|_{L^2(G)}$  and  $\left\| \frac{\partial p_k}{\partial x_n} + R_+(x, D', \tau)^* p_k + (1 + |\tau|)w_1 + v_k \right\|_{L^2(G)}$  are bounded. Therefore  $R_+(x, D', \tau)^* p_k$  and  $\frac{\partial p_k}{\partial x_n}$  are both bounded in  $L^2(0, \gamma; H^{-\frac{1}{2}, -1, \tau}(\mathbb{R}^n))$ .

We can then extract a subsequence, still denoted by  $\{(p_k, v_k)\}_{k=1}^\infty$ , such that

$$\begin{aligned} (p_k, v_k) &\rightharpoonup (p_\epsilon, v_\epsilon), \quad \text{weakly in } L^2(G) \times L^2(G), \\ \frac{\partial p_k}{\partial x_n} &\rightharpoonup \frac{\partial p_\epsilon}{\partial x_n}, \quad \text{weakly in } L^2(0, \gamma; H^{-1}(\mathbb{R}^n)), \\ \frac{\partial p_k}{\partial x_n} + R_+(x, D', \tau)^* p_k + (1 + |\tau|)w_1 + v_k &\rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon + (1 + |\tau|)w_1 + v_\epsilon, \\ &\text{weakly in } L^2(0, \gamma; H^{-\frac{1}{2}, -1}(\mathbb{R}^n)). \end{aligned}$$

However, since  $\|\frac{\partial p_k}{\partial x_n} + R_+(x, D', \tau)^* p_k + (1 + |\tau|)w_1 + v_k\|_{L^2(G)}^2$  stays bounded, we have

$$\frac{\partial p_k}{\partial x_n} + R_+(x, D', \tau)^* p_k + (1 + |\tau|)w_1 + v_k \rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon + (1 + |\tau|)w_1 + v_\epsilon,$$

weakly in  $L^2(G)$ . Then  $(p_\epsilon, v_\epsilon)$  is a minimizer of  $J_\epsilon$ , i.e.,  $(p_\epsilon, v_\epsilon) \in U$  and

$$J_\epsilon(p_\epsilon, v_\epsilon) = \min_{(p,v) \in U} J_\epsilon(p, v). \tag{2.133}$$

Writing the first order optimality conditions, we have

$$\langle \partial_p J_\epsilon(p_\epsilon, v_\epsilon), r \rangle = 0, \quad \langle \partial_v J_\epsilon(p_\epsilon, v_\epsilon), \tilde{r} \rangle = 0, \quad \forall r \in H^{\frac{1}{2},1}(G), \quad \forall \tilde{r} \in L^2(G). \tag{2.134}$$

Let us define  $q_\epsilon$  by

$$q_\epsilon = \frac{1}{\epsilon} \left( \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon + (1 + |\tau|)w_1 + v_\epsilon \right). \tag{2.135}$$

We obtain from (2.134) that for every  $r \in H^{\frac{1}{2},1}(G)$ , there holds

$$\int_G p_\epsilon \tilde{r} dx + \int_G q_\epsilon \overline{\left( \frac{\partial r}{\partial x_n} + R_+(x, D', \tau)^* r \right)} dx = 0 \tag{2.136}$$

and for every  $\tilde{r} \in L^2(G)$ , there holds

$$\int_G v_\epsilon \tilde{r} dx + \int_G q_\epsilon \tilde{r} dx = 0. \tag{2.137}$$

Then  $q_\epsilon$  satisfies the following problem:

$$L_+(x, D, \tau)q_\epsilon = \frac{\partial q_\epsilon}{\partial x_n} - R_+(x, D', \tau)q_\epsilon = p_\epsilon, \quad \text{in } G, \tag{2.138}$$

$$q_\epsilon = -m_\tau^2 v_\epsilon, \quad \text{in } G, \tag{2.139}$$

$$q_\epsilon(x', 0) = 0, \quad q_\epsilon(x', \gamma) = 0, \quad x' \in \mathbb{R}^n. \tag{2.140}$$

We can also write (2.138)–(2.140) in the form

$$L_+(x, D, \tau)q_\epsilon = (P + Q)(x, D, \tau)q_\epsilon = p_\epsilon, \quad \text{in } G, \tag{2.141}$$

$$q_\epsilon = -m_\tau^2 v_\epsilon, \quad \text{in } G, \tag{2.142}$$

$$q_\epsilon(x', 0) = 0, \quad q_\epsilon(x', \gamma) = 0, \quad x' \in \mathbb{R}^n. \tag{2.143}$$

Using Proposition 2.3 and (2.142), we obtain that there exists a constant  $C > 0$  such that

$$\|Qq_\epsilon\|_{L^2(G)} + \|Pq_\epsilon\|_{L^2(G)} + \|q_\epsilon\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)} \leq C\|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}. \tag{2.144}$$

Notice that  $Qq_\epsilon \in L^2(G)$  implies  $q_\epsilon \in L^2(0, \gamma; H^{\frac{1}{2},1}(\mathbb{R}^n))$ , which implies  $\frac{\partial q_\epsilon}{\partial x_n} \in L^2(G)$  from (2.138). Now, from the definition (2.135) of  $q_\epsilon$ , we see that  $p_\epsilon$  satisfies

$$\frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon = \epsilon q_\epsilon - (1 + |\tau|)w_1 - v_\epsilon,$$

which can be written as

$$(P - Q)(x, D, \tau)p_\epsilon = \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon = \epsilon q_\epsilon - (1 + |\tau|)w_1 - v_\epsilon. \tag{2.145}$$

Multiplying (2.145) by  $q_\epsilon$  in  $L^2(G)$  and using the boundary conditions on  $q_\epsilon$ , we obtain

$$-\int_G p_\epsilon \overline{L_+(x, D, \tau)q_\epsilon} dx = \epsilon \int_G |q_\epsilon|^2 dx - \int_G (1 + |\tau|)w_1 \overline{q_\epsilon} dx + \int_G m_\tau^2 v_\epsilon^2 dx,$$

and so

$$\int_G |p_\epsilon|^2 dx + \int_G m_\tau^2 |v_\epsilon|^2 dx + \epsilon \int_G |q_\epsilon|^2 dx = \int_G (1 + |\tau|)w_1 \overline{q_\epsilon} dx.$$

And by (2.144), we have

$$\begin{aligned} \|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}^2 &\leq \left( \int_G (1 + |\tau|)|w_1|^2 dx \right)^{\frac{1}{2}} \left( \int_G (1 + |\tau|)|q_\epsilon|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left( \int_G (1 + |\tau|)|w_1|^2 dx \right)^{\frac{1}{2}} \|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}. \end{aligned}$$

Thus, we obtain the first estimate on  $(p_\epsilon, v_\epsilon)$ ,

$$\|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)} \leq C \sqrt{1 + |\tau|} \|w_1\|_{L^2(G)}. \quad (2.146)$$

By (2.146), there exists a subsequence  $\{(p_{\epsilon_m}, v_{\epsilon_m})\}_{m=1}^\infty$  such that

$$\begin{aligned} (p_{\epsilon_m}, v_{\epsilon_m}) &\rightarrow (p, v), \quad \text{in } L^2(G) \times L^2(G), \\ x_n^2 p_{\epsilon_m} &\rightarrow x_n^2 p, \quad \text{in } H^{\frac{1}{2}, 1, \tau}(G), \\ q_\epsilon &\rightarrow q, \quad \text{in } H^{\frac{1}{2}, 1}(G). \end{aligned} \quad (2.147)$$

Using the above relations, we pass to the limit in (2.141)–(2.143). The pair  $(p, v, q) \in L^2(G) \times L^2(G) \times L^2(0, \gamma; H^{\frac{1}{2}, 1, \tau}(\mathbb{R}^n))$  satisfies the optimality system

$$L_+(x, D, \tau)^* p = (1 + |\tau|)w_1 + v, \quad \text{in } G, \quad (2.148)$$

$$L_+(x, D, \tau)q = p, \quad \text{in } G, \quad (2.149)$$

$$q = -m_\tau^2 v, \quad \text{in } G, \quad (2.150)$$

$$q(\cdot, 0) = q(\cdot, \gamma) = 0. \quad (2.151)$$

Using Proposition 2.3 and (2.150), we have

$$\|Qq\|_{L^2(G)} + \|Pq\|_{L^2(G)} + \|q\|_{H^{\frac{1}{4}, \frac{1}{2}, \tau}(G)} \leq C \|(p, m_\tau v)\|_{L^2(G)}. \quad (2.152)$$

Inequalities (2.146) and (2.147) imply

$$\|(p, m_\tau v)\|_{L^2(G)} \leq C \sqrt{1 + |\tau|} \|w_1\|_{L^2(G)}. \quad (2.153)$$

Observe that

$$p(\cdot, 0) = \frac{\partial}{\partial x_n} q(\cdot, 0).$$

By (2.148) and (2.149), we have

$$\mathfrak{G}(x, D, \tau)q = L_+(x, D, \tau)^* L_+(x, D, \tau)q = (Q^2 - P^2 + [Q, P])q = (1 + |\tau|)w_1 + v, \quad \text{in } G.$$

Let  $\theta \in C^\infty[0, \gamma]$ ,  $\theta(0) = 1$ ,  $\theta \equiv 0$  in a neighborhood of  $x_n = \gamma$ , and  $\chi(x_0, \dots, x_{n-1}) \in C_0^\infty(B(0, \delta'))$  such that  $\chi|_{B(0, \delta_0)} \equiv 1$ . Denote  $\tilde{\chi} = \chi\theta$ . Then function  $\tilde{\chi}q$  verifies

$$\mathfrak{G}(x, D, \tau)(\tilde{\chi}q) = (1 + |\tau|)\tilde{\chi}w_1 + \tilde{\chi}v - [\tilde{\chi}, \mathfrak{G}]q, \quad \text{in } G.$$

Multiplying this equation by  $P(x, D, \tau)(\tilde{\chi}q)$ , we have

$$\begin{aligned} \operatorname{Re}(\mathfrak{G}(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} &= (Q^2(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} + (P(\tilde{\chi}q), Q^2(\tilde{\chi}q))_{L^2(G)} - (P^2(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} \\ &\quad - (P(\tilde{\chi}q), P^2(\tilde{\chi}q))_{L^2(G)} + \operatorname{Re}([Q, P](\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} \\ &= (1 + |\tau|)(w_1, P(\tilde{\chi}q))_{L^2(G)} + (\tilde{\chi}v - [\tilde{\chi}, \mathfrak{G}]q, P(\tilde{\chi}q))_{L^2(G)}. \end{aligned} \quad (2.154)$$

By (2.152) and (2.150), we obtain

$$\|Q(\tilde{\chi}q)\|_{L^2(G)} + \|P(\tilde{\chi}q)\|_{L^2(G)} + \frac{\|q\|_{H^{\frac{1}{2}, 1, \tau}(G)}}{\sqrt{1 + |\tau|}} \leq C\|(p, m_\tau v)\|_{L^2(G)}. \quad (2.155)$$

Using (2.155) and Lemma A.4, we have

$$\begin{aligned} |([Q, P](\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)}| &\leq \|[Q, P](\tilde{\chi}q)\|_{L^2(G)}\|P(\tilde{\chi}q)\|_{L^2(G)} \\ &\leq C\|q\|_{H^{\frac{1}{2}, 1, \tau}(G)}\|P(\tilde{\chi}q)\|_{L^2(G)} \\ &\leq C\|q\|_{H^{\frac{1}{2}, 1, \tau}(G)}\|(p, m_\tau v)\|_{L^2(G)} \\ &\leq C\sqrt{1 + |\tau|}\|(p, m_\tau v)\|_{L^2(G)}^2. \end{aligned} \quad (2.156)$$

