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Generalized Green Correspondence of Graded Modules

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Abstract The author studies the Green correspondence and quasi-Green correspondence for indecomposable modules over strongly graded rings. The motivation is to investigate the influence of induction and restriction processes on indecomposability of graded modules.

Keywords Vertices, Sources, Graded rings, Graded modules, Green correspondence
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1 Introduction and Preliminaries

Green's work on indecomposable modules over a group ring KG in the modular case proved vital for solution of certain problems in the classification of finite simple groups and has inspired an extensive literature (see [1, 4, 5]). Green introduced new methods in modular representation theory based on his work on vertices and sources of indecomposable modules, defect groups, and the Green correspondence. Roughly speaking, the Green correspondence establishes further connection between indecomposable KG-modules and KH-modules with the same vertex P, where H is a subgroup of G containing $N_G(P)$. The idea is to achieve a group-theoretical reduction which transfers the study of indecomposable modules from G to a proper subgroup H of G. The Green correspondence has been generalized by several mathematicians (see [6, 7, 10]). Graded rings were introduced by Dade in [3] as a formal way to deal with finite group representation problems. Therefore, it is of course tempting to try to extend all group ring results of Green to graded rings.

In this paper, we establish and investigate the Green correspondence for indecomposable modules over a G-graded ring $R = \bigoplus_{\sigma\in G} R_\sigma$, where G is a finite group. We also study some generalizations of Green correspondence which possesses some properties of Green correspondence in case H does not contain $N_G(P)$.

Throughout this paper, G is a finite group with neutral element e, H is a subgroup of G, and $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly G-graded F-algebra, where F is a complete commutative local Noetherian ring whose residue class field F_0 has characteristic $p > 0$. We also assume that every R_σ $(\sigma \in G)$ is a finitely generated F-submodule of R. These assumptions imply that R and any finitely generated R-module are finitely generated as F-modules. Moreover, every finitely generated R-module satisfies the well-known Krull-Schmidt property (see [2]). For details on

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graded rings and modules, we refer to [11, 12]. If $M = \bigoplus_{\sigma\in G} M_\sigma$ is a G-graded R-module, then by $M^{(H)}$ we denote the R_e-module $M^{(H)} = \bigoplus_{\sigma\in H} M_\sigma$. All R-modules are finitely generated left modules. For any R-module V, we write V_H for the restriction of V to $R^{(H)} = \bigoplus_{\sigma\in H} R_\sigma$. If W is an $R^{(H)}$-module, then we shall write W^G for the induced R-module $R \bigotimes_{R^{(H)}} W$. An R-module V is said to be H-projective if V is a direct summand of $(V_H)^G$, and this is equivalent to the existence of an $R^{(H)}$-module W such that V is isomorphic to a direct summand of W^G. The following is a generalized version of Mackey's subgroup theorem.

Theorem 1.1 (see [9]) *Let L be a subgroup of G and T be a set of double coset representatives for (L, H) in G. Suppose that V is an $R^{(H)}$-module and, for every $g \in G$, $V^{(g)}$ is the $R^{(gHg^{-1})}$-module $V^{(g)} = R_g V \cong R_g \bigotimes_{R_e} V$. If for every $t \in T$, $V_t = (V^{(t)})_{tHt^{-1}\cap L}$, then*

$$(V^G)_L \cong \bigoplus_{t\in T} (V_t)^L, \quad \text{as } R^{(L)}\text{-modules.}$$

By a component of an R-module W we mean an R-module V which is isomorphic to a direct summand of W. In the sequel of this paper, we assume $p/|G|$.

Definition 1.1 (i) *A vertex of an indecomposable R-module V is a subgroup Q of G for which V is Q-projective and V is not H-projective for any proper subgroup H of Q.*

(ii) *If Q is a vertex of the R-module V and U is an indecomposable $R^{(Q)}$-module such that V is a component of U^G, then U is called a source of V corresponding to Q.*

Theorem 1.2 (see [9]) *Let V be an indecomposable R-module.*

(i) *If Q is a vertex of V and V is a component of $(V_L)^G$, where L is a subgroup of G, then there exists an $x \in G$ such that $xQx^{-1} \subseteq L$.*

