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Solutions and Multiple Solutions for $p(x)$-Laplacian Equations with Nonlinear Boundary Condition**

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Abstract The authors study the $p(x)$-Laplacian equations with nonlinear boundary condition. By using the variational method, under appropriate assumptions on the perturbation terms $f_1(x,u)$, $f_2(x,u)$ and $h_1(x)$, $h_2(x)$, such that the associated functional satisfies the "mountain pass lemma" and "fountain theorem" respectively, the existence and multiplicity of solutions are obtained. The discussion is based on the theory of variable exponent Lebesgue and Sobolev spaces.

Keywords $p(x)$-Laplacian, Nonlinear boundary condition, (PS) condition, Mountain pass lemma, Fountain theorem
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1 Introduction

In recent years, there has been a strong rise of interest in the study of various mathematical problems with variable exponent $p(x)$, which arises from eletrorheological fluids (see [1]) and elastic mechanics (see [2]). The application backgrounds also be traced in the book of Diening [3] and in the papers of Acerbi and Mingione [4, 5] and Mihăilescu, Rădulescu [6].

In this paper, we deal with the following nonlinear elliptic boundary value problem:

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f_1(x,u) - \mathrm{sgn}(u)h_1(x), & \text{in } \Omega, \\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = c(x)|u|^{q(x)-2}u + f_2(x,u) - \mathrm{sgn}(u)h_2(x), & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\nu}$ is outer normal derivative, $p(x) \in C(\overline{\Omega})$, $q(x) \in C(\partial\Omega)$, $p(x), q(x) > 1$ and $p(x) \neq q(y)$, $\forall\, x \in \Omega$, $y \in \partial\Omega$, $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathèodory functions, and the perturbations $h_1(x) \in L^{p'(x)}(\Omega) \cap L^\infty(\Omega)$, $h_2(x) \in L^{p'(x)}(\partial\Omega) \cap L^\infty(\partial\Omega)$ with $p'(x) = \frac{p(x)}{p(x)-1}$. Throughout this paper, we assume that $a(x)$, $b(x)$ and $c(x)$ satisfy $0 < a_1 \le a(x) \le a_2$, $0 < b_1 \le b(x) \le b_2$, $0 < c_1 \le c(x) \le c_2$, $\forall\, x \in \Omega$.

In the past decade, many people studied the nonlinear boundary value problems involving p-Laplacian. For example, if $a(x) = b(x) = c(x) \equiv 1$, $p(x) \equiv p$, $q(x) \equiv q$ (a constant) and

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$h_1(x) = h_2(x) \equiv 0$, then problem (1.1) becomes

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f_1(x,u), & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = |u|^{q-2}u + f_2(x,u), & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Bonder and Rossi [7] considered the existence of nontrivial solutions of problem (1.2) when $f_1(x,u) \equiv 0$, and discussed different cases when $f_2(x,u)$ is subcritical, critical and supercritical with respect to u. We also mention that Martinez and Rossi [8] studied the existence of solutions when $p = q$ and the perturbation terms $f_1(x,u)$ and $f_2(x,u)$ satisfy the Landesman-Lazer type conditions. Recently, Zhao and Zhao [9] studied the nonlinear boundary value problem, assumed that $f_1(x,u)$ satisfies the Ambrosetti-Rabinowitz type condition and got the multiple results. For other results involving problem (1.2), we refer readers to the references [10, 11].

When $p(x) \equiv p$, $q(x) \equiv q$ (a constant) and $h_1(x) = h_2(x) \equiv 0$, problem (1.1) becomes

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = f_1(x,u), & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = c(x)|u|^{q-2}u + f_2(x,u), & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

There are also many people who studied the p-Laplacian nonlinear boundary value problems involving (1.3). For example, Cîrstea and Rădulescu [12] used the weighted Sobolev space to discuss the existence and non-existence results, assuming that $f_1(x,u)$ is a special case in the problem (1.3), where Ω is an unbounded domain. Pflüger [13], by using the same technique, considered the existence and multiplicity of solutions when $b(x) \equiv 0$. The author showed the existence result when $f_1(x,u)$ and $f_2(x,u)$ are superlinear and satisfy the Ambrosetti-Rabinowitz type condition, and got the multiplicity of solutions when one of $f_1(x,u)$ and $f_2(x,u)$ is sublinear and the other one is superlinear. For other relative results to the problem (1.3), we refer to [14–17] and the references therein.

More recently, the study on the nonlinear boundary value problems with variable exponent has received considerable attention. For example, Deng [18] studied the eigenvalue of $p(x)$-Laplacian Steklov problem, and discussed the properties of the eigenvalue sequence under different conditions. Fan [19] discussed the boundary trace embedding theorems for variable exponent Sobolev spaces and some applications. Yao [20] constrained the two nonlinear perturbation terms $f_1(x,u)$ and $f_2(x,u)$ in appropriate conditions and got a number of results on existence and multiplicity of solutions. Motivated by Yao and problem (1.3), we consider the more general form of variable exponent boundary value problem (1.1). In this paper, we assume that the two nonlinear perturbation terms $f_1(x,u)$ and $f_2(x,u)$ do not satisfy the Ambrosetti-Rabinowitz type condition, and by using the "mountain pass lemma" and "fountain theorem", respectively, we get the existence and multiplicity of solutions of (1.1). These results extend some of the results in [19] and the classical results for the p-Laplacian in [7, 8, 10, 11].

This paper is organized as follows. In Section 2, we introduce some basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$, $L^{p(x)}(\partial\Omega)$ and generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ which are needed in the paper. In Section 3, we give the assumptions on $f_1(x,u)$ and $f_2(x,u)$, and state the main results of this paper. In Section 4, we prove the main results — Theorems 3.1 and 3.2.

2 Mathematical Preliminaries

In order to discuss problem (1.1), we need to state some properties of the space $W^{1,p(x)}(\Omega)$ which we call generalized Lebesgue-Sobolev spaces. Let Ω be a bounded domain of $\mathbb{R}^N$ and denote

$$C_+(\overline{\Omega}) = \{p(x) \mid p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall\, x \in \overline{\Omega}\}.$$

For $p(x) \in C_+(\overline{\Omega})$, we write

$$p^- = \min\{p(x); x \in \overline{\Omega}\}, \quad p^+ = \max\{p(x); x \in \overline{\Omega}\}.$$

We can also denote $C_+(\partial\Omega)$ and q^-, q^+ for any $q(x) \in C(\partial\Omega)$, and define

$$L^{p(x)}(\Omega) = \Big\{u \,\Big|\, u \text{ is a measureable real-valued function, } \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty\Big\},$$
$$L^{p(x)}(\partial\Omega) = \Big\{u \,\Big|\, u : \partial\Omega \to \mathbb{R} \text{ is measureable, } \int_{\partial\Omega} |u(x)|^{p(x)} \mathrm{d}\sigma_x < \infty\Big\},$$

with norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ defined by

$$|u|_{L^{p(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} \mathrm{d}x \le 1\Big\},$$
$$|u|_{L^{p(x)}(\partial\Omega)} = \inf\Big\{\tau > 0 : \int_{\partial\Omega} \Big|\frac{u(x)}{\tau}\Big|^{p(x)} \mathrm{d}\sigma_x \le 1\Big\},$$

where $\mathrm{d}\sigma_x$ is the surface measure on $\partial\Omega$. Therefore, $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$ and $(L^{p(x)}(\partial\Omega), |\cdot|_{L^{p(x)}(\partial\Omega)})$ become Banach spaces.