Next

$$\begin{aligned} &(Q^2(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} + (P(\tilde{\chi}q), Q^2(\tilde{\chi}q))_{L^2(G)} \\ &= -(PQ^2(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} + (Q^2P(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} \\ &= -(QPQ(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} + (Q^2P(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} + ([Q, P]Q(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} \\ &= ([Q, P](\tilde{\chi}q), Q(\tilde{\chi}q))_{L^2(G)} + (Q(\tilde{\chi}q), [Q, P]^*(\tilde{\chi}q))_{L^2(G)}. \end{aligned} \quad (2.157)$$

Hence, by (2.155) and Lemma A.4, we have

$$\begin{aligned} |\operatorname{Re}(Q^2(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)}| &\leq C\|q\|_{H^{\frac{1}{2}, 1, \tau}(G)}\|(p, m_\tau v)\|_{L^2(G)} \\ &\leq C\sqrt{1 + |\tau|}\|(p, m_\tau v)\|_{L^2(G)}^2. \end{aligned} \quad (2.158)$$

Finally

$$\begin{aligned} \operatorname{Re}(P^2(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} &= (P^2(\tilde{\chi}q), P(\tilde{\chi}q))_{L^2(G)} + (P(\tilde{\chi}q), P^2(\tilde{\chi}q))_{L^2(G)} \\ &= -(P^3(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} + (P^3(\tilde{\chi}q), \tilde{\chi}q)_{L^2(G)} - \|\chi p(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 \\ &= -\|\chi p(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (2.159)$$

Note that

$$\begin{aligned} |(\tilde{\chi}v - [\tilde{\chi}, \mathfrak{G}]q, P(\tilde{\chi}q))_{L^2(G)}| &\leq C(\|v\|_{L^2(G)} + \|Pq\|_{L^2(G)} + \|Qq\|_{L^2(G)})\|P(\tilde{\chi}q)\|_{L^2(G)} \\ &\leq C\|(p, m_\tau v)\|_{L^2(G)}^2. \end{aligned} \quad (2.160)$$

By (2.154)–(2.160), we have

$$\sqrt{1 + |\tau|}\|\chi p(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 \leq C(1 + |\tau|)\|(p, m_\tau v)\|_{L^2(G)}^2.$$

By (2.153), the right-hand side of this inequality can be estimated as follows:

$$\|\chi p(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 \leq C\sqrt{1 + |\tau|}\|(p, m_\tau v)\|_{L^2(G)}^2 \leq C(1 + |\tau|)^{\frac{3}{2}}\|w_1\|_{L^2(G)}^2. \quad (2.161)$$

The proof of Lemma 2.2 is complete.

We recall that

$$\text{supp } w(\cdot, x_n) \subset B(0, \delta), \quad \text{supp } g(\cdot, x_n) \subset B(0, \delta), \quad \forall x_n \in [0, \gamma]. \quad (2.162)$$

Then by Lemma A.5, for any  $\delta_2 > \delta$ ,

$$\|\chi_1 w(\cdot, x_n)\|_{L^2(\mathbb{R}^n \setminus B(0, \delta_2))} \leq \frac{C(\delta_2)}{1 + |\tau|} \|w(\cdot, x_n)\|_{L^2(\mathbb{R}^n)}, \quad (2.163)$$

$$\|\chi_1 g(\cdot, x_n)\|_{L^2(\mathbb{R}^n \setminus B(0, \delta_2))} \leq \frac{C(\delta_2)}{1 + |\tau|} \|g(\cdot, x_n)\|_{L^2(\mathbb{R}^n)}. \quad (2.164)$$

Taking the scalar products in  $L^2(G)$  of (2.130) by  $w_1$ , integrating by parts and using (2.163), we have

$$\begin{aligned} & (1 + |\tau|) \|w_1\|_{L^2(G)}^2 \\ &= (w_1, L_+(x, D, \tau)^* p - v)_{L^2(G)} \\ &= (L_+(x, D, \tau) w_1, p)_{L^2(G)} - (w_1, v)_{L^2(G)} + (\chi_1 \tilde{g}, p(\cdot, 0))_{L^2(\mathbb{R}^n)} \\ &= -(w_1, v)_{L^2(G)} + (\chi_1 \tilde{g}, p(\cdot, 0))_{L^2(\mathbb{R}^n)} + (\chi_1 z, p)_{L^2(G)} + ([L_+, \chi_1] w, p)_{L^2(G)} \\ &\leq C(\|z\|_{L^2(G)} \|p\|_{L^2(G)} + \|\tilde{g}\|_{L^2(\mathbb{R}^n)} \|m_\tau p(\cdot, 0)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \|w\|_{L^2(G)} \|m_\tau v\|_{L^2(G)} + \|w\|_{L^2(G)} \|p\|_{L^2(G)}). \end{aligned}$$

By (2.131), from this inequality, we obtain

$$\begin{aligned} (1 + |\tau|) \|w_1\|_{L^2(G)}^2 &\leq C(\|z\|_{L^2(G)} \|p\|_{L^2(G)} + \|\tilde{g}\|_{L^2(\mathbb{R}^n)} (1 + |\tau|)^{\frac{3}{4}} \|w_1\|_{L^2(G)} \\ &\quad + \|w\|_{L^2(G)} \|m_\tau v\|_{L^2(G)}) \\ &\leq C(\|z\|_{L^2(G)} \sqrt{1 + |\tau|} \|w_1\|_{L^2(G)} + \|\tilde{g}\|_{L^2(\mathbb{R}^n)} (1 + |\tau|)^{\frac{3}{4}} \|w_1\|_{L^2(G)} \\ &\quad + \sqrt{1 + |\tau|} \|w_1\|_{L^2(G)}^2 + \sqrt{1 + |\tau|} \|w\|_{L^2(G)}^2). \end{aligned}$$

This inequality and (2.112) imply

$$\begin{aligned} \|w_1\|_{L^2(G)} &\leq C \left( \frac{\|\chi_1 \tilde{g}\|_{L^2(\mathbb{R}^n)}}{(1 + |\tau|)^{\frac{1}{4}}} + \frac{\|z\|_{L^2(G)}}{\sqrt{1 + |\tau|}} + \frac{\|w\|_{L^2(G)}}{\sqrt{1 + |\tau|}} \right) \\ &\leq C \left( \frac{\|\chi_1 \tilde{g}\|_{L^2(\mathbb{R}^n)}}{(1 + |\tau|)^{\frac{1}{4}}} + \frac{\|\tilde{f}_1\|_{H^{-\frac{1}{2}, -1, \tau}(G)}}{\sqrt{1 + |\tau|}} + \frac{\|w\|_{L^2(G)}}{\sqrt{1 + |\tau|}} \right). \end{aligned} \quad (2.165)$$

By (2.129) and (2.165), we have

$$\sqrt{1 + |\tau|} \|w_1\|_{L^2(G)} \leq C \left( \frac{\|\chi_1 \tilde{g}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}}{(1 + |\tau|)^{\frac{1}{4}}} + \|\tilde{f}_1\|_{H^{-\frac{1}{2}, -1, \tau}(G)} + \|w\|_{L^2(G)} \right). \quad (2.166)$$

Taking into account (2.166), (2.128) and (2.97), we obtain (2.83). Really,

$$\begin{aligned} \sqrt{1 + |\tau|} \|w\|_{L^2(G)} &= \sqrt{1 + |\tau|} \sum_{\mu=0}^1 \|w_\mu\|_{L^2(G)} \\ &\leq C \sum_{\mu=0}^1 (\|\chi_\mu \tilde{f} + [\chi_\mu, L(x, D, \tau)] w\|_{H^{-\frac{1}{2}, -1, \tau}(G)} + (1 + |\tau|)^{-\frac{1}{4}} \|\chi_\mu \tilde{g}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)}) \end{aligned}$$

$$\leq C \left( \|\tilde{f}\|_{H^{-\frac{1}{2}, -1, \tau}(G)} + (1 + |\tau|)^{-\frac{1}{4}} \|\tilde{g}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + \frac{\|w\|_{H^{\frac{1}{2}, 1, \tau}(G)}}{(1 + |\tau|)^{1 - \frac{1}{\kappa}}} \right). \tag{2.167}$$

Then, from the energy estimate for solutions of problem (2.76)–(2.78), we have

$$\|w\|_{H^{\frac{1}{2}, 1}(G)} \leq C(\|g\|_{H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)} + (1 + |\tau|)\|w\|_{L^2(G)} + \|\tilde{f}\|_{H^{-\frac{1}{2}, -1, \tau}(G)}). \tag{2.168}$$

By (2.167) and (2.168), we obtain (2.83). The proof of Theorem 2.1 is complete.

### 3 Carleman Estimate for the Stokes System

Consider the Stokes system

$$P(D)y = \frac{\partial y}{\partial x_0} - \Delta y = \nabla p + f, \quad \text{in } Q, \tag{3.1}$$

$$\operatorname{div} y = 0, \quad \text{in } Q,$$

$$y = 0, \quad \text{on } (0, T) \times \Gamma,$$

$$y(0, x) = y_0. \tag{3.2}$$

We introduce the following spaces

$$H = \{u = (u_1, \dots, u_n) \in (L^2(\Omega))^n \mid \operatorname{div} u = 0, (u, \nu)|_{\partial\Omega} = 0\}, \quad n = 2, 3,$$

$$V = \{u = (u_1, \dots, u_n) \in (H_0^1(\Omega))^n \mid u \in H\}.$$

The proof of the following proposition can be found in the classical book [18].

**Proposition 3.1** (1) *Let  $y_0 \in H$  and  $f \in L^2(0, T; V')$ . Then there exists a unique solution  $y$  to problem (3.1)–(3.2) with  $y \in L^2(0, T; V) \cap C(0, T; H)$ ,  $\frac{\partial y}{\partial x_0} \in L^2(0, T; V')$ .*

(2) *Let  $y_0 \in V$ ,  $f \in L^2(0, T; H)$ . Then there exists a solution  $(y, p)$  to problem (3.1)–(3.2) with  $(y, p) \in C(0, T; V) \cap H^{1,2}(Q) \times L^2(0, T; H^1(\Omega))$ , and the following a priori estimate holds:*

$$\|(y, p)\|_{H^{1,2}(Q) \times L^2(Q)} \leq C(\|y_0\|_V + \|f\|_{L^2(0, T; H)}). \tag{3.3}$$

The goal of this section is to prove the following Carleman estimate for solutions of problem (3.1)–(3.2).

**Theorem 3.1** *Let  $\kappa = 8$ ,  $f \in L^2(0, T; H)$ ,  $y_0 \in V$  and  $y \in L^2(0, T; V) \cap H^{1,2}(Q)$  be a solution of (3.1)–(3.2). Then there exists a constant  $\hat{\lambda}$  such that for any  $\lambda > \hat{\lambda}$ , there exist constants  $C > 0$  and  $\hat{s}$  independent of  $s$ , such that*

$$\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{rot} y)e^{s\alpha}\|_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q)} \leq C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{rot} y)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q_\omega)}), \quad \forall s \geq \hat{s}. \tag{3.4}$$

The Carleman estimates for the Stokes system are applicable to the exact controllability problem. See [4, 6, 10, 11] as related works.

First, we prove the following simple proposition.