(ii) *The vertices of V form a unique conjugacy class of p-subgroups of G.*

(iii) *If U is a source of V corresponding to the vertex Q, then an $R^{(Q)}$-module U_1 is a source of V if and only if $U \cong U_1^{(g)}$ for some $g \in N_G(Q)$.*

Proposition 1.1 *Let W be an indecomposable R-module with vertex Q. Suppose that W is H-projective and*

$$W_H = \bigoplus_{i=1}^{n} V_i \quad (\text{a direct sum of indecomposable } R^{(H)}\text{-modules}).$$

If Q_i is a vertex of V_i for every $i = 1, 2, \cdots, n$, then

(i) *For every $i \in \{1, 2, \cdots, n\}$, Q_i is G-conjugate to a subgroup of Q.*

(ii) *There exists a $k \in \{1, 2, \cdots, n\}$ such that W is a component of V_k^G and Q_k is a vertex of W.*

Proof (i) It is a direct consequence of Theorem 1.1 that each V_i is $(x_iQx_i^{-1} \cap H)$-projective for some $x_i \in G$. Hence, Theorem 1.2 entails the existence of $y_i \in H$ such that $y_iQ_iy_i^{-1} \subseteq x_iQx_i^{-1} \cap H$. Thus $x_i^{-1}y_iQ_iy_i^{-1}x_i \subseteq Q$, or equivalently $x_i^{-1}y_iQ_i(x_i^{-1}y_i)^{-1} \subseteq Q$. It follows that $g_iQ_ig_i^{-1} \subseteq Q$, where $g_i = x_i^{-1}y_i$.

(ii) By definition, W is a component of $(W_H)^G = \bigoplus_{i=1}^{n}(V_i)^G$. It follows from the indecomposability of W that there exists a $k \in \{1, 2, \cdots, n\}$ such that W is a component of $(V_k)^G$. But V_k is Q_k-projective, thus [8, Corollary 3.6] entails that W is Q_k-projective. Hence W is a component of $(W_{Q_k})^G$. By Theorem 1.2, there exists a $y \in G$ such that $yQy^{-1} \subseteq Q_k$. It follows that $|Q| = |Q_k|$. Thus, by (i), $g_k Q_k g_k^{-1} = Q$.

The following properties of groups are well-known and very easy to prove.

Proposition 1.2 *If $C \subseteq D \subseteq S$ is a chain of subgroups of G, then the following statements are equivalent:*

(i) *C is G-conjugate to a subgroup of $gDg^{-1} \cap D$ for some $g \in G - S$.*

(ii) *C is S-conjugate to a subgroup of $gDg^{-1} \cap D$ for some $g \in G - S$.*

(iii) *C is S-conjugate to a subgroup of $gDg^{-1} \cap S$ for some $g \in G - S$.*

2 Green Correspondence of Graded Modules

Throughout this section, we assume that P is a p-subgroup of G. If the subgroup H of G contains $N_G(P)$, then we shall establish a bijection between the class of indecomposable $R^{(H)}$-modules with vertex P and the class of indecomposable R-modules with vertex P. Such bijection is known as the Green correspondence.

Lemma 2.1 *Let $Q \subseteq S$ be subgroups of G. If V is an $R^{(S)}$-module which is Q-projective, then V is the only possible component of $(V^G)_S$ which is not $(gQg^{-1} \cap S)$-projective for all $g \in G - S$.*

Proof If T is a left transversal for S in G, then

$$V^G = R \bigotimes_{R^{(S)}} V = \left(R^{(S)} \bigotimes_{R^{(S)}} V\right) \oplus \left(\bigoplus_{t \in T-S} R_t R^{(S)} \bigotimes_{R^{(S)}} V\right) = V \oplus V_1,$$

where

$$V_1 = \bigoplus_{t \in T-S} R_t R^{(S)} \bigotimes_{R^{(S)}} V.$$

Moreover, for every $t \in T - S$ and $x \in S$, there exist $t' \in T - S$ and $x' \in S$ such that $xt = t'x'$. Thus $R_x V_1 \subset V_1$, or equivalently V_1 is an $R^{(S)}$-module. It follows that V is a component of $(V^G)_S$. On the other hand, the Q-projectivity of V implies the existence of an $R^{(Q)}$-module W such that $W^S = V \oplus L$, where L is an $R^{(S)}$-module. Repeating the same argument as above, we may write $(L^G)_S = L \oplus L_1$ for some $R^{(S)}$-module L_1. It follows from Theorem 1.1 that there exists a set Y of double coset representatives of (Q, S) in G containing e such that