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with norm

$$\|u\| = \inf\Big\{\lambda > 0 \,\Big|\, \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} + \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\Big) \mathrm{d}x \le 1\Big\}.$$

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$\|u\|' = \inf\Big\{\lambda > 0 : \int_\Omega \Big(a(x)\Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} + b(x)\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\Big) \mathrm{d}x \le 1\Big\},$$

then from the assumptions of $a(x)$ and $b(x)$ it is easy to check that $\|\cdot\|'$ is an equivalent norm on $W^{1,p(x)}(\Omega)$. For simplicity, we denote

$$\Phi(u) = \int_\Omega (a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)}) \mathrm{d}x.$$

Hence we have (see [21])

(i) if $\Phi(u) \ge 1$, then $\xi_1 \|u\|^{p^-} \le \Phi(u) \le \xi_2 \|u\|^{p^+}$,

(ii) if $\Phi(u) \le 1$, then $\zeta_1 \|u\|^{p^+} \le \Phi(u) \le \zeta_2 \|u\|^{p^-}$,

where ξ_1, ξ_2 and ζ_1, ζ_2 are positive constants independent of u.

Proposition 2.1 (see [20]) (1) *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is a separable, uniform convex Banach space, and its conjugate space is* $L^{q(x)}(\Omega)$ *where* $\frac{1}{q(x)}+\frac{1}{p(x)}=1$. *For any* $u\in L^{p(x)}(\Omega)$ *and* $v\in L^{p(x)}(\Omega)$, *we have*

$$\Big|\int_\Omega uv\mathrm{d}x\Big|\le\Big(\frac{1}{p^-}+\frac{1}{q^-}\Big)|u|_{L^{p(x)}(\Omega)}|v|_{L^{q(x)}(\Omega)}.$$

(2) *If* $p_1,p_2\in C_+(\overline{\Omega})$ *and* $p_1(x)\le p_2(x)$ *for any* $x\in\overline{\Omega}$, *then* $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

(3) *If* $q(x)\in C_+(\overline{\Omega})$ *and* $q(x)<p^*(x)$, $\forall\, x\in\overline{\Omega}$, *then the embedding from* $W^{1,p(x)}(\Omega)$ *into* $L^{q(x)}(\Omega)$ *is compact and continuous, where*

$$p^*(x):=\begin{cases}\dfrac{Np(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\ge N.\end{cases}$$

Proposition 2.2 (see [20]) *If we denote*

$$p_*(x):=\begin{cases}\dfrac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\ge N,\end{cases}$$

then the embedding from $W^{1,p(x)}(\Omega)$ *into* $L^{q(x)}(\partial\Omega)$ *is compact and continuous, where* $q(x)\in C_+(\partial\Omega)$ *and* $q(x)<p_*(x)$, $\forall\, x\in\partial\Omega$.

Proposition 2.3 (see [22]) *Suppose that* $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathèodory function and satisfies*

$$|f(x,s)|\le a(x)+b|s|^{\frac{p_1(x)}{p_2(x)}},\quad \forall\, x\in\overline{\Omega},\ s\in\mathbb{R},$$

where $p_1,p_2\in C_+(\overline{\Omega})$, $a(x)\in L^{p_2(x)}(\Omega)$, $a(x)\ge 0$ *and* $b\ge 0$ *is a constant. Denote* N_f, *the Nemytsky operator, by*

$$N_f(u)(x)=f(x,u(x)).$$

Then N_f *is a continuous and bounded map from* $L^{p_1(x)}(\Omega)$ *to* $L^{p_2(x)}(\Omega)$.

Proposition 2.4 (see [20]) *If we denote*

$$\rho(u)=\int_\Omega |u(x)|^{p(x)}\mathrm{d}x,\quad \forall\, u\in L^{p(x)}(\Omega),$$

then

(1) $|u|_{L^{p(x)}(\Omega)}<1(=1;>1)\Leftrightarrow \rho(u)<1(=1;>1)$;

(2) $|u|_{L^{p(x)}(\Omega)}>1\Rightarrow |u|^{p^-}_{L^{p(x)}(\Omega)}\le\rho(u)\le|u|^{p^+}_{L^{p(x)}(\Omega)}$;

(3) $|u|_{L^{p(x)}(\Omega)}<1\Rightarrow |u|^{p^+}_{L^{p(x)}(\Omega)}\le\rho(u)\le|u|^{p^-}_{L^{p(x)}(\Omega)}$.

Proposition 2.5 (see [18]) *If we denote*

$$\rho(u)=\int_{\partial\Omega}|u(x)|^{p(x)}\mathrm{d}\sigma_x,\quad \forall\, u\in L^{p(x)}(\partial\Omega),$$

then

(1) $|u|_{L^{p(x)}(\partial\Omega)}>1\Rightarrow |u|^{p^-}_{L^{p(x)}(\partial\Omega)}\le\rho(u)\le|u|^{p^+}_{L^{p(x)}(\partial\Omega)}$;

(2) $|u|_{L^{p(x)}(\partial\Omega)}<1\Rightarrow |u|^{p^+}_{L^{p(x)}(\partial\Omega)}\le\rho(u)\le|u|^{p^-}_{L^{p(x)}(\partial\Omega)}$.

In this paper, we denote $X := W^{1,p(x)}(\Omega)$, $X^* := (W^{1,p(x)}(\Omega))^*$ the dual space, $\langle \cdot, \cdot \rangle$ the dual pair, and let "$\rightharpoonup$" represent weak convergence. By the assumptions on $a(x)$, $b(x)$ and $c(x)$ in Section 1, it is easy to check that the following assertions are true.

Proposition 2.6 (see [23]) *If we denote*

$$I(u) = \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x, \quad \forall\, u \in X,$$

then $I \in C^1(X, \mathbb{R})$ and the derivative operator of I, denoted by L, is

$$\langle L(u), v\rangle = \int_\Omega (a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv)\mathrm{d}x, \quad \forall\, u, v \in X,$$

and we have

(i) *$L : X \to X^*$ is a continuous, bounded and strictly monotone operator;*

(ii) *L is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in X and $\limsup\limits_{n\to\infty}(L(u_n)-L(u), u_n-u) \le 0$, then $u_n \to u$ in X;*

(iii) *$L : X \to X^*$ is a homeomorphism.*

Proposition 2.7 (see [18]) *If we denote*

$$\phi(u) = \int_{\partial\Omega} \frac{1}{q(x)}(c(x)|u|^{q(x)})\mathrm{d}\sigma_x, \quad \forall\, u \in X,$$

where $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p_*(x)$, $\forall\, x \in \overline{\Omega}$, then $\phi \in C^1(X, \mathbb{R})$ and the derivative operator J of ϕ is

$$\langle J(u), v\rangle = \int_{\partial\Omega} c(x)|u|^{q(x)-2}uv\mathrm{d}\sigma_x, \quad \forall\, u, v \in X,$$

and we have that $\phi : X \to R$ and $J : X \to X^*$ are sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u_0$ in X implies $J(u_n) \to J(u_0)$.

Let X be a reflexive and separable Banach space. Then there are $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{\mathrm{span}\{e_j \mid j = 1, 2, \cdots\}}, \quad X^* = \overline{\mathrm{span}\{e_j^* \mid j = 1, 2, \cdots\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

For convenience, we write

$$X_j = \mathrm{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{2.1}$$

Theorem 2.1 (Fountain Theorem) (see [24]) *Assume*

(A_1) *X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional and the subspaces X_k, Y_k and Z_k are defined by* (2.1).