**Proposition 3.2** *Let  $u \in H^{1,2}(Q) \cap L^2(0, T; H_0^1(\Omega))$ . Then  $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)$ .*

**Proof** Let  $\vec{n}(x_1, \dots, x_n) \in C^1(\bar{\Omega})$  be a smooth vector field such that  $\vec{n} = \nu$  on  $\partial\Omega$ . Since the function  $\sum_{i=1}^n n_i \partial_{x_i} u \in L^2(0, T; H^1(\Omega))$ , we have  $\frac{\partial u}{\partial \nu} \in L^2(0, T; H^{\frac{1}{2}}(\partial\Omega))$ . In order to show that  $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{4}}(0, T; L^2(\partial\Omega))$ , we observe that by using a partition of unity and a local change of variables, it suffices to consider a situation when  $\Omega = \{(x_1, \dots, x_n) \mid x_n > 0\}$  and the function  $u$  has a support in  $B(0, \delta) \cap \{(x_1, \dots, x_n) \mid x_n \geq 0\}$ . Denote by  $\widehat{u}$  the Fourier transform with respect to the variables  $x_0, \dots, x_{n-1}$ . Then

$$\begin{aligned} \|u\|_{H^{\frac{1}{4}}(0, T; L^2(\partial\Omega))}^2 &= - \int_{\mathbb{R}_+^{n+1}} \frac{\partial}{\partial x_n} \left| \frac{\partial \widehat{u}}{\partial x_n} \right|^2 \sqrt{1 + |\xi_0|} d\xi_0 \cdots d\xi_{n-1} dx_n \\ &= - \int_{\mathbb{R}_+^{n+1}} (\partial_{x_n x_n}^2 \widehat{u} \overline{\partial_{x_n} \widehat{u}} + \partial_{x_n} \widehat{u} \partial_{x_n x_n}^2 \overline{\widehat{u}}) \sqrt{1 + |\xi_0|} d\xi_0 \cdots d\xi_{n-1} dx_n \\ &\leq \int_{\mathbb{R}_+^{n+1}} (|\partial_{x_n x_n}^2 \widehat{u}|^2 + (1 + |\xi_0|) |\partial_{x_n} \widehat{u}|^2) d\xi_0 \cdots d\xi_{n-1} dx_n. \end{aligned} \quad (3.5)$$

Integrating by parts and taking into account the Dirichlet boundary conditions we have

$$\int_{\mathbb{R}_+^{n+1}} (1 + |\xi_0|) |\partial_{x_n} \widehat{u}|^2 d\xi_0 \cdots d\xi_{n-1} dx_n = \int_{\mathbb{R}_+^{n+1}} (1 + |\xi_0|) \widehat{u} \overline{\partial_{x_n x_n}^2 \widehat{u}} d\xi_0 \cdots d\xi_{n-1} dx_n.$$

Applying the Cauchy-Bynakovskii inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}_+^{n+1}} (1 + |\xi_0|) |\partial_{x_n} \widehat{u}|^2 d\xi_0 \cdots d\xi_{n-1} dx_n \\ &\leq \int_{\mathbb{R}_+^{n+1}} (|\partial_{x_n x_n}^2 \widehat{u}|^2 + (1 + |\xi_0|)^2 |\widehat{u}|^2) d\xi_0 \cdots d\xi_{n-1} dx_n. \end{aligned} \quad (3.6)$$

Combining (3.5) and (3.6), we have

$$\|u\|_{H^{\frac{1}{4}}(0, T; L^2(\partial\Omega))} \leq C \|u\|_{H^{1,2}(Q)}.$$

The proof of the proposition is finished.

**Proof of Theorem 3.1** Applying the operator  $\text{rot}$  to equation (3.1), we have

$$\frac{\partial \text{rot } y}{\partial x_0} - \Delta \text{rot } y = \text{rot } f, \quad \text{in } Q. \quad (3.7)$$

Next, we apply the Carleman estimate (2.38) to (3.7). There exists an  $s_0 > 0$  such that

$$\begin{aligned} s \int_Q \varphi |\text{rot } y|^2 e^{2s\alpha} dx &\leq C \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} \text{rot } y e^{s\widehat{\alpha}}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \text{rot } y e^{s\widehat{\alpha}}\|_{L^2(\Sigma)}^2 \right. \\ &\quad \left. + \int_Q |f|^2 e^{2s\alpha} dx + \int_{Q_\omega} s \varphi |\text{rot } y|^2 e^{2s\alpha} dx \right), \quad \forall s \geq s_0, \end{aligned} \quad (3.8)$$

where we set  $\widehat{\alpha}(x_0) = \alpha(x)|_{\partial\Omega}$  by (2.8). Since  $\text{div } y = 0$ , we have  $-\Delta y = \text{rot rot } y$ . Setting  $u = y e^{s\alpha}$ , we obtain

$$-e^{s\alpha} \Delta e^{-s\alpha} u = e^{s\alpha} \text{rot rot } y = \text{rot}(e^{s\alpha} \text{rot } y) + [e^{s\alpha}, \text{rot}] \text{rot } y.$$

Notice that

$$[e^{s\alpha}, \text{rot}] \text{rot } y(x) = s \frac{c(\tilde{x})}{\ell(x_0)^\kappa} (e^{s\alpha} \text{rot } y), \quad (3.9)$$

where  $c \in (C^1(\overline{\Omega}))^3$  is some function.

Applying the Carleman estimate for elliptic equations obtained in [12] and using (3.9), we have

$$\begin{aligned} & \sqrt{\frac{s}{\ell^\kappa(x_0)}} \|ye^{s\alpha}\|_{L^2(\Omega)} \\ & \leq C \left( \|\operatorname{rot} ye^{s\alpha}\|_{L^2(\Omega)} + \sqrt{\frac{s}{\ell^\kappa(x_0)}} \|ye^{s\alpha}\|_{L^2(\omega)} \right), \quad \forall s \geq s_1, \forall x_0 \in [0, T], \end{aligned} \tag{3.10}$$

where the constants  $C$  and  $s_0$  are independent of  $s, x_0$ . Combining (3.8) and (3.10), we obtain

$$\begin{aligned} & \int_Q s\varphi |\operatorname{rot} y|^2 e^{2s\alpha} dx + s^2 \|\varphi ye^{s\alpha}\|_{L^2(Q)}^2 \\ & \leq C \left( s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{L^2(\Sigma)}^2 + \int_Q |f|^2 e^{2s\alpha} dx \right. \\ & \quad \left. + \int_{Q_\omega} s\varphi |\operatorname{rot} y|^2 e^{2s\alpha} dx + s^2 \|\varphi ye^{s\alpha}\|_{L^2(Q_\omega)}^2 \right), \quad \forall s \geq \max\{s_0, s_1\}. \end{aligned} \tag{3.11}$$

We need to estimate the first term on the right-hand side of (3.11). Denote

$$(w, q) = \ell(ye^{s\hat{\alpha}(x_0)}, pe^{s\hat{\alpha}(x_0)}).$$

The pair  $(w, q)$  solves the following initial/boundary value problem:

$$P(D)w = \frac{\partial w}{\partial x_0} - \Delta w = \nabla q + fle^{s\hat{\alpha}} + s\hat{\alpha}'lw + ye^{s\hat{\alpha}}\ell', \quad \text{in } Q, \tag{3.12}$$

$$\operatorname{div} w = 0, \quad \text{in } Q,$$

$$w = 0, \quad \text{on } (0, T) \times \Gamma,$$

$$w(0, \cdot) = 0. \tag{3.13}$$

By Proposition 3.1 and the fact that  $\hat{\alpha}(x_0) \leq \alpha(x)$  for all  $x \in Q$ , for solutions to problem (3.12)–(3.13), we have

$$\begin{aligned} \|(w, q)\|_{H^{1,2}(Q) \times L^2(0,T;H^1(\Omega))} & \leq C(\|fe^{s\hat{\alpha}}\|_{L^2(Q)} + \|sye^{s\hat{\alpha}}\|_{L^2(Q)}) \\ & \leq C(\|fe^{s\alpha}\|_{L^2(Q)} + \|sye^{s\alpha}\|_{L^2(Q)}). \end{aligned} \tag{3.14}$$

Using Proposition 3.2, we observe that there exists a constant  $C > 0$  such that

$$\left\| \frac{\partial w}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)} \leq C \|w\|_{H^{1,2}(Q)}. \tag{3.15}$$

Fix  $\kappa = 8$ . Then

$$\|s\varphi^{\frac{3}{4} + \frac{3}{2\kappa}} ye^{s\alpha}\|_{L^2(Q)} \leq C \|s\varphi ye^{s\alpha}\|_{L^2(Q)}.$$

Combining (3.15) with (3.14), we have

$$\left\| \frac{\partial w}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)} \leq C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q)}). \tag{3.16}$$

Hence

$$\begin{aligned} & s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{L^2(\Sigma)}^2 \\ & \leq Cs^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 \leq Cs^{-\frac{1}{2}} \|w\|_{H^{1,2}(Q)}^2 \\ & \leq Cs^{-\frac{1}{2}} (\|fe^{s\alpha}\|_{L^2(Q)}^2 + \|s\varphi ye^{s\alpha}\|_{L^2(Q)}^2). \end{aligned} \tag{3.17}$$



The first two terms on the right-hand side of (3.11) can be estimated by the right-hand side of (3.17). Observe that the term  $Cs^{-\frac{1}{2}}\|s\varphi ye^{s\alpha}\|_{L^2(Q)}^2$  can be absorbed by the left-hand side of (3.11) for all sufficiently large  $s$ . This proves the statement of the theorem.

#### 4 Observability Estimate for a Parabolic System with Parameter

Consider the system of parabolic equations

$$P(D)y = \frac{\partial y}{\partial x_0} - \Delta y - \frac{1}{\varepsilon} \nabla \operatorname{div} y = f, \quad \text{in } Q, \quad (4.1)$$

$$y = 0, \quad \text{on } (0, T) \times \Gamma, \quad (4.2)$$

$$y(0, x) = y_0. \quad (4.3)$$

Here  $\varepsilon$  is a positive parameter. The goal of this section is to obtain an observability estimate for system (4.1)–(4.3), which is uniform with respect to the small parameter  $\varepsilon$ . We have

**Theorem 4.1** *Let  $\kappa = 8$ ,  $f \in L^2(Q)$ ,  $y_0 \in H_0^1(\Omega)$ , and  $y \in L^2(0, T; H_0^1(\Omega)) \cap H^{1,2}(Q)$  be a solution of (4.1)–(4.3). Then there exists a constant  $\widehat{\lambda}$  such that for any  $\lambda > \widehat{\lambda}$ , there exist constants  $C > 0$  and  $\widehat{s}$  independent of  $s$ , such that*

$$\begin{aligned} & \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{div} y)e^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{rot} y)e^{s\alpha}\|_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q)} \\ & \leq C(\|fe^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{div} y)e^{s\alpha}\|_{L^2(Q_\omega)} \\ & \quad + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{rot} y)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q_\omega)}), \quad \forall s \geq \widehat{s}. \end{aligned} \quad (4.4)$$

**Proof** Applying the operators  $\operatorname{rot}$  and  $\operatorname{div}$  to equation (4.1), we have

$$\frac{\partial \operatorname{rot} y}{\partial x_0} - \Delta \operatorname{rot} y = \operatorname{rot} f, \quad \text{in } Q, \quad (4.5)$$

$$\frac{\partial \operatorname{div} y}{\partial x_0} - \left(1 + \frac{1}{\varepsilon}\right) \Delta \operatorname{div} y = \operatorname{div} f, \quad \text{in } Q. \quad (4.6)$$

Next, we apply the Carleman estimate (2.38) to (4.5) and (4.6). There exists an  $s_0 > 0$  such that

$$\begin{aligned} s \int_Q \varphi |\operatorname{rot} y|^2 e^{2s\alpha} dx & \leq C \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} \operatorname{rot} ye^{s\widehat{\alpha}}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \operatorname{rot} ye^{s\widehat{\alpha}}\|_{L^2(\Sigma)}^2 \right. \\ & \quad \left. + \int_Q |f|^2 e^{2s\alpha} dx + \int_{Q_\omega} s\varphi |\operatorname{rot} y|^2 e^{2s\alpha} dx \right), \quad \forall s \geq s_0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} s \int_Q \varphi |\operatorname{div} y|^2 e^{2s\alpha} dx & \leq C \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} \operatorname{div} ye^{s\widehat{\alpha}}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \operatorname{div} ye^{s\widehat{\alpha}}\|_{L^2(\Sigma)}^2 \right. \\ & \quad \left. + \int_Q |f|^2 e^{2s\alpha} dx + \int_{Q_\omega} s\varphi |\operatorname{div} y|^2 e^{2s\alpha} dx \right), \quad \forall s \geq s_0, \end{aligned} \quad (4.8)$$

where  $\widehat{\alpha}(x_0) = \alpha(x)|_{\partial\Omega}$ . Using the formula  $\Delta y = -\operatorname{rot} \operatorname{rot} y + \nabla \operatorname{div} y$  and setting  $u = ye^{s\alpha}$ , we obtain

$$\begin{aligned} e^{s\alpha} \Delta e^{-s\alpha} u & = e^{s\alpha} (-\operatorname{rot} \operatorname{rot} y + \nabla \operatorname{div} y) \\ & = -\operatorname{rot}(e^{s\alpha} \operatorname{rot} y) + \nabla(e^{s\alpha} \operatorname{div} y) - [e^{s\alpha}, \operatorname{rot}] \operatorname{rot} y + [e^{s\alpha}, \nabla] \operatorname{div} y. \end{aligned}$$

Note that

$$[e^{s\alpha}, \text{rot}] \text{rot } y(x) = s \frac{c(\tilde{x})}{\ell(x_0)^\kappa} (e^{s\alpha} \text{rot } y), \tag{4.9}$$

where  $c \in (C^1(\overline{\Omega}))^3$  is some function.