$$V \oplus V_1 \oplus L \oplus L_1 \cong \bigoplus_{y \in Y} (W_y)^S.$$

Obviously, if $Y_1 = Y - \{e\}$, then

$$\bigoplus_{y \in Y_1} (W_y)^S \cong V_1 \oplus L_1, \quad (W_e)^S = W^S \cong V \oplus L.$$

Moreover, for every $y \in Y_1$, it is obvious that $y \in G - S$ and $(W_y)^S$ is $(yQy{-}1) \cap S$-projective. Thus V_1 is $(yQy{-}1) \cap S$-projective. This completes the proof.

Lemma 2.2 *Let $Q \subseteq S$ be subgroups of G. If Q is a p-subgroup of G and U is an indecomposable R-module with vertex Q, then*

(i) *there exists an indecomposable $R^{(S)}$-module V with vertex Q such that U is a component of V^G.*

(ii) *U_S has an indecomposable component L with vertex Q.*

Proof (i) Let W be an $R^{(Q)}$-module such that U is a component of $W^G = (W^S)^G$. If

$$W^S = \bigoplus_{i=1}^{n} V_i \quad \text{(a direct sum of indecomposable } R^{(S)}\text{-modules)},$$

then there exists a $V_j = V$ $(j \in \{1, 2, \cdots, n\})$ such that U is a component of V^G. Moreover, Proposition 1.1 entails that the vertex Q_0 of V is S-conjugate to a subgroup of Q. On the other hand, since U is Q_0-projective, then Q is G-conjugate to a subgroup of Q_0. It follows that Q_0 is S-conjugate to Q. Invoking Theorem 1.2, we obtain that Q is a vertex of V as well.

(ii) If

$$U_S = \bigoplus_{i=1}^{r} L_i \quad \text{(a sum of indecomposable } R^{(S)}\text{-modules)},$$

then Proposition 1.2 entails that the vertex Q_i of L_i is G-conjugate to a subgroup of Q for every $i \in \{1, 2, \cdots, r\}$. Write

$$U_Q = \bigoplus_{i=1}^{n} T_i \quad \text{(a sum of indecomposable } R^{(Q)}\text{-modules)}.$$

It follows from Proposition 1.1 that there exists a $k \in \{1, 2, \cdots, n\}$ such that the vertex E of T_k is G-conjugate to Q. Because E is a subgroup of Q, we infer that $E = Q$.

Finally, since $U_Q = \bigoplus_{i=1}^{r} (L_i)_Q$, there exists a $j \in \{1, 2, \cdots, r\}$ such that T_k is a component of $(L_j)_Q$. Thus $E = Q$ is S-conjugate to Q_j. Hence Q is a vertex of L_j.

Lemma 2.3 *Let $Q \subseteq D \subseteq S$ be subgroups of G and U be an indecomposable R-module with vertex Q.*

(i) *If Q is G-conjugate to a subgroup of $xDx^{-1} \cap D$ for some $x \in G - S$, then for any indecomposable component L of U_S there exists a $g \in G - S$ such that L is $gDg^{-1} \cap S$-projective.*

(ii) *If Q is not G-conjugate to a subgroup of $xDx^{-1} \cap D$ for every $x \in G - S$, then there exists a unique indecomposable component V of U_S which is not $gDg^{-1} \cap S$-projective for all $g \in G - S$. Furthermore, V is the only indecomposable component of U_S with vertex Q.*

Proof We first observe that by Lemma 2.2 there exists an indecomposable $R^{(S)}$-module V with vertex Q such that U is a component of V^G. It follows that V is D-projective (see [8]), and Lemma 2.1 entails that V is the only possible component of $(V^G)_S = U_S$ which is not $gDg^{-1} \cap S$-projective for every $g \in G - S$.

(i) Direct consequence of the previous observation and Proposition 1.1.