Suppose that, for every $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that

(A_2) $\inf\{\varphi(u); u \in Z_k, \|u\| = \gamma_k\} \to \infty$, *as $k \to \infty$;*

(A$_3$) $\max\{\varphi(u); u \in Y_k, \|u\| = \rho_k\} \leq 0$;

(A$_4$) *φ satisfies $(\mathrm{PS})_c$ condition for every $c > 0$.*

Then φ has a sequence of critical values tending to $+\infty$.

Theorem 2.2 (Dual Fountain Theorem) (see [24]) *Assume that* (A$_1$) *is satisfied and there is a $k_0 > 0$ such that, for each $k \geq k_0$, there exists $\rho_k > \gamma_k > 0$ such that*

(B$_1$) $\inf\{\varphi(u); u \in Z_k, \|u\| = \rho_k\} \geq 0$;

(B$_2$) $b_k := \max\{\varphi(u); u \in Y_k, \|u\| = \gamma_k\} < 0$;

(B$_3$) $d_k := \inf\{\varphi(u); u \in Z_k, \|u\| \leq \rho_k\} \to 0$, *as* $k \to \infty$;

(B$_4$) *φ satisfies $(\mathrm{PS})_c^*$ condition for every $c \in [d_{k_0}, 0)$.*

Then φ has a sequence of negative critical values converging to 0.

Remark 2.1 That φ satisfies $(\mathrm{PS})_c^*$ condition means that any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $\varphi(u_{n_j}) \to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$ contains a subsequence converging to a critical point of φ.

3 Assumptions and Statement of Main Results

For a variational approach, the functional associated to problem (1.1) is

$$\begin{aligned}\varphi(u) = &\int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x - \int_{\partial\Omega} \frac{1}{q(x)}(c(x)|u|^{q(x)})\mathrm{d}\sigma_x \\ &- \int_\Omega F_1(x,u)\mathrm{d}x - \int_{\partial\Omega} F_2(x,u)\mathrm{d}\sigma_x + \int_\Omega h_1(x)|u|\mathrm{d}x + \int_{\partial\Omega} h_2(x)|u|\mathrm{d}\sigma_x.\end{aligned}$$

Then $\varphi : X \to \mathbb{R}$, where F_1 and F_2 denote the primitive functions of f_1 and f_2, i.e., $F_i(x,u) = \int_0^u f_i(x,s)\mathrm{d}s$, $i = 1, 2$. Obviously, we can give some appropriate assumptions on f_1 and f_2 such that $\varphi(u) \in C^1(X, \mathbb{R})$, and

$$\begin{aligned}\langle \varphi'(u), v\rangle = &\int_\Omega (a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv)\mathrm{d}x - \int_{\partial\Omega}(c(x)|u|^{q(x)-2}uv)\mathrm{d}\sigma_x \\ &- \int_\Omega f_1(x,u)v\mathrm{d}x - \int_{\partial\Omega} f_2(x,u)v\mathrm{d}\sigma_x + \int_\Omega \mathrm{sgn}(u)h_1(x)v\mathrm{d}x + \int_{\partial\Omega}\mathrm{sgn}(u)h_2(x)v\mathrm{d}\sigma_x\end{aligned}$$

for any $u, v \in X$. Then we know that the weak solution of (1.1) corresponds to the critical point of the functional φ.

Now we state below the assumptions for problem (1.1), and let $\mho_1 = \Omega$, $\mho_2 = \partial\Omega$.

(i) For any $\eta \in [q^-, q^+]$ and almost all $x \in \mho_i$, we have

$$\lim_{|u|\to\infty} \frac{\eta F_i(x,u) - f_i(x,u)u}{|u|^{\mu(x)}} = 0$$

with $\mu(x) \in C_+(\overline{\Omega})$ and $\mu^+ < p^-$;

(ii) There exist $\vartheta_i > 0$, $i = 1, 2$, such that for almost all $x \in \mho_i$ and all $u \in \mathbb{R}$, we have

$$|f_i(x,u) - \mathrm{sgn}(u)h_i(x)| \leq \vartheta_i |u|^{\beta_i(x)-1}$$

with $\beta_1(x) \in C_+(\overline{\Omega})$, $\beta_1(x) < p^*(x)$, $\forall\, x \in \Omega$, and $\beta_2(x) \in C_+(\partial\Omega)$, $\beta_2(x) < p_*(x)$, $\forall\, x \in \partial\Omega$;

(iii) For almost all $x \in \mho_i$, $i = 1, 2$, we have

$$\liminf_{|u|\to\infty} \frac{F_i(x,u)}{|u|^{\beta_i(x)}} > 0;$$

(iv) $f_i(x, -u) = -f_i(x, u)$, $\forall\, x \in \mho_i$, $u \in \mathbb{R}$, $i = 1, 2$.

Since $h_1(x) \in L^{p'(x)}(\Omega) \cap L^\infty(\Omega)$ and $h_2(x) \in L^{p'(x)}(\partial\Omega) \cap L^\infty(\partial\Omega)$, the functional we defined above is of class $C^1(X, \mathbb{R})$ under condition (ii); moreover, φ is even if (iv) holds. Our main results are as follows.

Theorem 3.1 *Suppose that hypotheses* (i)–(iii) *hold. We have*

(1) *if* $p^+ < q^-, \beta_i^-$, $i = 1, 2$, *then problem* (1.1) *has at least one nontrivial solution;*

(2) *if* $q^+, \beta_i^+ < p^-$, $i = 1, 2$, *then problem* (1.1) *has at least one solution.*

Theorem 3.2 *Suppose that hypotheses* (i)–(iii) *hold. We have*

(1) *if* $\beta_i^+ < q^-$ *and* $p(x) < \beta_i(x)$, $\forall\, x \in \mho_i$, $i = 1, 2$, *then problem* (1.1) *has a sequence of solutions* u_k *such that* $\varphi(u_k) \to +\infty$, *as* $k \to +\infty$*;*

(2) *if* $q^+ < \beta_i^-$ *and* $\beta_i(x) < p(x)$, $\forall\, x \in \mho_i$, $i = 1, 2$, *then problem* (1.1) *has a sequence of solutions* v_k *such that* $\varphi(v_k) < 0$, $\varphi(v_k) \to 0$, *as* $k \to +\infty$.

4 Proof of Theorems 3.1 and 3.2

Before proving Theorem 3.1, we will prove the following two lemmas. Thereafter, for simplicity, we will denote by C the positive constants, and they may be different despite the same appearance.

Lemma 4.1 *If the assumptions in Theorem* 3.1(1) *hold, then* φ *satisfies* $(\mathrm{PS})_c$ *condition with* $c > 0$.