Applying the Carleman estimate for elliptic equations obtained in [12] and using (4.9), we have

$$\sqrt{\frac{s}{\ell^\kappa(x_0)}} \|ye^{s\alpha}\|_{L^2(\Omega)} \leq C \left( \|\text{rot } ye^{s\alpha}\|_{L^2(\Omega)} + \|\text{div } ye^{s\alpha}\|_{L^2(\Omega)} + \sqrt{\frac{s}{\ell^\kappa(x_0)}} \|ye^{s\alpha}\|_{L^2(\omega)} \right) \tag{4.10}$$

for all  $s \geq s_1$  and  $x_0 \in [0, T]$ , where the constants  $C$  and  $s_0$  are independent of  $s, x_0$ . Therefore, combining (4.7), (4.8) and (4.10), we have

$$\begin{aligned} & \int_Q s\varphi(|\text{rot } y|^2 + |\text{div } y|^2)e^{2s\alpha} dx + s^2 \|\varphi ye^{s\alpha}\|_{L^2(Q)}^2 \\ & \leq C \left( s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{L^2(\Sigma)}^2 + \int_Q |f|^2 e^{2s\alpha} dx \right. \\ & \quad \left. + \int_{Q_\omega} s\varphi |\text{rot } y|^2 e^{2s\alpha} dx + \int_{Q_\omega} s\varphi |\text{div } y|^2 e^{2s\alpha} dx + s^2 \|\varphi ye^{s\alpha}\|_{L^2(Q_\omega)}^2 \right) \end{aligned} \tag{4.11}$$

for all  $s \geq \max\{s_0, s_1\}$ . We need to estimate the first term on the right-hand side of (4.11). Denote  $w = \ell ye^{s\hat{\alpha}(x_0)}$ . The function  $w$  solves the following initial/boundary value problem:

$$P(D)w = \frac{\partial w}{\partial x_0} - \Delta w - \frac{1}{\varepsilon} \nabla \text{div } w = fle^{s\hat{\alpha}} + s\hat{\alpha}' \ell w + ye^{s\hat{\alpha}} \ell' \ell, \quad \text{in } Q, \tag{4.12}$$

$$\text{div } w = 0, \quad \text{in } Q,$$

$$w = 0, \quad \text{on } (0, T) \times \Gamma,$$

$$w(0, \cdot) = 0. \tag{4.13}$$

Using standard a priori estimates for a parabolic equations and the fact  $\hat{\alpha}(x_0) \leq \alpha(x)$  for all  $x \in Q$ , we have the following estimate for solutions of (4.12)–(4.13):

$$\begin{aligned} \|w\|_{H^{1,2}(Q)} & \leq C (\|fe^{s\hat{\alpha}}\|_{L^2(Q)} + \|s\varphi^{\frac{3}{4} + \frac{3}{2\kappa}} ye^{s\hat{\alpha}}\|_{L^2(Q)}) \\ & \leq C (\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi^{\frac{3}{4} + \frac{3}{2\kappa}} ye^{s\alpha}\|_{L^2(Q)}). \end{aligned} \tag{4.14}$$

Using Proposition 3.2, we observe that there exists a constant  $C > 0$  such that

$$\left\| \frac{\partial w}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)} \leq C \|w\|_{H^{1,2}(Q)}. \tag{4.15}$$

Fix  $\kappa = 8$ . Then

$$\|s\varphi^{\frac{3}{4} + \frac{3}{2\kappa}} ye^{s\alpha}\|_{L^2(Q)} \leq C \|s\varphi ye^{s\alpha}\|_{L^2(Q)}.$$

Combining (4.15) with (4.14), we have

$$\left\| \frac{\partial w}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)} \leq C (\|fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi ye^{s\alpha}\|_{L^2(Q)}). \tag{4.16}$$

Hence

$$\begin{aligned} & s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{L^2(\Sigma)}^2 \\ & \leq C s^{-\frac{1}{2}} \left\| \varphi^{-\frac{1}{4} + \frac{1}{\kappa}} \frac{\partial y}{\partial \nu} e^{s\hat{\alpha}} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 \leq C s^{-\frac{1}{2}} \|w\|_{H^{1,2}(Q)}^2 \\ & \leq C s^{-\frac{1}{2}} (\|fe^{s\alpha}\|_{L^2(Q)}^2 + \|s\varphi ye^{s\alpha}\|_{L^2(Q)}^2). \end{aligned} \tag{4.17}$$

The first two terms on the right-hand side of (4.11) can be estimated by the right-hand side of (4.17). Observe that the term  $Cs^{-\frac{1}{2}}\|s\varphi ye^{s\alpha}\|_{L^2(Q)}^2$  can be absorbed by the left-hand side of (4.11) for all sufficiently large  $s$ . This proves the statement of the theorem.

### Appendix Calculus for Pseudodifferential Operators with a Parameter

Let  $\mathcal{O}$  be a domain in  $\mathbb{R}^n$ .

**Definition A.1** We say that the symbol  $a(x', \xi', s) \in C^0(\overline{\mathcal{O}} \times \mathbb{R}^{n+1})$  belongs to the class  $C_{cl}^k S^{\frac{\kappa}{2}, \kappa, s}(\mathcal{O})$ , if

- (1) There exists a compact set  $K \subset\subset \mathcal{O}$ , such that  $a(x', \xi', s)|_{\mathcal{O} \setminus K} = 0$ ,
- (2) For any  $\beta = (\beta_0, \dots, \beta_n)$ , there exists a constant  $C_\beta$  such that

$$\left\| \frac{\partial^{\beta_0}}{\partial \xi_0^{\beta_0}} \cdots \frac{\partial^{\beta_{n-1}}}{\partial \xi_{n-1}^{\beta_{n-1}}} \frac{\partial^{\beta_n}}{\partial s^{\beta_n}} a(\cdot, \xi', s) \right\|_{C^k(\overline{\mathcal{O}})} \leq C_\beta \left( |\xi_0| + s^2 + \sum_{i=1}^{n-1} \xi_i^2 \right)^{\frac{\kappa - |\beta|}{2}},$$

where  $|\beta| = 2\beta_0 + \sum_{j=1}^n \beta_j$  and  $M(\xi', s) \geq 1$ ,

- (3) For any  $N \in \mathbb{N}_+$ , the symbol  $a$  can be represented as

$$a(x', \xi', s) = \sum_{j=1}^N a_j(x', \xi', s) + R_N(x', \xi', s),$$

where the functions  $a_j$  have the following properties:

$$a_j(x', \tau^2 \xi_0, \tau \xi_1, \dots, \tau \xi_{n-1}, \tau s) = \tau^{\kappa - j} a_j(x', \xi', s)$$

for all  $\tau > 1$  and  $(x', \xi', s) \in \{(x', \xi', s) \mid x' \in K, M(\xi', s) > 1\}$ , and

$$\left\| \frac{\partial^{\beta_0}}{\partial \xi_0^{\beta_0}} \cdots \frac{\partial^{\beta_{n-1}}}{\partial \xi_{n-1}^{\beta_{n-1}}} \frac{\partial^{\beta_n}}{\partial s^{\beta_n}} a_j(\cdot, \xi', s) \right\|_{C^k(\overline{\mathcal{O}})} \leq C_\beta \left( |\xi_0| + s^2 + \sum_{i=1}^{n-1} \xi_i^2 \right)^{\frac{\kappa - j - |\beta|}{2}}$$

for all  $\beta$  and  $(\xi', s)$  such that  $M(\xi', s) \geq 1$ , and the term  $R_N$  satisfies the estimate

$$\|R_N(\cdot, \xi', s)\|_{C^k(\overline{\mathcal{O}})} \leq C_N \left( |\xi_0| + s^2 + \sum_{i=1}^{n-1} \xi_i^2 \right)^{\frac{\kappa - N}{2}}$$

for all  $(\xi', s)$  such that  $M(\xi', s) \geq 1$ .

For the symbol  $a$ , we introduce the seminorm

$$\begin{aligned} \pi_{C^k}(a) &= \sum_{j=1}^{\widehat{N}} \sup_{|\beta| \leq \widehat{N}} \sup_{M(\xi', s) \geq 1} \left\| \frac{\partial^{\beta_0}}{\partial \xi_0^{\beta_0}} \cdots \frac{\partial^{\beta_{n-1}}}{\partial \xi_{n-1}^{\beta_{n-1}}} \frac{\partial^{\beta_n}}{\partial s^{\beta_n}} a_j(\cdot, \xi', s) \right\|_{C^k(\overline{\mathcal{O}})} (1 + M(\xi', s))^{-\kappa + j + |\beta|} \\ &\quad + \sup_{M(\xi', s) \leq 1} \|a(\cdot, \xi', s)\|_{C^\kappa(\overline{\mathcal{O}})}. \end{aligned}$$

Let  $\{\omega_j\}_{j=1}^\infty$  be a sequence of eigenfunctions of the operator  $\Delta$  on  $\mathcal{M} = \{(\xi_0, \dots, \xi_n) \mid \xi_0^2 + \sum_{j=1}^n \xi_j^2 = 1\}$ , and let  $\{\lambda_j\}_{j=1}^\infty$  be a sequence of corresponding eigenvalues. Assume that

$$(\omega_i, \omega_j)_{L^2(\mathcal{M})} = \delta_{i,j}.$$

The following asymptotic formula is established in [8]:

$$\lambda_j = cj^{\frac{2}{n}} + o(j^{\frac{2}{n}}), \quad \text{as } j \rightarrow +\infty.$$

For each  $k$ , thanks to the standard elliptic estimate for the Laplace operator, we have

$$\|\omega_j\|_{H^{2k}(\mathcal{M})} \leq C_k \lambda_j^k. \tag{A.1}$$

Therefore, by the Sobolev embedding theorem, we have

$$\|\omega_j\|_{C^0(\mathcal{M})} \leq C \lambda_j^n, \quad \forall j \in \{1, \dots, \infty\}. \tag{A.2}$$

We extend the function  $\omega_j$  on the set  $\{\xi \mid \xi_0^2 + \sum_{i=1}^n \xi_i^4 \leq 1\}$  as a smooth function and set

$$\omega_j(\xi) = \omega_j\left(\frac{\xi_0}{M^2(\xi)}, \frac{\xi_1}{M(\xi)}, \dots, \frac{\xi_n}{M(\xi)}\right), \quad M(\xi) = \left(\xi_0^2 + \sum_{k=1}^n \xi_k^4\right)^{\frac{1}{4}}.$$

We introduce the pseudodifferential operator

$$\tilde{\omega}_j(D)w = \int_{\mathbb{R}^{n+1}} \omega_j(\xi) F_{(x',s) \rightarrow \xi} w e^{i\langle (x',s), \xi \rangle} d\xi.$$

Here, in order to distinguish the Fourier transforms with respect to different variables, we will use the following notations:

$$F_{(x',s) \rightarrow \xi} u = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{\mathbb{R}^{n+1}} e^{-i\left(\sum_{j=0}^{n-1} x_j \xi_j + s \xi_n\right)} u(x', s) dx' ds,$$

$$F_{x_n \rightarrow \xi_n} u = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-ix_n \xi_n} u(x_n) dx_n.$$

First, we define the operator  $A(x', D', s)$  for functions in  $C_0^\infty(\mathcal{O})$ :

$$A(x', D', s)u = \int_{\mathbb{R}^n} a(x', \xi', s) \hat{u}(\xi') dx'.$$

The following lemma allows us to extend the definition of the operator  $A$  on Sobolev spaces.

