

On the Normal Subgroup with Exactly Two G -Conjugacy Class Sizes***

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Abstract Let G be a finite group with a non-central Sylow r -subgroup R , $Z(G)$ the center of G , and N a normal subgroup of G . The purpose of this paper is to determine the structure of N under the hypotheses that N contains R and the G -conjugacy class size of every element of N is either 1 or m . Particularly, it is shown that N is Abelian if $N \cap Z(G) = 1$ and the G -conjugacy class size of every element of N is either 1 or m .

Keywords Normal subgroups, Conjugacy class sizes, Nilpotent groups

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1 Introduction

All groups considered in this paper are finite. Let G be a group, π a set of some primes and x an element of G . x^G denotes the conjugacy class containing x , $|x^G|$ denotes the size of x^G , x_π and $x_{\pi'}$ denote π -component and π' -component of x , respectively. Moreover, we write G_π for a Hall π -subgroup of G , $G_{\pi'}$ for a Hall π' -subgroup of G , and n_π for the π -part of n whenever n is a positive integer.

In 1904, Burnside proved that if a group G has a conjugacy class with prime power size, then G is not simple (see [3, Corollary II, p. 322]). Since then, many authors have investigated the relationship between the structure of a group and its conjugacy class sizes (for example, [1, 2, 4, 5, 8–14]). Among these results, a classic result by Itô [8] asserts that a group G is nilpotent if $|x^G| = 1$ or m for every $x \in G$. Recently, Beltrán and Felipe [2] proved that every Hall p' -subgroup of a p -solvable group is nilpotent if $|x^G| = 1$ or m for every p' -element x of G . On the other hand, the structure of a normal subgroup N of a group G was given if N is the union of some G -conjugacy classes (see [9–12]). Now, we are interested in the following question: Let G be a finite group and let N be a normal subgroup of G . If $|x^G| = 1$ or m for every element $x \in N$, is N nilpotent?

Our main result is the following theorem.

Theorem 1.1 *Let G be a finite group with a non-central Sylow r -subgroup R and N a normal subgroup of G containing R . If $|x^G| = 1$ or m for every element x of N , then N is nilpotent.*

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2 Preliminaries

We first list some lemmas which are useful in the proof of our main result.

Lemma 2.1 (see [5, Lemma 1.1]) *Let N be a normal subgroup of a group G and x an element of G . Then*

- (a) $|x^N|$ divides $|x^G|$;
- (b) $|(Nx)^{G/N}|$ divides $|x^G|$.

Corollary 2.1 *Let N be a normal subgroup of a group G and $N \cap Z(G) = 1$. If $|x^G| = 1$ or m for every element x of N , then N is Abelian.*

Proof By the assumption of this corollary, we may assume $|N| = km + 1$, where k is the number of noncentral G -conjugacy classes contained in N . It follows that $(|N|, m) = 1$. By Lemma 2.1, we deduce that N is Abelian.

Lemma 2.2 (see [7, Theorem 33.4]) *Let G be a group. A prime p does not divide any conjugacy class size of G if and only if G has a central Sylow p -subgroup.*

Lemma 2.3 *Let π be a set of some primes and N be a normal subgroup of a group G . If $\bar{x} = xN$ is a π -element, then there exists a π -element x^* of G such that $\bar{x} = \overline{x^*}$.*

Proof Let $o(\bar{x}) = n_0$ and $o(x) = n \cdot m$ such that n is a π -number and $(n, m) = 1$. Then $n_0 \mid n$ and $x^{n_0} \in N$. Since $(n, m) = 1$, there exist integers u and v such that $un + vm = 1$. It follows from $x = x^{un} \cdot x^{vm}$ that $xN = (x^m)^v N$. It is clear that $x^* = (x^m)^v$ is a π -element.

Lemma 2.4 (see [6, Theorem 1]) *Let G be a group acting transitively on a set Ω with $|\Omega| > 1$. Then there exist a prime p and a p -element $x \in G$ such that x acts without fixed point on Ω .*

3 Proof of Theorem 1.1

Now, we are equipped to prove the main result.

Assume that Theorem 1.1 is not true. Let G be a counterexample with minimal order, and $Z(G)$ be the center of the group G . Without loss of generality, we may replace N by $NZ(G)$. Therefore we may assume that $Z(G) \leq N$. We will complete the proof by the following steps.