Proof Suppose $\{u_n\}_{n\geq 1} \subset X$, and for every $c > 0$,

$$\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0, \quad \text{in } X^*, \text{ as } n \to \infty.$$

Then for n large enough, we can find $M_1 > 0$ such that

$$|\varphi(u_n)| \leq M_1. \tag{4.1}$$

By assumption (ii), we know that the Carathèodory functions $f_i(x, u)$ $(i = 1, 2)$ are in the subcritical growth. So, through Propositions 2.1–2.3, we know that if we denote

$$\begin{aligned} H(u) &= \int_\Omega F_1(x,u)\mathrm{d}x + \int_{\partial\Omega} F_2(x,u)\mathrm{d}\sigma_x - \int_\Omega h_1(x)|u|\mathrm{d}x - \int_{\partial\Omega} h_2(x)|u|\mathrm{d}\sigma_x \\ &= \int_\Omega (F_1(x,u) - h_1(x)|u|)\mathrm{d}x + \int_{\partial\Omega} (F_2(x,u) - h_2(x)|u|)\mathrm{d}\sigma_x, \end{aligned}$$

then $H(u)$ is weakly continuous and has a derivative operator, denoted by ψ, and we see that

$$\langle \psi(u), v\rangle = \int_\Omega f_1(x,u)v\mathrm{d}x + \int_{\partial\Omega} f_2(x,u)v\mathrm{d}\sigma_x - \int_\Omega \mathrm{sgn}(u)h_1(x)v\mathrm{d}x - \int_{\partial\Omega} \mathrm{sgn}(u)h_2(x)v\mathrm{d}\sigma_x$$

is compact (see [11]). By Propositions 2.6 and 2.7, we deduce that $\varphi' = L - J - \psi$ is also of type (S_+). Since $\varphi'(u_n) \to 0$, we have $\langle \varphi'(u_n), u_n\rangle \to 0$. In particular, the sequence $\{\langle \varphi'(u_n), u_n\rangle\}_{n\geq 1}$ is bounded. Thus there exists $M_2 > 0$ such that

$$|\langle \varphi'(u_n), u_n\rangle| \leq M_2. \tag{4.2}$$

We claim that the sequence $\{u_n\}_{n\geq 1}$ is bounded. If it is not true, by passing a subsequence if necessary, we may assume $\|u_n\| \to +\infty$. Without loss of generality, we assume $\|u_n\| \geq 1$ and $|u_n| \to +\infty$ as $n \to +\infty$ for any $x \in \Omega$.

From (4.1) and (4.2), we have

$$\begin{aligned} M_1 &\geq \varphi(u_n) = I(u_n) - \phi(u_n) - H(u_n) \\ &\geq \frac{1}{p^+}\Phi(u_n) - \frac{1}{q^-}\int_{\partial\Omega}(c(x)|u_n|^{q(x)})\mathrm{d}\sigma_x - H(u_n), \end{aligned} \tag{4.3}$$

$$M_2 \geq -\langle \varphi'(u_n), u_n\rangle = -\Phi(u_n) + \int_{\partial\Omega}(c(x)|u_n|^{q(x)})\mathrm{d}\sigma_x + \langle \psi(u_n), u_n\rangle. \tag{4.4}$$

Combining (4.3) and (4.4), we have

$$\begin{aligned} q^- M_1 + M_2 &\geq \Big(\frac{q^-}{p^+} - 1\Big)\Phi(u_n) - q^- H(u_n) + \langle \psi(u_n), u_n\rangle \\ &\geq \Big(\frac{q^-}{p^+} - 1\Big)\xi_1\|u_n\|^{p^-} - \int_\Omega (q^- F_1(x,u_n) - f_1(x,u_n)u_n)\mathrm{d}x \\ &\quad - \int_{\partial\Omega}(q^- F_2(x,u_n) - f_2(x,u_n)u_n)\mathrm{d}\sigma_x \\ &\quad + (q^- - 1)\Big(\int_\Omega h_1(x)|u_n|\mathrm{d}x + \int_{\partial\Omega} h_2(x)|u_n|\mathrm{d}\sigma_x\Big). \end{aligned} \tag{4.5}$$

By virtue of assumption (i), let $\eta = q^-$ and n be large enough. Then for almost all $x \in \mho_i$, we have $q^- F_i(x,u_n) - f_i(x,u_n)u_n \leq \varepsilon |u_n|^{\mu(x)}$. Thus by Hölder's inequality, (4.5) becomes

$$\begin{aligned} q^- M_1 + M_2 &\geq \Big(\frac{q^-}{p^+} - 1\Big)\xi_1\|u_n\|^{p^-} - \varepsilon\Big(\int_\Omega |u_n|^{\mu(x)}\mathrm{d}x + \int_{\partial\Omega}|u_n|^{\mu(x)}\mathrm{d}\sigma_x\Big) \\ &\quad - 2(q^- - 1)(|h_1|_{L^{p'(x)}(\Omega)}|u_n|_{L^{p(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)}|u_n|_{L^{p(x)}(\partial\Omega)}) \\ &\geq \Big(\frac{q^-}{p^+} - 1\Big)\xi_1\|u_n\|^{p^-} - C\varepsilon\|u_n\|^{\mu^+} \\ &\quad - 2(q^- - 1)C(|h_1|_{L^{p'(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)})\|u_n\|. \end{aligned}$$

The last inequality follows from the compact embedding in Propositions 2.1 and 2.2. Since $q^- > p^+$, we have $\frac{q^-}{p^+} - 1 > 0$ and $p^- > \mu^+$, which implies that the sequence $\{u_n\}_{n\geq 1} \subset X$ is bounded. It is a contradiction to the supposition. Therefore, we have proved that $\{u_n\}_{n\geq 1} \subset X$ is bounded. We may assume $u_n \rightharpoonup u_0$ in X as $n \to +\infty$. Note $\varphi' = L - J - \psi$. Then we have

$$\varphi'(u_n) = L(u_n) - J(u_n) - \psi(u_n) \to 0.$$

From Proposition 2.6, we know that L is a homeomorphism. Besides, by Proposition 2.7 and the fact that ψ is compact, we have

$$u_n \to L^{-1}(J(u_0) + \psi(u_0)), \quad \text{in } X, \text{ as } n \to +\infty.$$

Thus $u_n \to u_0$ in X. Therefore, φ satisfies $(\mathrm{PS})_c$ condition.

Lemma 4.2 *Under the assumptions in Theorem* 3.1(1)*, there exists* $\gamma_0 > 0$ *such that for all* $0 < \gamma < \gamma_0$ *we have* $\inf\{\varphi(u) : \|u\| = \gamma\} > 0$.

Proof By hypothesis (ii), for almost all $x \in \mho_i$ and all $u \in \mathbb{R}$, we have

$$f_i(x,u) \le \mathrm{sgn}(u) h_i(x) + \vartheta_i |u|^{\beta_i(x)-1},$$

which implies

$$F_i(x,u) \le h_i(x)|u| + \frac{\vartheta_i}{\beta_i(x)}|u|^{\beta_i(x)}, \quad i = 1,2.$$

So, if we assume $\|u\| < 1$ small enough such that $\xi_1\|u\|^{p^-} > \zeta_1 \|u\|^{p^+}$, we have

$$\begin{aligned}
\varphi(u) &= I(u) - \phi(u) - H(u)\\
&\ge \frac{1}{p^+}\Phi(u) - \frac{c_2}{q^-}\int_{\partial\Omega} |u|^{q(x)} \mathrm{d}\sigma_x - H(u)\\
&\ge \frac{1}{p^+}\min\{\xi_1\|u\|^{p^-}, \zeta_1\|u\|^{p^+}\} - \frac{Cc_2}{q^-}\|u\|^{q^-} - \frac{C\vartheta_1}{\beta_1^-}\|u\|^{\beta_1^-} - \frac{C\vartheta_2}{\beta_2^-}\|u\|^{\beta_2^-}\\
&\ge \frac{\zeta_1}{p^+}\|u\|^{p^+} - \frac{Cc_2}{q^-}\|u\|^{q^-} - \frac{C\vartheta_1}{\beta_1^-}\|u\|^{\beta_1^-} - \frac{C\vartheta_2}{\beta_2^-}\|u\|^{\beta_2^-}.
\end{aligned}$$

Since $p^+ < q^-$ and $p^+ < \beta_i^-$, $i = 1,2$, there exists $\gamma_0 > 0$ small enough, such that for all $0 < \gamma < \gamma_0$ we have $\inf\{\varphi(u) : \|u\| = \gamma\} > 0$.