(ii) By Lemma 2.2, $U_S = (V^G)_S$ has at least one indecomposable component, say, W, with vertex Q. If W is $gDg^{-1} \cap S$-projective for some $g \in G - S$, then the vertex Q of W is S-conjugate to a subgroup of $xDx^{-1} \cap S$ for some $x \in G - S$. This contradicts the assumption. Hence W is not $gDg{-}1 \cap S$-projective for all $g \in G - S$. Therefore $W \cong V$ as asserted.

The following theorem generalizes the well-known correspondence theorem of Green for indecomposable modules over group rings (see [4]). The one-to-one correspondence established in this theorem is called the Green correspondence.

Theorem 2.1 *Let P be a p-subgroup of G such that $N_G(P) \subseteq H$. Then there is a one-to-one correspondence between the isomorphism classes $[U]$ of finitely generated indecomposable R-modules with vertex P and the isomorphism classes $[V]$ of finitely generated indecomposable $R^{(H)}$-modules with vertex P. Moreover, $[U]$ corresponds to $[V]$ if and only if either U is a component of V^G or V is a component of U_H.*

Proof The proof consists of two main steps.

Step 1 We assert that for every indecomposable R-module U with vertex P there exists a unique indecomposable component $V = f(U)$ of U_H with vertex P. Moreover, U is a component of V^G and V is the unique indecomposable component of U_H which is not $gPg^{-1} \cap H$-projective for all $g \in G - H$. Indeed, $N_G(P) \subseteq H$ implies $gPg^{-1} \cap P \neq P$ for every $g \in G - H$. Hence P is not G-conjugate to a subgroup of $gPg^{-1} \cap P$ for every $g \in G - H$. Thus Lemma 2.3(ii) with $D = Q = P$ and $S = H$ proves our assertion.

Step 2 We claim that, for any indecomposable $R^{(H)}$-module V with vertex P, there exists a unique indecomposable component U of V^G with vertex P such that V is a component of U_H. Actually, if

$$V^G = \bigoplus_{i=1}^{n} U_i \quad \text{(a direct sum of indecomposable } R\text{-modules)},$$

then Lemma 2.1 implies the existence of a unique $U \in \{U_1, \cdots, U_n\}$ such that V is component of U_H. Applying Lemma 2.3 with $D = P$ and $S = H$, one can deduce that U is a unique indecomposable component of V^G with vertex P. This completes the proof of our claim.

Let $[U]$ denote the isomorphism class of U. Then combining these two steps one easily observes that the correspondence $f : [U] \leftrightarrow [f(U)] = [V]$ provides a bijection between the isomorphism classes of indecomposable R-modules and $R^{(H)}$-modules with vertex P.

Corollary 2.1 *With notations and assumptions as in Theorem 2.1, if V is an indecomposable $R^{(H)}$-module with vertex P, then*

(i) *$f^{-1}(V)$ is a unique indecomposable component of V^G which is not $xPx^{-1} \cap P$-projective for all $x \in G - H$.*

(ii) *$f^{-1}(V)$ and V have the same source.*

Proof A direct consequence of the proof of Theorem 2.1.

3 Quasi-Green Correspondence

In this section, we investigate a few remarks about quasi-Green correspondence which appears originally in [10]. The notations and assumptions are as in Section 2.

Definition 3.1 (see [10]) *Let U be an indecomposable R-module and V be an indecomposable $R^{(H)}$-module. We say that V (if it exists) is the quasi-Green correspondent of U in H if $U_H \cong V \oplus V'$ and*

(i) *U is a component of V^G,*

(ii) *U is not a component of V'^G.*

The well-known "3 out of 3" lemma states that if U is an indecomposable R-module with vertex $P \subseteq H$, then there exists an indecomposable $R^{(H)}$-module V satisfying

(i) hPh^{-1} is a vertex of V for some $h \in H$,

(ii) U is a component of V^G,

(iii) V is a component of U_H.

The proof is just a minor modification of the proof of Theorem 5 in [1] for modules over group rings. The following essential proposition is a routine consequence of the "3 out of 3" lemma.