**Lemma A.1** *Let  $a(x', \xi_0, \dots, \xi_{n-1}, s) \in C_{cl}^0 S^{\frac{1}{2}, 1, s}(\mathcal{O})$ . Then  $A \in \mathcal{L}(H_0^{\frac{1}{2}, 1, s}(\mathcal{O}); L^2(\mathcal{O}))$  and  $\|A\|_{\mathcal{L}(H_0^{\frac{1}{2}, 1, s}(\mathcal{O}); L^2(\mathcal{O}))} \leq C(\pi_{C^0}(a))$ .*

**Proof** Thanks to Definition A.1(3), it suffices to consider the case where

$$a(x', \tau^2 \xi_0, \tau \xi_1, \dots, \tau \xi_{n-1}, \tau s) = \tau a(x', \xi_0, \dots, \xi_{n-1}, s), \quad \forall \tau > 1. \tag{A.3}$$

The operator

$$\tilde{A}(x', D)v = \int_{\{\xi_0^2 + \sum_{i=1}^n \xi_i^4 \leq 1\}} a(x', \xi_0, \dots, \xi_n) F_{(x',s) \rightarrow \xi} v(\xi) e^{i\langle x, \xi \rangle} d\xi$$

is a continuous operator from  $L^2(\mathcal{O} \times \mathbb{R})$  into  $L^2(\mathcal{O} \times \mathbb{R})$  with the norm estimated as

$$\|\tilde{A}(x', D)\|_{\mathcal{L}(L^2(\mathcal{O} \times \mathbb{R}), L^2(\mathcal{O} \times \mathbb{R}))} \leq C(\pi_{C^0}(a)).$$

Consider the symbol  $b(x', \xi_0, \dots, \xi_n) = a(x', \xi_0, \dots, \xi_n) \left( \xi_0^2 + \sum_{j=1}^n \xi_j^4 \right)^{-\frac{1}{4}}$ . Then by (A.3),

$$b(x', \tau^2 \xi_0, \tau \xi_1, \dots, \tau \xi_n) = b(x', \xi_0, \xi_1, \dots, \xi_n), \quad \forall \tau \geq 1.$$

We can represent the symbol  $b$  as

$$b(x', \xi) = \sum_{j=1}^{\infty} b_j(x') \omega_j \left( \frac{\xi_0}{M^2(\xi)}, \frac{\xi_1}{M(\xi)}, \dots, \frac{\xi_n}{M(\xi)} \right), \quad b_j(x') = (b(x', \xi), \omega_j(\xi))_{L^2(\mathcal{M})}.$$

Observe  $b_j(x') = (\Delta_{\xi}^k b(x', \xi), \omega_j(\xi))_{L^2(\mathcal{M})} \lambda_j^{-k}$ . So

$$\|b_j\|_{C^0(\overline{\mathcal{O}})} \leq C_m \lambda_j^{-m}, \quad \forall m \in \{1, \dots, \infty\}. \quad (\text{A.4})$$

By (A.2) and (A.4),

$$\|B(x', D)v\|_{L^2(\mathcal{O} \times \mathbb{R})} \leq \sum_{j=1}^{\infty} \|b_j\|_{C^0(\overline{\mathcal{O}})} \|\tilde{\omega}_j(D)\| \|v\|_{L^2(\mathcal{O} \times \mathbb{R})} \leq \sum_{j=1}^{\infty} C_m \lambda_j^{-m} \lambda_j^n \|v\|_{L^2(\mathcal{O} \times \mathbb{R})}.$$

Taking  $m = 3n$ , we have

$$\|B(x', D)v\|_{L^2(\mathcal{O} \times \mathbb{R})} \leq \sum_{j=1}^{\infty} C_m \lambda_j^{-2n} \|v\|_{L^2(\mathcal{O} \times \mathbb{R})}.$$

Therefore, the operator

$$A^b(x', D)v = \int_{\{\xi_0^2 + \sum_{i=1}^n \xi_i^4 \geq 1\}} a(x', \xi) F_{(x', s) \rightarrow \xi} v e^{i(x, \xi)} d\xi$$

is a continuous operator from  $H_0^{\frac{1}{2}, 1}(\mathcal{O} \times \mathbb{R})$  into  $L^2(\mathcal{O} \times \mathbb{R})$  with the norm satisfying the estimate

$$\|A^b\|_{\mathcal{L}(H_0^{\frac{1}{2}, 1}(\mathcal{O} \times \mathbb{R}), L^2(\mathcal{O} \times \mathbb{R}))} \leq \sum_{j=1}^{\infty} C_m \lambda_j^{-2n}.$$

Next, we observe that for the function  $v(x) = u(x_0, \dots, x_{n-1})w(x_n)$ , there holds

$$\begin{aligned} \|\tilde{A}(x', D)v\|_{L^2(\mathcal{O} \times \mathbb{R})} &= \sqrt{2\pi} \|A(x', D_0, \dots, D_{n-1}, \xi_n) u F_{x_n \rightarrow \xi_n} w\|_{L^2(\mathcal{O} \times \mathbb{R})} \\ &\leq C(\pi_{C^0}(a)) \left( \int_{-\infty}^{\infty} \|u\|_{H^{\frac{1}{2}, 1, \xi_n}(\mathcal{O})}^2 |F_{x_n \rightarrow \xi_n} w|^2 d\xi_n \right)^{\frac{1}{2}}. \end{aligned} \quad (\text{A.5})$$

We take a sequence  $\{w_j(x_n)\}_1^{\infty}$  such that  $F_{x_n \rightarrow \xi_n} w_j(\xi_n)$  has a compact support and  $|F_{x_n \rightarrow \xi_n} w_j|^2 \rightarrow \delta(\xi_n - s)$ , where  $s \in \mathbb{R}$  is an arbitrary point. Since the function  $\xi_n \rightarrow \|A(x', D', \xi_n)u\|_{L^2(\mathcal{O})}$  is continuous, we have

$$\begin{aligned} \|A(x', D_0, \dots, D_{n-1}, \xi_n) u \hat{w}\|_{L^2(\mathcal{O} \times \mathbb{R})}^2 &= \int_{\mathbb{R}} \|A(x', D_0, \dots, D_{n-1}, \xi_n) u\|_{L^2(\mathcal{O})}^2 |F_{x_n \rightarrow \xi_n} w_j|^2 d\xi_n \\ &\rightarrow \|A(x', D_0, \dots, D_{n-1}, s)u\|_{L^2(\mathcal{O})}^2. \end{aligned}$$

This fact and (A.5) imply

$$\|A(x', D', s)u\|_{L^2(\mathcal{O})} \leq C(\pi_{C^0}(a)) \|u\|_{H^{\frac{1}{2}, 1, s}(\mathcal{O})}$$

for almost all  $s$ . Since the norm of the operator  $A$  is a continuous function of  $s$ , we have this inequality for all  $s$ .

The following theorem provides an estimate for a commutator of a Lipschitz function and the pseudodifferential operator  $\tilde{\omega}_j$ .

**Theorem A.1** *Let  $f \in W_\infty^1(\mathcal{O})$  be a function with compact support. Then*

$$\|[f, \tilde{\omega}_j]\|_{\mathcal{L}(L^2(\mathcal{O}), H^{\frac{1}{2}, 1, s}(\mathcal{O}))} \leq C \|f\|_{W_\infty^1(\mathcal{O})} \lambda_j^{4n},$$

where the constant  $C$  is independent of  $j$ .

The proof of this theorem is similar to the proof of the corollary in [15, p. 309].

**Lemma A.2** *Let  $a(x', \xi_0, \dots, \xi_{n-1}, s) \in C_{cl}^1 S^{\frac{1}{2}, 1, s}(\mathcal{O})$ . Then  $A(x', D', s)^* = A^*(x', D', s) + R$ , where  $A^*$  is the pseudodifferential operator with symbol  $\overline{a(x', \xi_0, \dots, \xi_{n-1}, s)}$ , and  $R \in \mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$  satisfies*

$$\|R\|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \leq C \pi_{C^1}(a).$$

**Proof** Thanks to Definition A.1(3), it suffices to consider the case when

$$a(x', \tau^2 \xi_0, \tau \xi_1, \dots, \tau \xi_{n-1}, \tau s) = \tau a(x', \xi_0, \dots, \xi_{n-1}, s), \quad \forall \tau > 1.$$

The symbol  $a(x', \xi)$  can be represented as

$$a(x', \xi) = \sum_{j=1}^{\infty} a_j(x') M(\xi) \tilde{\omega}_j(\xi).$$

Consider the operator

$$\tilde{A}(x', D) = \sum_{j=1}^{\infty} a_j(x') M(D) \tilde{\omega}_j(D), \quad M(D)w = \int_{\mathbb{R}^{n+1}} M(\xi) \widehat{w} e^{i(x, \xi)} d\xi.$$

Then

$$\begin{aligned} \tilde{A}(x', D)^* &= \sum_{j=1}^{\infty} (a_j(x') M(D) \tilde{\omega}_j(D))^* = \sum_{j=1}^{\infty} M(D) \tilde{\omega}_j(D) \overline{a_j(x')} \\ &= \sum_{j=1}^{\infty} \overline{a_j(x')} M(D) \tilde{\omega}_j(D) + \sum_{j=1}^{\infty} [M(D) \tilde{\omega}_j(D), \overline{a_j(x')}] \end{aligned}$$

Observe that  $\sum_{j=1}^{\infty} \overline{a_j(x')} M(D) \tilde{\omega}_j(D)$  is the operator with symbol  $\overline{a(x', \xi_0, \dots, \xi_n)} \in C_{cl}^1 S^{\frac{1}{2}, 1, s}(\mathcal{O})$ .