Proof of Theorem 3.1(1) We claim that there exists $h \in X$ such that $\varphi(h) < 0$. By virtue of hypothesis (iii), there exists an $M_3 > 0$ such that, for all $x \in \mho_i$, $i = 1,2$ and $|u| > M_3$, we have $F_i(x,u) > C|u|^{\beta_i(x)}$. We choose $w \in X \setminus \{0\}$ and let $t > 1$ large enough such that $|tw| > M_3$. Then we have $F_i(x,tw) > C|tw|^{\beta_i(x)}$, $i = 1,2$. Therefore

$$\begin{aligned}
\varphi(tw) =& I(tw) - \phi(tw) - H(tw)\\
\le& \frac{t^{p^+}}{p^-}\Phi(w) - \int_{\partial\Omega} \frac{1}{q(x)}(c(x)|tw|^{q(x)})\mathrm{d}\sigma_x - C\int_\Omega |tw|^{\beta_1(x)}\mathrm{d}x\\
&- C\int_{\partial\Omega}|tw|^{\beta_2(x)}\mathrm{d}\sigma_x + \int_\Omega h_1(x)|tw|\mathrm{d}x + \int_{\partial\Omega} h_2(x)|tw|\mathrm{d}\sigma_x\\
\le& \frac{t^{p^+}}{p^-}\Phi(w) - \frac{c_1 t^{q^-}}{q^+}\int_{\partial\Omega}|w|^{q(x)}\mathrm{d}\sigma_x - Ct^{\beta_1^-}\int_\Omega |w|^{\beta_1(x)}\mathrm{d}x\\
&- Ct^{\beta_2^-}\int_{\partial\Omega}|w|^{\beta_2(x)}\mathrm{d}\sigma_x + t\Big(\int_\Omega h_1(x)|w|\mathrm{d}x + \int_{\partial\Omega} h_2(x)|w|\mathrm{d}\sigma_x\Big).
\end{aligned}$$

Since $1 < p^+ < \beta_i^-$, $i = 1,2$ and $1 < p^+ < q^-$, we have $\varphi(tw) \to -\infty$ as $t \to +\infty$. So we choose a t_0 large enough such that $\varphi(t_0 w) < 0$, and set $h = t_0 w$. Then h is the desired element. Since $\varphi(0) = 0$, from Lemmas 4.1 and 4.2 we see that φ satisfies the condition of mountain pass theorem (see [24]). So φ admits at least one nontrivial critical point. It is the nontrivial solution of (1.1).

Proof of Theorem 3.1(2) Under the conditions $q^+, \beta_i^+ < p^-$, $i = 1, 2$, we claim that φ is coercive. In fact, we assume $\|u\|$ large enough. Similarly to the proof of Lemma 4.2, we have

$$\varphi(u) = I(u) - \phi(u) - H(u) \geq \frac{\xi_1}{p^-}\|u\|^{p^-} - \frac{Cc_2}{q^-}\|u\|^{q^+} - \frac{C\vartheta_1}{\beta_1^-}\|u\|^{\beta_1^+} - \frac{C\vartheta_2}{\beta_2^-}\|u\|^{\beta_2^+}.$$

So φ is coercive. Besides, φ is sequentially weakly lower semicontinuous. Thus φ is bounded below and φ attains its infimum in X, i.e., $\varphi(u_0) = \inf\limits_{u\in X} \varphi(u)$ and u_0 is a critical point of φ, which is a weak solution of (1.1).

Remark 4.1 In the proof of Theorem 3.1(2), we can not guarantee that u_0 is nontrivial. In fact, we can also apply [25, Theorem 3.5] to get the weak solution because, when $p^- > q^+$, we can also prove that φ satisfies (PS) condition.

Now we prove Theorem 3.2 below. Firstly, we state the following useful lemma.

Lemma 4.3 (see [20]) *If $\alpha(x) \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$, $\forall\, x \in \overline{\Omega}$ and $\lambda(x) \in C_+(\partial\Omega)$, $\lambda(x) < p_*(x)$, $\forall\, x \in \partial\Omega$, and denote*

$$\begin{aligned}\alpha_k &= \sup\{|u|_{L^{\alpha(x)}(\Omega)}; \|u\| = 1, u \in Z_k\},\\ \lambda_k &= \sup\{|u|_{L^{\lambda(x)}(\partial\Omega)}; \|u\| = 1, u \in Z_k\},\end{aligned}$$

then $\lim\limits_{k\to\infty} \alpha_k = 0$, $\lim\limits_{k\to\infty} \lambda_k = 0$.

Proof of Theorem 3.2(1) We will prove that φ satisfies the conditions of Theorem 2.1. Obviously, because of the assumptions of (iv), φ is an even functional and satisfies $(\mathrm{PS})_c$ condition (see Lemma 4.1). We will prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that (A_2) and (A_3) hold. Let $u \in Z_k$ with $\|u\|$ appropriate large such that $\xi_1\|u\|^{p^-} < \zeta_1\|u\|^{p^+}$. Through the assumptions of (ii), we have

$$\begin{aligned}\varphi(u) &= I(u) - \phi(u) - H(u)\\ &\geq \frac{1}{p^+}\Phi(u) - \frac{c_2}{q^-}\int_{\partial\Omega}|u|^{q(x)}\mathrm{d}\sigma_x - \frac{\vartheta_1}{\beta_1^-}\int_\Omega |u|^{\beta_1(x)}\mathrm{d}x - \frac{\vartheta_2}{\beta_2^-}\int_{\partial\Omega}|u|^{\beta_2(x)}\mathrm{d}\sigma_x\\ &\geq \frac{1}{p^+}\min\{\xi_1\|u\|^{p^-}, \zeta_1\|u\|^{p^+}\} - \frac{c_2}{q^-}\max\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\}\\ &\quad - \frac{\vartheta_1}{\beta_1^-}\max\{|u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}\} - \frac{\vartheta_2}{\beta_2^-}\max\{|u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\}\\ &\geq \frac{\xi_1}{p^+}\|u\|^{p^-} - C(q^-, \beta_1^-, \beta_2^-)\max\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)},\\ &\quad |u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\}.\end{aligned}$$

If

$$\max\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}, |u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\} = |u|^{q^+}_{L^{q(x)}(\partial\Omega)},$$

then we have

$$\varphi(u) \geq \frac{\xi_1}{p^+}\|u\|^{p^-} - C(q^-, \beta_1^-, \beta_2^-)\lambda_k^{q^+}\|u\|^{q^+}.$$

Choose $\gamma_k = (q^+ C(q^-, \beta_1^-, \beta_2^-)\frac{\lambda_k^{q^+}}{\xi_1})^{\frac{1}{p^- - q^+}}$. For any $u \in Z_k$ with $\|u\| = \gamma_k$, we have

$$\varphi(u) \geq \xi_1\Big(\frac{1}{p^+} - \frac{1}{q^+}\Big)\gamma_k^{p^-}.$$

Since $\lambda_k \to 0$ as $k \to \infty$ and $1 < p^- \leq p^+ < q^- < q^+$, we have $\frac{1}{p^+} - \frac{1}{q^+} > 0$ and $\gamma_k \to \infty$. Thus, for sufficiently large k, we have $\varphi(u) \to \infty$ with $u \in Z_k$ and $\|u\| = \gamma_k$ as $k \to \infty$. In other cases, similarly, we can deduce

$$\varphi(u) \to \infty, \quad \text{since } \lambda_k \to 0,\ \alpha_k \to 0,\ k \to \infty,$$

and note that from $p(x) < \beta_i(x)$, $\forall\, x \in \overline{\Omega}$, we have $p^- < \beta_i^- \leq \beta_i^+$. So (A$_2$) holds.