Proposition 3.1 *Suppose that the p-subgroup P is contained in H. Let U be an indecomposable R-module with vertex P having quasi-Green correspondent V in H. If $gPg^{-1} \subseteq H$ for some $g \in G$, then there exists an $h \in H$ such that $gPg^{-1} = hPh^{-1}$. In particular, P is a vertex of V.*

Proposition 3.2 *Suppose that $N_G(P) \subseteq H$. Let U be an indecomposable R-module with vertex P and $f(U)$ be the Green correspondent of U in H. Then $f(U)$ is the quasi-Green correspondent of U in H if and only if $gPg^{-1} \subseteq H$ implies $g \in H$.*

Proof If $f(U)$ is the quasi-Green correspondent of U in H and $gPg^{-1} \subseteq H$ for some $g \in G$, then Proposition 3.1 entails that $gPg^{-1} = hPh^{-1}$ for some $h \in H$. Thus $g \in H$ is obvious.

Conversely, suppose that $gPg^{-1} \subseteq H$ implies $g \in H$. Invoking Theorem 2.1, we see that $U_H = f(U) \oplus Y$ and $f(U)$ is the unique indecomposable component of U_H which is not $gPg^{-1} \cap H$-projective for all $g \in G - H$. But, for every $g \in G - H$, $|gPg^{-1} \cap H| < |P|$. Thus if U is a direct summand of Y^G, then [8, Corollary 3.6] implies that U is $gPg^{-1} \cap H$-projective. This contradicts the assumption that P is a vertex of U. Hence U is not a direct summand of Y^G. Since U is a component of $f(U)^G$ by Theorem 2.1, $f(U)$ is the quasi-Green correspondent of U in H.

Proposition 3.2 entails that, in general, the Green correspondent of a module need not be its quasi-Green correspondent. Thus the quasi-Green correspondent is not a generalization of the Green correspondent. In an attempt to generalize the notion of Green-correspondent, the following convenient definition is suggested in [7].

Definition 3.2 *Suppose that H contains a p-subgroup P of G. Let $[U]$ be the isomorphism* *class of an indecomposable R-module U with vertex P. The isomorphism class $[T_P(U)] = [V]$ of an indecomposable $R^{(H)}$-module V with vertex P is called the quasi-Green correspondent of $[U]$ in H with respect to P, if*

(i) *$U_H \cong V \oplus M \oplus N$, where M and N are $R^{(H)}$-modules,*

(ii) *U is a direct summand of V^G,*

(iii) *U is not a direct summand of M^G,*

(iv) *the indecomposable direct summands of N have vertices not H-conjugate to P.*

We say that the quasi-Green correspondence from G to H with respect to P (denoted by T_P) exists, if the correspondence $[U] \to [V] = [T_P(U)]$ is a bijection between the isomorphism classes of indecomposable R-modules and $R^{(H)}$-modules with vertex P. For simplicity, the quasi-Green correspondence T_P from G to H with respect to P will be called the generalized Green correspondence. Invoking Theorem 2.1, one can just imitate the proof of [7, Proposition 1.5] to prove the following essential result.

Theorem 3.1 *With notation as in Definition* 3.2*, if $N_G(P) \subseteq H$, then a Green correspondence provides a quasi-Green correspondence with respect to P. In other words, if U is an indecomposable R-module with vertex P, then there exists a quasi-Green correspondent $T_P(U)$ of U in H with respect to P such that $T_P(U) \cong f(U)$, where $f(U)$ is the Green correspondent of U in H.*

Remark 3.1 (i) In fact, it is not hard to extend all results in [7] to the situation considered here. For instance, the converse of Theorem 3.1 is also true (see [7, Theorem 1.6]). Thus, a generalized Green correspondence coincides with a Green correspondence, provided $N_G(P) \subseteq H$.

(ii) The quasi-Green correspondent with respect to P of an indecomposable $R^{(H)}$-module V with vertex P had been defined in [7] to be an indecomposable R-module $U = S_p(V)$ with vertex P satisfying certain properties. Furthermore, we say that the quasi-Green correspondence from H to G with respect to P exists, if the correspondence $[V] \to [U] = [S_P(V)]$ is a bijection. The most important result in this situation states that "Let G be a p-solvable group and $P \subseteq H$. If the quasi-Green correspondence S_P from H to G with respect to P exists, then $[N_G(P) : N_H(P)]$ is a power of P". The same statement for group ring has been proved in [7], and one can easily observe that the generalization to graded rings is quite possible.

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