Let us estimate the norm of the operator  $\sum_{j=1}^{\infty} [M(D) \tilde{\omega}_j(D), \overline{a_j(x')}]$ . By Theorem A.1,

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} [\overline{a_j(x')}, M(D) \tilde{\omega}_j(D)] \right\|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} &\leq C_m \sum_{j=1}^{\infty} \|a_j\|_{C^1(\overline{\mathcal{O}})} \lambda_j^{\tilde{\kappa}(n)} \leq C_m \sum_{j=1}^{\infty} \lambda_j^{\tilde{\kappa}(n)} \pi_{C^1}(a) \lambda_j^{-m} \\ &\leq C \pi_{C^1}(a). \end{aligned}$$

Denote  $v = u(x_0, \dots, x_{n-1})w(x_n)$ ,  $\tilde{v} = \tilde{u}(x_0, \dots, x_{n-1})\tilde{w}(x_n)$ . We have

$$(\tilde{A}(x', D)v, \tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})} = (v, \tilde{A}(x', D)^* \tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})} = (v, \tilde{A}^*(x', D)\tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})} + (v, R\tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})}.$$

On the other hand,

$$\begin{aligned} (\tilde{A}(x', D)v, \tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})} &= 2\pi \int_{\mathbb{R}} (A(x', D', \xi_n)u, \tilde{u})_{L^2(\mathcal{O})} w \overline{w} d\xi_n \\ &= 2\pi \int_{\mathbb{R}} (u, A(x', D', \xi_n)^* \tilde{u})_{L^2(\mathcal{O})} w \overline{w} d\xi_n. \end{aligned}$$

Taking into account that  $(v, A^*(x', D)\tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})} = \int_{\mathbb{R}} (u, A^*(x', D', \xi_n)\tilde{u})_{L^2(\mathcal{O})} w \overline{w} d\xi_n$ , we have

$$\begin{aligned} \left| \int_{\mathbb{R}} (u, (A(x', D', \xi_n)^* - A^*(x', D', \xi_n))\tilde{u})_{L^2(\mathcal{O})} w \overline{w} d\xi_n \right| &= |(v, R\tilde{v})_{L^2(\mathcal{O} \times \mathbb{R})}| \\ &\leq C \|v\|_{L^2(\mathcal{O} \times \mathbb{R})} \|\tilde{v}\|_{L^2(\mathcal{O} \times \mathbb{R})}. \end{aligned}$$

We take a sequence  $\{w_j\}_{j=1}^{\infty}$  such that  $F_{x_n \rightarrow \xi_n} w_j$  has a compact support and  $|F_{x_n \rightarrow \xi_n} w_j|^2 \rightarrow \delta(\xi_n - s)$ , where  $s \in \mathbb{R}$  is an arbitrary point. Since the function  $\xi_n \rightarrow \|A(x', D', \xi_n)u\|_{L^2(\mathcal{O})}$  is continuous, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}} (u, (A(x', D', \xi_n)^* - A^*(x', D', \xi_n))\tilde{u})_{L^2(\mathcal{O})} |w_j|^2 d\xi_n \right| \\ &\rightarrow |(u, (A(x', D', s)^* - A^*(x', D', s))\tilde{u})_{L^2(\mathcal{O})}|. \end{aligned}$$

Since

$$|(u, (A(x', D', s)^* - A^*(x', D', s))\tilde{u})_{L^2(\mathcal{O})}| \leq C \|u\|_{L^2(\mathcal{O})} \|\tilde{u}\|_{L^2(\mathcal{O})},$$

the statement of the lemma is proved.

**Lemma A.3** *Let  $a(x', \xi_0, \dots, \xi_{n-1}, s) \in C_{\text{cl}}^1 S_{\frac{1}{2}}^{\hat{j}, s}(\mathcal{O})$ , where  $\hat{j} \in \{0, 1\}$ , and  $b(x', \xi_0, \dots, \xi_{n-1}, s) \in C_{\text{cl}}^1 S_{\frac{1}{2}}^{\mu, s}(\mathcal{O})$ . Then  $A(x', D', s)B(x', D', s) = C(x', D', s) + R_0$ , where  $C(x', D', s)$  is the operator with symbol  $a(x', \xi_0, \dots, \xi_{n-1}, s)b(x', \xi_0, \dots, \xi_{n-1}, s)$  and  $R_0 \in \mathcal{L}(H_0^{\frac{\mu+\tau}{2}, \mu+\tau, s}(\mathcal{O}), H^{\frac{\tau+1}{2}, \tau+1, s}(\mathcal{O}))$  for any  $\tau \in [-1, 0]$  if  $\hat{j} = 0$ , and  $R_0 \in \mathcal{L}(H_0^{\frac{\mu}{2}, \mu, s}(\mathcal{O}), L^2(\mathcal{O}))$  if  $\hat{j} = 1$ . Moreover,*

$$\begin{aligned} \|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2}, \mu, s}(\mathcal{O}), L^2(\mathcal{O}))} &\leq C \pi_{C^1}(a) \pi_{C^1}(b) \quad \text{for } \hat{j} = 1, \\ \|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2}, \mu+\tau, s}(\mathcal{O}), H^{\frac{\tau+1}{2}, \tau+1, s}(\mathcal{O}))} &\leq C (\pi_{C^1}(a) \pi_{C^1}(b)) \quad \text{for } \hat{j} = 0. \end{aligned}$$

**Proof** We set

$$A(x', D) = \sum_{j=1}^{\infty} a_j(x') M^{\hat{j}}(D) \tilde{\omega}_j(D), \quad B(x', D) = \sum_{j=1}^{\infty} b_j(x') M^{\mu}(D) \tilde{\omega}_j(D).$$

Observe

$$\begin{aligned} A(x', D)B(x', D) &= \sum_{m,k=1}^{\infty} a_m(x') b_k(x') M^{\hat{j}+\mu}(D) \tilde{\omega}_m(D) \tilde{\omega}_k(D) \\ &\quad + \sum_{m,k=1}^{\infty} a_m(x') [M^{\hat{j}} \tilde{\omega}_m, b_k] M^{\mu}(D) \tilde{\omega}_k(D). \end{aligned}$$

Since  $C(x', D) = \sum_{m,k=1}^{\infty} a_m(x') b_k(x') M^{\hat{j}+\mu}(D) \tilde{\omega}_m(D) \tilde{\omega}_k(D)$ , and for  $\hat{j} = 1$ ,

$$\begin{aligned} \|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2}, \mu}(\mathcal{O}), L^2(\mathcal{O}))} &= \left\| \sum_{m,k=1}^{\infty} a_m(x') [M \tilde{\omega}_m, b_k] M^{\mu}(D) \tilde{\omega}_k(D) \right\|_{\mathcal{L}(H_0^{\frac{\mu}{2}, \mu}(\mathcal{O}), L^2(\mathcal{O}))} \\ &\leq \sum_{m,k=1}^{\infty} \|a_m\|_{C^1(\overline{\mathcal{O}})} \| [M \tilde{\omega}_m, b_k] \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \| \tilde{\omega}_k(D) \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \\ &\leq C_l \sum_{m,k=1}^{\infty} \lambda_m^{-l} \| [M \tilde{\omega}_m, b_k] \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \lambda_k^{\tilde{\kappa}(n)}, \end{aligned}$$

by applying Theorem A.1, we obtain

$$\begin{aligned} \|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2}, \mu}(\mathcal{O}), L^2(\mathcal{O}))} &\leq \sum_{m,k=1}^{\infty} \|a_m\|_{C^1(\overline{\mathcal{O}})} \| [M \tilde{\omega}_m, b_k] \|_{\mathcal{L}(L^2, L^2)} \| \tilde{\omega}_k(D) \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \\ &\leq C_l \sum_{m,k=1}^{\infty} \lambda_m^{-l} \|b_k\|_{C^1(\overline{\mathcal{O}})} \lambda_m^{\tilde{\kappa}_1(n)} \lambda_k^{\tilde{\kappa}(n)} \leq C_{l, l_1} \sum_{m,k=1}^{\infty} \lambda_m^{-l} \lambda_k^{-l_1} \lambda_m^{\tilde{\kappa}_1(n)} \lambda_k^{\tilde{\kappa}(n)} \\ &\leq C_{l, l_1} \sum_{k=1}^{\infty} \lambda_m^{-l} \lambda_m^{\tilde{\kappa}_1(n)} \sum_{m=1}^{\infty} \lambda_k^{-l_1} \lambda_k^{\tilde{\kappa}(n)} < \infty. \end{aligned} \quad (\text{A.6})$$

Let  $v = v_j = u(x_0, \dots, x_{n-1}) w_j(x_n)$ . We take a sequence  $\{w_j\}_{j=1}^{\infty}$  such that  $F_{x_n \rightarrow \xi_n} w_j$  has a compact support and  $|F_{x_n \rightarrow \xi_n} w_j|^2 \rightarrow \delta(\xi_n - s)$ , where  $s \in \mathbb{R}$  is arbitrary. Then for any  $u \in H_0^{\frac{1}{2} + \frac{\mu}{2}, 1 + \mu}(\mathcal{O})$ ,

$$\begin{aligned} &\|A(x', D)B(x', D)v_j - C(x', D)v_j\|_{L^2(\mathcal{O} \times \mathbb{R})}^2 \\ &= 2\pi \int_{\mathbb{R}} \|(A(x', D', \xi_n)B(x', D', \xi_n) - C(x', D', \xi_n))u\|_{L^2(\mathcal{O})}^2 |F_{x_n \rightarrow \xi_n} w_j|^2 d\xi_n \\ &\leq C \|v_j\|_{H_0^{\frac{\mu}{2}, \mu}(\mathbb{R}^{n+1})}^2. \end{aligned} \quad (\text{A.7})$$

Passing to the limit in (A.7) as  $j \rightarrow +\infty$ , we obtain

$$\|(A(x', D', s)B(x', D', s) - C(x', D', s))u\|_{L^2(\mathcal{O})}^2 \leq C \|u\|_{H_0^{\frac{\mu}{2}, \mu, s}(\mathcal{O})}^2.$$

Let  $\hat{j} = 0$ . Then

$$\begin{aligned} &\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2}, \mu+\tau}(\mathcal{O}), H_0^{\frac{\tau+1}{2}, \tau+1, s}(\mathcal{O}))} \\ &\leq \sum_{m,k=1}^{\infty} \|a_m[\tilde{\omega}_m, b_k] M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}), H_0^{\frac{\tau+1}{2}, \tau+1}(\mathcal{O}))} \| \tilde{\omega}_k(D) \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))}. \end{aligned} \quad (\text{A.8})$$

In order to estimate the norm  $\|a_m[\tilde{\omega}_m, b_k] M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}), H_0^{\frac{\tau+1}{2}, \tau+1}(\mathcal{O}))}$ , we observe

$$M^{\tau+1} a_m[\tilde{\omega}_m, b_k] M^{-\tau} = a_m M^{\tau+1} [\tilde{\omega}_m, b_k] M^{-\tau} + [M^{\tau+1}, a_m] [\tilde{\omega}_m, b_k] M^{-\tau}.$$

For the second term in this equality, we have

$$\begin{aligned} \| [M^{\tau+1}, a_m] \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} &\leq \|b_k\|_{C^1(\overline{\mathcal{O}})} \| \tilde{\omega}_m \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))}, \\ \| [M^{\tau+1}, a_m] \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} &\leq C \|a_m\|_{C^1(\overline{\mathcal{O}})}. \end{aligned} \quad (\text{A.9})$$



In order to estimate the first term, we observe  $[\tilde{\omega}_m, b_k]^* = -[\tilde{\omega}_m, b_k]$ . Then

$$[\tilde{\omega}_m, b_k] \in \mathcal{L}(L^2(\mathbb{R}^n), H^{\frac{1}{2},1}(\mathbb{R}^n)), \quad [\tilde{\omega}_m, b_k] \in \mathcal{L}(H^{-\frac{1}{2},-1}(\mathbb{R}^n), L^2(\mathbb{R}^n)).$$

Using an interpolation argument, we get

$$[\tilde{\omega}_m, b_k] \in \mathcal{L}(H^{-\frac{\gamma}{2},-\gamma}(\mathbb{R}^n), H^{\frac{1-\gamma}{2},1-\gamma}(\mathbb{R}^n)), \quad \forall \gamma \in [0, 1], \quad (\text{A.10})$$

$$\|[\tilde{\omega}_m, b_k]\|_{\mathcal{L}(H^{-\frac{\gamma}{2},-\gamma}(\mathbb{R}^n), H^{\frac{1-\gamma}{2},1-\gamma}(\mathbb{R}^n))} \leq \|[\tilde{\omega}_m, b_k]\|_{\mathcal{L}(L^2(\mathbb{R}^n), H^{\frac{1}{2},1}(\mathbb{R}^n))}, \quad \forall \gamma \in [0, 1]. \quad (\text{A.11})$$

Applying (A.9)–(A.11) to (A.8), we obtain

$$\begin{aligned} \|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau}(\mathcal{O}), H_0^{\frac{\tau+1}{2},\tau+1}(\mathcal{O}))} &\leq C_l \sum_{m,k=1}^{\infty} \lambda_m^{-l} \|b_k\|_{C^1(\bar{\mathcal{O}})} \lambda_m^{\tilde{\kappa}_1(n)} \lambda_k^{\tilde{\kappa}(n)} \\ &\leq C_{l,l_1} \sum_{m,k=1}^{\infty} \lambda_m^{-l} \lambda_k^{-l_1} \lambda_m^{\tilde{\kappa}_1(n)} \lambda_k^{\tilde{\kappa}(n)} \\ &\leq C_{l,l_1} \sum_{k=1}^{\infty} \lambda_m^{-l} \lambda_m^{\tilde{\kappa}_1(n)} \sum_{m=1}^{\infty} \lambda_k^{-l_1} \lambda_k^{\tilde{\kappa}(n)} < \infty. \end{aligned} \quad (\text{A.12})$$

We finish the proof of the lemma by using similar arguments as in case  $\hat{j} = 1$ .