Similar to the proof of Theorem 3.1, by virtue of the hypotheses of (iii), there exists an $M_3 > 0$ such that, for all $x \in \mho_i$, $i = 1, 2$ and $|u| > M_3$, we have

$$F_i(x, u) > C|u|^{\beta_i(x)}.$$

On the other hand, when $|u| \leq M_3$, from assumption (ii), we have $|F_i(x, u)| \leq C$. Thus the two inequalities above imply

$$F_i(x, u) > C|u|^{\beta_i(x)} - C.$$

Let $u \in Y_k$. We have

$$\begin{aligned}\varphi(u) =& I(u) - \phi(u) - H(u)\\ \leq& \frac{1}{p^-}\Phi(u) - \frac{c_1}{q^+}\int_{\partial\Omega}|u|^{q(x)}\mathrm{d}\sigma_x - C\Big(\int_{\Omega}|u|^{\beta_1(x)}\mathrm{d}x + \int_{\partial\Omega}|u|^{\beta_2(x)}\mathrm{d}\sigma_x\Big)\\ &+ \int_{\Omega}h_1(x)|u|\mathrm{d}x + \int_{\partial\Omega}h_2(x)|u|\mathrm{d}\sigma_x + C\\ \leq& \frac{1}{p^-}\max\{\xi_2\|u\|^{p^+}, \zeta_2\|u\|^{p^-}\} - \frac{c_1}{q^+}\min\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\}\\ &+ 2(|h_1|_{L^{p'(x)}(\Omega)}|u_n|_{L^{p(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)}|u_n|_{L^{p(x)}(\partial\Omega)}) + C.\end{aligned}$$

If $\max\{\xi_2\|u\|^{p^+}, \zeta_2\|u\|^{p^-}\} = \xi_2\|u\|^{p^+}$, $\min\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\} = |u|^{q^+}_{L^{q(x)}(\partial\Omega)}$, then we have

$$\varphi(u) \leq \frac{\xi_2}{p^-}\|u\|^{p^+} - \frac{c_1}{q^+}|u|^{q^+}_{L^{q(x)}(\partial\Omega)} + 2(|h_1|_{L^{p'(x)}(\Omega)}|u_n|_{L^{p(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)}|u_n|_{L^{p(x)}(\partial\Omega)}) + C.$$

Since $\dim Y_k < \infty$, all norms are equivalent in Y_k. So we get

$$\varphi(u) \leq \frac{\xi_2}{p^-}\|u\|^{p^+} - \frac{c_1}{q^+}\|u\|^{q^+} + 2(|h_1|_{L^{p'(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)})\|u\| + C.$$

Also note $1 < p^+ < q^- \leq q^+$. Then we get $\varphi(u) \to -\infty$ as $\|u\| \to \infty$. About other cases, the proofs are similar and we omit them here. So (A$_3$) holds. From the proof of (A$_2$) and (A$_3$), we can choose $\rho_k > \gamma_k > 0$. Thus we complete the proof of Theorem 3.2(1).

Proof of Theorem 3.2(2) We use Theorem 2.2 to prove it. We need to prove that φ satisfies the $(\mathrm{PS})_c^*$ condition, and there exist $\rho_k > \gamma_k > 0$ such that for k large enough we have (B$_1$)–(B$_3$). By the hypotheses of (ii), for almost all $x \in \mho_i$ and all $u \in \mathbb{R}$, we have

$$f_i(x, u) \geq \mathrm{sgn}(u)h_i(x) - \vartheta_i|u|^{\beta_i(x)-1},$$

which implies

$$F_i(x,u) \geq h_i(x)|u| - \frac{\vartheta_i}{\beta_i(x)}|u|^{\beta_i(x)}, \quad i=1,2.$$

For $u \in Y_k$ with $\|u\| < 1$ appropriate small such that $\xi_2\|u\|^{p^+} < \zeta_2\|u\|^{p^-}$, we have

$$\begin{aligned}
\varphi(u) &= I(u) - \phi(u) - H(u) \\
&\leq \frac{1}{p^-}\Phi(u) - \frac{c_1}{q^+}\int_{\partial\Omega}|u|^{q(x)}\mathrm{d}\sigma_x - \int_\Omega (F_1(x,u) - h_1(x)|u|)\mathrm{d}x \\
&\quad - \int_{\partial\Omega}(F_2(x,u) - h_2(x)|u|)\mathrm{d}\sigma_x \\
&\leq \frac{1}{p^-}\Phi(u) - \frac{c_1}{q^+}\int_{\partial\Omega}|u|^{q(x)}\mathrm{d}\sigma_x + \frac{\vartheta_1}{\beta_1^-}\int_\Omega |u|^{\beta_1(x)}\mathrm{d}x + \frac{\vartheta_2}{\beta_2^-}\int_{\partial\Omega}|u|^{\beta_2(x)}\mathrm{d}\sigma_x \\
&\leq \frac{\zeta_2}{p^-}\|u\|^{p^-} - \frac{c_1}{q^+}\min\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\} + \frac{\vartheta_1}{\beta_1^-}\max\{|u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}\} \\
&\quad + \frac{\vartheta_2}{\beta_2^-}\max\{|u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\}.
\end{aligned}$$

If $\min\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\} = |u|^{q^+}_{L^{q(x)}(\partial\Omega)}$ and

$$\max\{|u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\} = |u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)},$$

noting that $\dim Y_k < \infty$ implies all norms are equivalent in Y_k, we see that the above inequality becomes

$$\begin{aligned}
\varphi(u) &\leq \frac{\zeta_2}{p^-}\|u\|^{p^-} - \frac{c_1}{q^+}|u|^{q^+}_{L^{q(x)}(\partial\Omega)} + C(\beta_1^-, \beta_2^-)|u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)} \\
&= \frac{\zeta_2}{p^-}\|u\|^{p^-} - \frac{c_1}{q^+}\|u\|^{q^+} + C(\beta_1^-, \beta_2^-)\|u\|^{\beta_1^+}.
\end{aligned}$$

If $p^- < \beta_1^+$, then we get

$$\varphi(u) \leq C\|u\|^{p^-} - \frac{c_1}{q^+}\|u\|^{q^+}.$$

Let $\|u\| = \gamma_k$. If we choose γ_k small enough, and note that $p^-, \beta_i^- > q^+$, $i=1,2$, we have $\varphi(u) < 0$. If $p^- \geq \beta_1^+$, we have

$$\varphi(u) \leq C\|u\|^{\beta_1^+} - \frac{c_1}{q^+}\|u\|^{q^+}.$$

We have the same result. So (B_2) holds. For other cases, from $\beta_i(x) < p(x)$, $\forall x \in \overline{\Omega}$, $i=1,2$, we can check that (B_2) also holds.

Now we prove that (B_1) holds. Let $u \in Z_k$ with $\|u\|$ appropriate small such that $\zeta_1\|u\|^{p^+} < \xi_1\|u\|^{p^-}$. Similarly to the proof of Theorem 3.2(1), we have

$$\begin{aligned}
\varphi(u) &= I(u) - \phi(u) - H(u) \\
&\geq \frac{\zeta_1}{p^+}\|u\|^{p^+} - C(q^-, \beta_1^-, \beta_2^-)\max\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}, \\
&\quad |u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\}.
\end{aligned}$$

If

$$\max\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}, |u|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}, |u|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |u|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\} = |u|^{q^+}_{L^{q(x)}(\partial\Omega)},$$

then we have

$$\varphi(u) \geq \frac{\zeta_1}{p^+}\|u\|^{p^+} - C(q^-, \beta_1^-, \beta_2^-)\lambda_k^{q^+}\|u\|^{q^+}.$$

Choosing $\rho_k = (C(q^-, \beta_1^-, \beta_2^-)\lambda_k^{q^+}\frac{p^+}{\zeta_1})^{\frac{1}{p^+-q^+}}$ with $\|u\| = \rho_k$, we have

$$\varphi(u) \geq \frac{\zeta_1}{p^+}\rho_k^{p^+} - \frac{\zeta_1}{p^+}\rho_k^{p^+} = 0.$$

Since $q^+ < \beta_i^-$ and $\beta_i(x) < p(x)$, $\forall\, x \in \overline{\Omega}$, we have $p^+ \geq p^- > \beta_i^- > q^+$, $i = 1, 2$, and because $\lambda_k \to 0$, we have $\rho_k \to 0$. For the other cases we can get the same result. From the proof above, we can choose $\rho_k > \gamma_k > 0$. Note $Y_k \cap Z_k \neq \emptyset$. So

$$\max_{u\in Y_k, \|u\|\leq\gamma_k} \varphi(u) < 0 \leq \inf_{u\in Z_k, \|u\|\leq\rho_k} \varphi(u).$$

On the other hand, from the proof of (B_1), we know that for $u \in Z_k$ and $\|u\| \leq \rho_k$ small enough

$$\varphi(u) \geq -C(q^-, \beta_1^-, \beta_2^-)\lambda_k^{q^+}\|u\|^{q^+}$$

holds, since $\lambda_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$. This implies that (B_3) holds. Finally, we prove $(\mathrm{PS})_c^*$ condition. Consider a sequence $\{u_{n_j}\}_{n_j\geq1} \subset X$ such that

$$n_j \to \infty, \quad u_{n_j} \subset Y_{n_j}, \quad \varphi(u_{n_j}) \to c \quad \text{and} \quad (\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0.$$

We claim that the sequence $\{u_{n_j}\}_{n\geq1}$ is bounded. If it is not true, by passing a subsequence if necessary, we may assume $\|u_{n_j}\| \to +\infty$. Without loss of generality, we assume $\|u_{n_j}\| \geq 1$ and $|u_{n_j}| \to +\infty$ as $n_j \to +\infty$ for any $x \in \Omega$. Similarly to Lemma 4.1, we have

$$\begin{aligned} M_1 &\geq -\varphi(u_{n_j}) = -I(u_{n_j}) + \phi(u_{n_j}) + H(u_{n_j}) \\ &\geq -\frac{1}{p^-}\Phi(u_{n_j}) + \frac{1}{q^+}\int_{\partial\Omega}(c(x)|u_{n_j}|^{q(x)})\mathrm{d}\sigma_x + H(u_{n_j}), \end{aligned} \tag{4.6}$$

$$M_2 \geq \langle\varphi'(u_{n_j}), u_{n_j}\rangle = \Phi(u_{n_j}) - \int_{\partial\Omega}(c(x)|u_{n_j}|^{q(x)})\mathrm{d}\sigma_x - \langle\psi(u_{n_j}), u_{n_j}\rangle. \tag{4.7}$$

Combining (4.6) and (4.7), we have

$$\begin{aligned} q^+M_1 + M_2 &\geq \Big(1 - \frac{q^+}{p^-}\Big)\Phi(u_{n_j}) + q^+H(u_{n_j}) - \langle\psi(u_{n_j}), u_{n_j}\rangle \\ &\geq \Big(1 - \frac{q^+}{p^-}\Big)\xi_1\|u_{n_j}\|^{p^-} - \int_\Omega(q^+F_1(x, u_{n_j}) - f_1(x, u_{n_j})u_{n_j})\mathrm{d}x \\ &\quad - \int_{\partial\Omega}(q^+F_2(x, u_{n_j}) - f_2(x, u_{n_j})u_{n_j})\mathrm{d}\sigma_x \\ &\quad + (q^+ - 1)\Big(\int_\Omega h_1(x)|u_{n_j}|\mathrm{d}x + \int_{\partial\Omega} h_2(x)|u_{n_j}|\mathrm{d}\sigma_x\Big). \end{aligned} \tag{4.8}$$

By virtue of assumption (i), let $\eta = q^+$ and n be large enough. Then for almost all $x \in \mho_i$ we have

$$q^- F_i(x, u_{n_j}) - f_i(x, u_{n_j})u_{n_j} \le \varepsilon |u_{n_j}|^{\mu(x)}.$$

Thus, by Hölder's inequality, (4.8) becomes

$$\begin{aligned} q^- M_1 + M_2 \ge& \Big(1 - \frac{q^+}{p^-}\Big)\xi_1 \|u_{n_j}\|^{p^-} - \varepsilon\Big(\int_\Omega |u_{n_j}|^{\mu(x)}\mathrm{d}x + \int_{\partial\Omega} |u_{n_j}|^{\mu(x)}\mathrm{d}\sigma_x\Big) \\ &- 2(q^+ - 1)(|h_1|_{L^{p'(x)}(\Omega)}|u_{n_j}|_{L^{p(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)}|u_{n_j}|_{L^{p(x)}(\partial\Omega)}) \\ \ge& \Big(1 - \frac{q^+}{p^-}\Big)\xi_1 \|u_{n_j}\|^{p^-} - C\varepsilon \|u_{n_j}\|^{\mu^+} \\ &- 2(q^+ - 1)C(|h_1|_{L^{p'(x)}(\Omega)} + |h_2|_{L^{p'(x)}(\partial\Omega)})\|u_{n_j}\|. \end{aligned}$$

The last inequality follows from the compact embedding in Propositions 2.1 and 2.2. Since $p^- > q^+$, we have

$$1 - \frac{q^+}{p^-} > 0 \quad \text{and} \quad p^- > \mu^+,$$

which implies that the sequence $\{u_{n_j}\}_{n_j \ge 1} \subset X$ is bounded. It is a contradiction to the supposition. Therefore, we have proved that $\{u_{n_j}\}_{n_j \ge 1} \subset X$ is bounded. By passing a subsequence if necessary, we can assume $u_{n_j} \rightharpoonup u_0$ in X as $n_j \to +\infty$. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u_0$. Hence

$$\begin{aligned} \lim_{n_j \to \infty} \varphi'(u_{n_j})(u_{n_j} - u_0) &= \lim_{n_j \to \infty} \varphi'(u_{n_j})(u_{n_j} - v_{n_j}) + \lim_{n_j \to \infty} \varphi'(u_{n_j})(v_{n_j} - u_0) \\ &= \lim_{n_j \to \infty} (\varphi|_{Y_{n_j}})'(u_{n_j})(u_{n_j} - u_0) = 0. \end{aligned}$$

Noting that $\varphi' = L - J - \psi$ is of type (S_+), we have

$$u_{n_j} \to u_0 \quad \text{and} \quad \varphi'(u_{n_j}) \to \varphi'(u_0).$$

Next, we prove $\varphi'(u_0) = 0$. Taking arbitrary $w_k \in Y_k$, when $n_j \ge k$, we have

$$\begin{aligned} \langle \varphi'(u_0), w_k \rangle &= \langle \varphi'(u_0) - \varphi'(u_{n_j}), w_k \rangle + \langle \varphi'(u_{n_j}), w_k \rangle \\ &= \langle \varphi'(u_0) - \varphi'(u_{n_j}), w_k \rangle + \langle (\varphi|_{Y_{n_j}})'(u_{n_j}), w_k \rangle. \end{aligned}$$

Taking limit on the right-hand side of the equation above, we obtain

$$\langle \varphi'(u_0), w_k \rangle = 0, \quad \forall\, w_k \in Y_k.$$

So we have $\varphi'(u_0) = 0$. Therefore, φ satisfies $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$. Thus we complete the proof.