The direct consequence of Lemma A.3 is the following commutator estimate.

**Lemma A.4** *Let  $a(x', \xi', s) \in C_{\text{cl}}^1 S^{\frac{1}{2},1,s}(\mathcal{O})$  and  $b(x', \xi', s) \in C_{\text{cl}}^1 S^{0,0,s}(\mathcal{O})$ . Then  $[A, B] \in \mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))$ , and*

$$\|[A, B]\|_{\mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))} \leq C(\pi_{C^0}(a)\pi_{C^0}(b) + \pi_{C^0}(a)\pi_{C^1}(b) + \pi_{C^1}(a)\pi_{C^0}(b)).$$

**Proof** By Lemma A.3, we have

$$A(x', D', s)B(x', D', s) = C(x', D', s) + R_0, \quad B(x', D', s)A(x', D', s) = C(x', D', s) + \tilde{R}_0,$$

where  $R_0, \tilde{R}_0 \in \mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$ . Since  $[A, B] = R_0 - \tilde{R}_0$ , we immediately obtain the statement of the lemma.

**Lemma A.5** *Let  $a(x', \xi', s) \in C_{\text{cl}}^1 S^{\frac{1}{2},1,s}(\mathcal{O})$  be a symbol with compact support in  $\mathcal{O}$ . Let  $u \in H^{\frac{1}{2},1,s}(\mathcal{O})$ ,  $\text{supp } u \subset B(0, \delta)$ , and  $\delta' > \delta$ . Then there exists a constant  $C(\delta', \delta, \pi_{C^1}(a))$  such that*

$$\|A(x', D', s)u\|_{H^{\frac{1}{2},1,s}(\mathcal{O} \setminus B(0, \delta'))} \leq C\|u\|_{H^{\frac{1}{2},1,s}(\mathcal{O})}.$$

**Proof** Consider the operator

$$\tilde{A}(x', D) = \sum_{j=1}^{\infty} a_j(x')M(D)\tilde{\omega}_j(D).$$

By (A.1), for any multiindex  $\beta$ , we have

$$\left| \frac{\partial^\beta}{\partial \xi^\beta} \omega_j(\xi) \right| \leq C_\beta \lambda_j^{\tilde{\kappa}_2|\beta|}.$$

Hence, by Lemma 2.2 (see [17, Chapter II]), we have

$$\|M(D)\tilde{\omega}_j(D)v\|_{H^{\frac{1}{2},1}(\mathcal{O} \setminus B(0, \delta')) \times [-1,1] \setminus [-\frac{1}{2}, \frac{1}{2}]} \leq C\lambda_j^{\tilde{\kappa}}\|v\|_{H^{\frac{1}{2},1}(B(0, \delta) \times [-\frac{1}{4}, \frac{1}{4}])}$$

for every function  $w$  such that  $\text{supp } w \subset B(0, \delta) \times [-\frac{1}{4}, \frac{1}{4}]$ . Therefore

$$\begin{aligned} & \|\tilde{A}(x', D)v\|_{H^1(\mathcal{O} \setminus B(0, \delta') \times [-1, 1] \setminus [-\frac{1}{2}, \frac{1}{2}])} \\ & \leq \sum_{j=1}^{\infty} \|a_j(x')M(D)\tilde{\omega}_j(D)v\|_{H^1(\mathcal{O} \setminus B(0, \delta') \times [-1, 1] \setminus [-\frac{1}{2}, \frac{1}{2}])} \\ & \leq \sum_{j=1}^{\infty} \|a_j\|_{C^1(\overline{\mathcal{O}})} \|M(D)\tilde{\omega}_j(D)v\|_{H^1(\mathcal{O} \setminus B(0, \delta') \times [-1, 1] \setminus [-\frac{1}{2}, \frac{1}{2}])} \\ & \leq C \sum_{j=1}^{\infty} \lambda_j^{-m} \lambda_j^{\widehat{\kappa}} \|v\|_{H^{\frac{1}{2}, 1}(B(0, \delta) \times [-\frac{1}{4}, \frac{1}{4}])}. \end{aligned} \tag{A.13}$$

Let  $v = v_j = u(x_0, \dots, x_{n-1})w_j(x_n)$ . We take a sequence  $\{w_j\}_{j=1}^{\infty}$  such that  $F_{x_n \rightarrow \xi_n} w_j$  has a compact support and  $|F_{x_n \rightarrow \xi_n} w_j|^2 \rightarrow \delta(\xi_n - s_0)$ , where  $s_0 \in \mathbb{R}$  is an arbitrary point. Then

$$\begin{aligned} \|\tilde{A}(x', D)v\|_{H^1(\mathcal{O} \setminus B(0, \delta') \times [-1, 1] \setminus [-\frac{1}{2}, \frac{1}{2}])}^2 &= \int_{-\frac{1}{4}}^{\frac{1}{4}} \|A(x', D', s)u\|_{H^{1, s}(\mathcal{O} \setminus B(0, \delta'))}^2 |F_{x_n \rightarrow \xi_n} w_j|^2 d\xi_n \\ &\rightarrow \|A(x, D', s_0)u\|_{H^{1, s_0}(\mathcal{O} \setminus B(0, \delta'))}^2 \end{aligned} \tag{A.14}$$

and

$$\|v_j\|_{H^{\frac{1}{2}, 1}(B(0, \delta) \times [-\frac{1}{4}, \frac{1}{4}])} \rightarrow \|u\|_{H^{\frac{1}{2}, 1, s_0}(B(0, \delta))}, \quad \text{as } j \rightarrow +\infty.$$

These relations and (A.13) prove the statement of the lemma.

We shall use the following variant of Gårding’s inequality.

**Lemma A.6** *Let  $p(x', \xi', s) \in C_{\text{cl}}^1 S^{\frac{1}{2}, 1, s}(\mathcal{O})$  be a symbol with compact support in  $\mathcal{O}$ . Let  $u \in H^{\frac{1}{2}, 1, s}(\mathcal{O})$ ,  $\text{supp } u \subset B(0, \delta)$ , and  $\delta' > \delta$  be such that  $\overline{B(0, \delta')} \subset \mathcal{O}$  and  $\text{Re } p(x', \xi', s) > \widehat{C}|s|M(\xi', s)$  for any  $x \in B(0, \delta')$ . Then there exist positive constants  $C$  and  $C_1$  independent of  $s$ , such that*

$$\text{Re}(Pu, u) \geq C|s|\|u\|_{H^{\frac{1}{4}, \frac{1}{2}, s}(\mathcal{O})}^2 - C_1\|u\|_{L^2(\mathcal{O})}^2.$$

**Proof** Let  $\chi \in C_0^\infty(B(0, \delta'))$  be a function such that  $\chi|_{B(0, \delta)} = 1$ . Consider the pseudo-differential operator  $A(x', D', s)$  with symbol  $A(x', \xi', s) = (\text{Re } p(x', \xi', s) - \chi \frac{\widehat{C}}{2}|s|M(\xi', s))^{\frac{1}{2}} \in C_{\text{cl}}^1 S^{\frac{1}{4}, \frac{1}{2}, s}(\mathcal{O})$ . Then, according to Lemma A.3, we have

$$A(x', D', s)^* A(x', D', s) = \text{Re } p(x, \xi', s) - \chi \frac{\widehat{C}}{2}|s|M(\xi', s) + R,$$

where  $R \in \mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))$ . Therefore

$$\text{Re}(Pu, u)_{L^2(\mathcal{O})} = \|A(x', D, s)u\|_{L^2(\mathcal{O})}^2 - ((1 - \chi)M(D)u, u)_{L^2(\mathbb{R}^n)} + \frac{\widehat{C}}{2}\|u\|_{H^{\frac{1}{4}, \frac{1}{2}, s}(\mathcal{O})}^2 + (Ru, u)_{L^2(\mathcal{O})}.$$

Observing that  $|(Ru, u)_{L^2(\mathcal{O})}| \leq C(\pi_{C^1}(p))\|u\|_{L^2(\mathcal{O})}^2$ , and by Lemma 2.2 (see also [17, Chapter II]), we have  $|((1 - \chi)M(D)u, u)_{L^2(\mathbb{R}^n)}| \leq C\|u\|_{L^2(\mathcal{O})}^2$ . We obtain the statement of the lemma.

Consider the following system of equations:

$$\frac{\partial W}{\partial x_n} + K(x, D', s)W = F, \quad \text{in } G, \tag{A.15}$$

$$W(x', 0) = g, \tag{A.16}$$

where  $W = (w_1, \dots, w_m)$ ,  $F = (f_1, \dots, f_m)$ ,  $g = (g_1, \dots, g_m)$ . Let  $K(x, D', s)$  be an  $m \times m$  matrix pseudodifferential operator such that

$$K_{ij}(x, \xi', s) \in C_{cl}^1 S^{\frac{1}{2}, 1, s}(G). \tag{A.17}$$

Assume that there exists a constant  $C > 0$  such that

$$\text{Spec } K(x, \xi', s) \subset \left\{ z \in \mathbb{C} : \text{Re } z \geq C \left( \xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + s^4 \right)^{\frac{1}{4}} \right\} \tag{A.18}$$

for all  $(x, \xi', s) \in G \times \mathbb{R}^{n+1} \cap \{(\xi', s) \mid M(\xi', s) \geq 1\}$ , and

$$\text{the matrix } K(x, \xi', s) \text{ is independent of } x \text{ outside a ball } B(0, \delta). \tag{A.19}$$

**Lemma A.7** *Suppose that assumptions (A.17)–(A.19) hold true. Then*

(1) *For each  $g \in H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)$  and  $F \in L^2(G)$ , there exists a unique solution  $W \in H^{\frac{1}{2}, 1, s}(G)$  to problem (A.15)–(A.16), and the following a priori estimate holds true:*

$$\|W\|_{H^{\frac{1}{2}, 1, s}(G)} + \|W\|_{L^\infty(0, \gamma; H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \tag{A.20}$$

(2) *For each  $g \in L^2(\mathbb{R}^n)$  and  $F \in H^{-\frac{1}{2}, -1, s}(G)$  with  $\text{supp } F \subset\subset G$ , there exists a unique solution  $W \in L^2(G)$  to problem (A.15)–(A.16), and the following a priori estimate holds true:*

$$\|W\|_{L^2(G)} + \|W\|_{L^\infty(0, \gamma; H^{-\frac{1}{4}, -\frac{1}{2}, s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{-\frac{1}{4}, -\frac{1}{2}, s}(\mathbb{R}^n)} + \|F\|_{H^{-\frac{1}{2}, -1, s}(G)}). \tag{A.21}$$

**Remark A.1** For the initial data  $g \in L^2(\mathbb{R}^n)$  and  $F \in H^{-\frac{1}{2}, -1, s}(G)$  with  $\text{supp } F \subset\subset G$ , we understand the solution of problem (A.15)–(A.16) in the following way:

$$\left( W, \left( -\frac{\partial}{\partial x_n} + K(x, D', s)^* \right) \Phi \right)_{L^2(G)} = (g, \Phi(\cdot, 0))_{L^2(\mathbb{R}^n)} + (F, \Phi)_{L^2(G)}$$

for any function  $\Phi \in H^{\frac{1}{2}, 1, s}(G)$ ,  $\Phi(\cdot, \gamma) = 0$ .