Remark 4.2 In the proof of Theorem 3.2(2), if $\beta_i^+ < p^-$, $i = 1, 2$, we can get the further result that the sequence of nontrivial solutions of (1.1), denoted as $\{v_k\}_{k \ge 1}$, is bounded. In fact, we have

$$\begin{aligned}
0 &= \langle \varphi'(v_k), v_k\rangle = \Phi(v_k) - \int_{\partial\Omega} c(x)|v_k|^{q(x)} \mathrm{d}\sigma_x - \langle \psi(v_k), v_k\rangle \\
&\geq \Phi(v_k) - \int_{\partial\Omega} c(x)|v_k|^{q(x)} \mathrm{d}\sigma_x - \Big(\int_{\Omega} \vartheta_1 |v_k|^{\beta_1(x)} \mathrm{d}x + \int_{\partial\Omega} \vartheta_2 |v_k|^{\beta_2(x)} \mathrm{d}\sigma_x\Big) \\
&\geq \min\{\xi_1 \|v_k\|^{p^-}, \zeta_1 \|v_k\|^{p^+}\} - c_2 \max\{|v_k|^{q^+}_{L^{q(x)}(\partial\Omega)}, |v_k|^{q^-}_{L^{q(x)}(\partial\Omega)}\} \\
&\quad - \vartheta_1 \max\{|v_k|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |v_k|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}\} - \vartheta_2 \max\{|v_k|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |v_k|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\} \\
&\geq \min\{\xi_1 \|v_k\|^{p^-}, \zeta_1 \|v_k\|^{p^+}\} - C(c_2, \vartheta_1, \vartheta_2) \max\{|v_k|^{q^+}_{L^{q(x)}(\partial\Omega)}, |v_k|^{q^-}_{L^{q(x)}(\partial\Omega)}, \\
&\quad |v_k|^{\beta_1^+}_{L^{\beta_1(x)}(\Omega)}, |v_k|^{\beta_1^-}_{L^{\beta_1(x)}(\Omega)}, |v_k|^{\beta_2^+}_{L^{\beta_2(x)}(\partial\Omega)}, |v_k|^{\beta_2^-}_{L^{\beta_2(x)}(\partial\Omega)}\}.
\end{aligned}$$

Note

$$q^+ < \beta_i^- < \beta_i^+ < p^-, \quad i = 1, 2.$$

Similarly to the proof above, we conclude that $\{v_k\}_{k\geq 1}$ is bounded in $W^{1,p(x)}(\Omega)$.

References

[1] Růžička, M., Electrorheological Fulids Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.

[2] Zhikov, V. V., Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR. Izv.*, **29**(1), 1987, 33–66.

[3] Diening, L., Theoretical and numerical results for electrorheological fluids, Ph.D Thesis, University of Frieburg, Germany, 2002.

[4] Acerbi, E. and Mingione, G., Regularity results for stationary electro-rheological fluids, *Arch. Rational Mech. Anal.*, **164**(3), 2002, 213–259.

[5] Acerbi, E. and Mingione, G., Regularity results for a class of functionals with non-standard growth, *Arch. Rational Mech. Anal.*, **156**(2), 2001, 121–140.

[6] Mihăilescu, M. and Rădulescu, V., A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London, Ser. A*, **462**, 2006, 2625–2641.

[7] Bonder, J. F. and Rossi, J. D., Existence results for the p-Laplacian with nonlinear boundary conditions, *J. Math. Anal. Appl.*, **263**(1), 2001, 195–223.

[8] Martínez, S. and Rossi, J. D., Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance, *Electron. J. Diff. Eqs.*, **2003**(27), 2003, 1–14.

[9] Zhao, J. H. and Zhao, P. H., Infinity many weak solutions for a p-Laplacian with nonlinear boundary conditions, *Electron. J. Diff. Eqs.*, **2007**(90), 2007, 1–14.

[10] Martínez, S. and Rossi, J. D., On the Fučik spectrum and a resonance problem for the p-Laplacian with a nonlinear boundary condition, *Nonlinear Anal.*, **59**(6), 2004, 813–848.

[11] Song, S. Z. and Tang, C. L., Resonance problems for the p-Laplacian with nonlinear boundary condition, *Nonlinear Anal.*, **64**(9), 2006, 2007–2021.

[12] St. Cîrstea, F.-C. and Rădalescu, V., Existence and nonexistence results for quasilinear problem with nonlinear boundary condition, *J. Math. Anal. Appl.*, **244**(1), 2000, 169–183.

[13] Pflüger, K., Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition, *Electron. J. Diff. Eqs.*, **10**, 1998, 1–13.

[14] Cîrstea, F., Motreanu, D. and Radulescu, V., Weak solutions of quasilinear problems with nonlinear boundary condition, *Nonlinear Anal.*, **43**(5), 2001, 623–636.

[15] Kandilakis, D. A. and Lyberopoulos, A. N., Indefinite quasilinear elliptic problems with subcritical and supercritical nonlinearities on unbounded domains, *J. Diff. Eqs.*, **230**(1), 2006, 337–361.

[16] Mezei, I. I. and Varga, C., Multiplicity result for a double eigenvalue quasilinear problem on unbounded domain, *Nonlinear Anal.*, **69**(11), 2008, 4099–4105.

[17] Pflüger, K., Nonlinear boundary value problems in weighted Sobolev spaces, *Nonlinear Anal.*, **30**(2), 1997, 1263–1270.

[18] Deng, S. G., Eigenvalues of the $p(x)$-Laplacian Steklov problem, *J. Math. Anal. Appl.*, **339**(2), 2008, 925–937.

[19] Fan, X. L., Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.*, **339**(2), 2008, 1395–1412.

[20] Yao, J. H., Solutions for Neumann boundary value problem involving $p(x)$-Laplace operators, *Nonlinear Anal.*, **68**(5), 2008, 1271–1283.

[21] Liu, W. L. and Zhao, P. H., Existence of positive solutions for $p(x)$-Laplacian equations in unbounded domains, *Nonlinear Anal.*, **69**(10), 2008, 3358–3371.

[22] Fan, X. L. and Zhang, Q. H., Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.*, **52**(8), 2003, 1843–1852.

[23] Fan, X. L., $p(x)$-Laplacian equations in R^N with periodic data and nonperiodic perturbations, *J. Math. Anal. Appl.*, **341**(1), 2008, 103–119.

[24] Willem, M., Minimax Theorems, Birkhauser, Basel, 1996.

[25] Chang, K. C., Variational methods for nondifferentiable functionals and their applications to partial differntial equations, *J. Math. Anal. Appl.*, **80**, 1981, 102–129.

[26] Lou, H. W., On singular sets of local solutions to p-Laplace equations, *Chin. Ann. Math.*, **29B**(5), 2008, 521–530.