**Proof** Set

$$\tilde{\Lambda}^{\frac{1}{2}}(D', s)w = \int_{\mathbb{R}^n} \left( 1 + \xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + s^4 \right)^{\frac{1}{8}} \widehat{w} e^{i\langle x', \xi' \rangle} d\xi'.$$

Let  $K_1(x, \xi', s)$  be the principal symbol of the operator  $K$ . Consider the matrix

$$P(x, \xi', s) = \int_0^\infty e^{-tK_1^*} e^{-tK_1} dt. \tag{A.22}$$

By (A.18), the integral on the right-hand side of (A.22) is convergent. Then for the principal symbol of the operator  $K$ , we have

$$PK_1 + K_1^*P = I.$$

For some positive constant  $C_1 > 0$ , we also observe

$$(P\vec{v}, \vec{v}) \geq C_1 |\vec{v}|^2, \quad \forall \vec{v} \in \mathbb{R}^m. \tag{A.23}$$

In order to show the solvability of (A.15)–(A.16), we will first consider regularized problems. For  $\epsilon \in ]0, 1[$ , let us consider a family of the Friedrichs mollifiers  $(\mathcal{J}_\epsilon)_\epsilon$  with  $\mathcal{J}_\epsilon \in \mathcal{S}^{-\infty}(\mathbb{R}^n)$  (see [16]). ( $\mathcal{S}^p(\mathbb{R}^n)$  is the class of pseudodifferential operators of order  $p$  on  $\mathbb{R}^n$ .) It is known that

$$\begin{aligned} \mathcal{J}_\epsilon u &\rightarrow u \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } \epsilon \rightarrow +0, \\ \sup_{\epsilon \in [0,1]} \|\mathcal{J}_\epsilon\|_{\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} &\leq C, \\ \{[A, \mathcal{J}_\epsilon] : 0 < \epsilon < 1\} &\text{ is a bounded set in } \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)) \end{aligned}$$

for any  $A(x, D, s) \in C^0([0, 1]; C_{\text{cl}}^1 S^{\frac{1}{2}, 1, s}(\mathbb{R}^n))$ . Let  $\epsilon \in (0, 1)$ . We consider the following Cauchy problem for an ordinary differential equation in a Banach space:

$$\frac{\partial W_\epsilon}{\partial x_n} + \mathcal{J}_\epsilon^* K(x, D', s) \mathcal{J}_\epsilon W_\epsilon = \mathcal{J}_\epsilon^* F, \quad \text{in } G, \quad (\text{A.24})$$

$$W_\epsilon(x', 0) = g. \quad (\text{A.25})$$

For any  $g \in H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^n)$  and  $F \in L^2(G)$ , there exists a unique solution  $W_\epsilon$  to problem (A.24)–(A.25) with  $W_\epsilon \in L^\infty(0, \gamma; H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n))$  and  $\frac{\partial W_\epsilon}{\partial x_n} \in L^2(G)$ .

Simple computations provide

$$\begin{aligned} &\frac{d}{dx_n} (P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= \left( P\tilde{\Lambda}^{\frac{1}{2}} \frac{\partial W_\epsilon}{\partial x_n}, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon \right)_{L^2(\mathbb{R}^n)} + \left( P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} \frac{\partial W_\epsilon}{\partial x_n} \right)_{L^2(\mathbb{R}^n)} + (P'\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= (P\tilde{\Lambda}^{\frac{1}{2}} (-\mathcal{J}_\epsilon^* K \mathcal{J}_\epsilon W_\epsilon + \mathcal{J}_\epsilon^* F), \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} (-\mathcal{J}_\epsilon^* K \mathcal{J}_\epsilon W_\epsilon + \mathcal{J}_\epsilon^* F))_{L^2(\mathbb{R}^n)} \\ &\quad + (P'\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= - (P\mathcal{J}_\epsilon^* K \mathcal{J}_\epsilon \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &\quad - (P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \mathcal{J}_\epsilon^* K \mathcal{J}_\epsilon \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F)_{L^2(\mathbb{R}^n)} + (P'\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= - ((P\mathcal{J}_\epsilon^* K \mathcal{J}_\epsilon + \mathcal{J}_\epsilon^* K^* \mathcal{J}_\epsilon P) \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &\quad + (P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F)_{L^2(\mathbb{R}^n)} + (P'\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= - ((PK + K^*P) \mathcal{J}_\epsilon \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \mathcal{J}_\epsilon \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &\quad - (\mathcal{J}_\epsilon K^* [\mathcal{J}_\epsilon, P] \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F)_{L^2(\mathbb{R}^n)} + (P'\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Here  $P'$  is the pseudodifferential operator with symbol  $\frac{\partial}{\partial x_n} p(x, \xi', s)$ . Note that

$$|(\mathcal{J}_\epsilon K^* [\mathcal{J}_\epsilon, P] \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)}| \leq C \|W_\epsilon\|_{H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)}, \quad (\text{A.26})$$

$$|(P'\tilde{\Lambda}^{\frac{1}{2}} W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)}| \leq C \|W_\epsilon\|_{H^{\frac{1}{4}, \frac{1}{2}, s}(\mathbb{R}^n)}. \quad (\text{A.27})$$

Here and below, all constants  $C$  are independent of  $\epsilon \in (0, 1)$ . After simple computations, we obtain

$$\begin{aligned} &(P\tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon^* F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} = ([P\tilde{\Lambda}^{\frac{1}{2}}, \mathcal{J}_\epsilon^*] F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\tilde{\Lambda}^{\frac{1}{2}} F, \mathcal{J}_\epsilon \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= ([P\tilde{\Lambda}^{\frac{1}{2}}, \mathcal{J}_\epsilon^*] F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (F, \tilde{\Lambda}^{\frac{1}{2}} P^* \mathcal{J}_\epsilon \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} \\ &= ([P\tilde{\Lambda}^{\frac{1}{2}}, \mathcal{J}_\epsilon^*] F, \tilde{\Lambda}^{\frac{1}{2}} W_\epsilon)_{L^2(\mathbb{R}^n)} + (F, \tilde{\Lambda}^{\frac{1}{2}} P^* \tilde{\Lambda}^{\frac{1}{2}} \mathcal{J}_\epsilon W_\epsilon)_{L^2(\mathbb{R}^n)} + (F, \tilde{\Lambda}^{\frac{1}{2}} P^* [\mathcal{J}_\epsilon, \tilde{\Lambda}^{\frac{1}{2}}] W_\epsilon)_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Therefore, for any positive  $\delta$ , we have

$$\begin{aligned} & |(P\tilde{\Lambda}^{\frac{1}{2}}\mathcal{J}_\epsilon F, \tilde{\Lambda}^{\frac{1}{2}}W_\epsilon)_{L^2(\mathbb{R}^n)}| \\ & \leq C(\delta)\|F(\cdot, x_n)\|_{L^2(\mathbb{R}^n)}^2 + C\|\tilde{\Lambda}^{\frac{1}{2}}W_\epsilon\|_{L^2(\mathbb{R}^n)}^2 + \delta\|\mathcal{J}_\epsilon W_\epsilon\|_{H^{\frac{1}{2},1,s}(\mathbb{R}^n)}^2. \end{aligned} \quad (\text{A.28})$$

Similarly, we have

$$\begin{aligned} & |(P\tilde{\Lambda}^{\frac{1}{2}}W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}}\mathcal{J}_\epsilon^* F)_{L^2(\mathbb{R}^n)}| \\ & \leq C(\delta)\|F(\cdot, x_n)\|_{L^2(\mathbb{R}^n)}^2 + C\|\tilde{\Lambda}^{\frac{1}{2}}W_\epsilon\|_{L^2(\mathbb{R}^n)}^2 + \delta\|\mathcal{J}_\epsilon W_\epsilon\|_{H^{\frac{1}{2},1,s}(\mathbb{R}^n)}^2. \end{aligned} \quad (\text{A.29})$$

Using (A.26)–(A.29), we obtain

$$\begin{aligned} & \frac{d}{dx_n}(P\tilde{\Lambda}^{\frac{1}{2}}W_\epsilon, \tilde{\Lambda}^{\frac{1}{2}}W_\epsilon)_{L^2(\mathbb{R}^n)} + C_1\|\mathcal{J}_\epsilon W_\epsilon\|_{H^{\frac{1}{2},1,s}(\mathbb{R}^n)}^2 \\ & \leq C(\|\tilde{\Lambda}^{\frac{1}{2}}W_\epsilon\|_{L^2(\mathbb{R}^n)}^2 + \|F(\cdot, x_n)\|_{L^2(\mathbb{R}^n)}^2). \end{aligned} \quad (\text{A.30})$$

Applying Gronwall's inequality, we obtain

$$\|W_\epsilon\|_{L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))} + \|\mathcal{J}_\epsilon W_\epsilon\|_{L^2(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \quad (\text{A.31})$$

Using (A.31) from (A.15), we obtain the estimate for  $\frac{\partial W_\epsilon}{\partial x_n}$ :

$$\left\| \frac{\partial W_\epsilon}{\partial x_n} \right\|_{L^2(G)} \leq C(\|g\|_{H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \quad (\text{A.32})$$

Inequalities (A.31) and (A.32) imply

$$\|W_\epsilon\|_{H^{\frac{1}{2},1,s}(G)} + \|W_\epsilon\|_{L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \quad (\text{A.33})$$

We can now extract a subsequence still denoted by  $\epsilon$ , such that

$$W_\epsilon \rightharpoonup W, \quad \text{weakly-}^* \text{ in } L^\infty(0,\gamma;L^2(\mathbb{R}^n)), \quad (\text{A.34})$$

$$\frac{\partial W_\epsilon}{\partial x_n} \rightharpoonup \frac{\partial W}{\partial x_n}, \quad \text{weakly in } L^2(G), \quad (\text{A.35})$$

$$\mathcal{J}_\epsilon W_\epsilon \rightharpoonup W, \quad \text{weakly in } L^2(0,\gamma;H^{\frac{1}{2},1}(\mathbb{R}^n)), \quad (\text{A.36})$$

$$\mathcal{J}_\epsilon W_\epsilon \rightharpoonup W, \quad \text{weakly-}^* \text{ in } L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)). \quad (\text{A.37})$$

Of course,  $W \in H^{\frac{1}{2},1}(G)$  and  $W$  is a solution of (A.15)–(A.16). From (A.33) we obtain (A.20).

Now we prove (2) of this lemma. Since the space  $L^2(G)$  is dense in the space  $\{F \in H^{-\frac{1}{2},-1,s}(G) \mid \text{supp } F \subset\subset G\}$ , in order to prove (2), it suffices to establish the a priori estimate (A.21). Let  $\Phi$  be a solution to the following boundary value problem:

$$\left( -\frac{\partial}{\partial x_n} + K^*(x, D', s) \right) \Phi = W, \quad \text{in } G, \quad \Phi(\cdot, \gamma) = 0. \quad (\text{A.38})$$

By the definition of a weak solution, we have

$$\|W\|_{L^2(G)}^2 = (g, \Phi(\cdot, \gamma))_{L^2(\mathbb{R}^n)} + (F, \Phi)_{L^2(G)}. \quad (\text{A.39})$$

By (1) of this lemma, the solution to (A.38) satisfies the estimate

$$\|\Phi\|_{H^{\frac{1}{2},1,s}(G)} + \|\Phi\|_{L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))} \leq C\|W\|_{L^2(G)}. \quad (\text{A.40})$$

Using in equality (A.39) estimate (A.40), we obtain (A.21).

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