Step 1 $N_p \not\leq Z(G)$ for any prime divisor $p(\neq r)$ of $|N|$.

If not, there exists a prime divisor $q(\neq r)$ of the order of N such that $N_q \leq Z(G)$ and thus $N_q \trianglelefteq G$. Consider the quotient groups G/N_q and N/N_q . For convenience, we use “ \sim ” to work in the factor group mod N_q . Obviously, \tilde{R} is a non-central Sylow r -subgroup of \tilde{G} , and \tilde{N} is a normal subgroup of \tilde{G} containing \tilde{R} . Let \tilde{x} be an element of \tilde{N} and y an element of G . We may assume that x is a q' -element of N by Lemma 2.3. If $\tilde{x}\tilde{y} = \tilde{y}\tilde{x}$, then $[x, y] \in N_q$. So y normalizes the group $\langle x \rangle \times N_q$, and hence $[x, y] \in \langle x \rangle$. Consequently, $[x, y] = 1$. So $C_{\tilde{G}}(\tilde{x}) = \widetilde{C_G(x)}$. Therefore, $|\tilde{x}^{\tilde{G}}| = |\tilde{G} : C_{\tilde{G}}(\tilde{x})| = |\tilde{G} : \widetilde{C_G(x)}| = |G : C_G(x)| = 1$ or m . This means that the hypotheses of the theorem are inherited by factor group \tilde{G} and \tilde{N} . We conclude that \tilde{N} is nilpotent by the minimal choice of G . Since $N_q \leq Z(G)$ while $Z(G) \leq Z(N)$, N is nilpotent, a contradiction.

In the following, we will consider the quotient groups $G/Z(G)$ and $N/Z(G)$. For convenience, we use “ $\bar{}$ ” to work in the factor group mod $Z(G)$.

Step 2 For any $1 \neq \bar{x} \in \bar{N}$, we have

- (i) If $o(\bar{x})$ is a power of a prime p , then $C_N(x)_{p'} \leq Z(C_G(x))$.
- (ii) If $o(\bar{x})$ is not a prime power, then $C_N(x) \leq Z(C_G(x))$.

If $o(\bar{x})$ is a power of p , then by Lemma 2.3 we may assume that x is a p -element. For any p' -element $y \in N \cap C_G(x) = C_N(x)$, since $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$, the assumption of the theorem implies $C_G(xy) = C_G(x)$, and hence $C_G(x) \subseteq C_G(y)$. It follows that $y \in Z(C_G(x))$, and therefore $C_N(x)_{p'} \leq Z(C_G(x))$.

If $o(\bar{x})$ is not a prime power, then $o(x)$ is also not a prime power. So we may assume $x = x_1 x_2 \cdots x_s$, where the order of each x_i is a power of a prime p_i and x_i commutes pairwise with $p_i \neq p_j$ ($i, j = 1, 2, \dots, s$ and $s \geq 2$). Since $o(\bar{x})$ is not a prime power, there at least exist two elements beyond $Z(G)$ among x_i ($1 \leq i \leq s$), say, x_1 and x_2 . Noticing that x, x_1 and x_2 are non-central elements and $C_G(x) = C_G(x_1) \cap C_G(x_2 \cdots x_s) = C_G(x_2) \cap C_G(x_1 x_3 \cdots x_s)$, we have $C_G(x) = C_G(x_1) = C_G(x_2)$ by the assumption of the theorem. On the other hand, (i) implies that $C_N(x)_{p'_1} \leq Z(C_G(x))$ and $C_N(x)_{p'_2} \leq Z(C_G(x))$, while $C_N(x)_{p'_1} C_N(x)_{p'_2} = C_N(x)$. So $C_N(x) \leq Z(C_G(x))$.

Step 3 Let $1 \neq \bar{x}, \bar{y} \in \bar{N}$ and $o(\bar{x})$ be not a prime power. If $C_G(x) \neq C_G(y)$, then $C_N(x) \cap C_N(y) = Z(G)$.

By (ii), we have $C_N(x) \leq Z(C_G(x))$. If there exists an element a such that $1 \neq a \in C_N(x) \cap C_N(y)$ but $a \notin Z(G)$, then $C_G(x) = C_G(a)$, and therefore $C_N(x) = C_N(a)$. Also, $a \in C_N(y)$ implies $y \in C_N(a) = C_N(x) \leq Z(C_G(x))$. It follows that $C_G(x) \subseteq C_G(y)$. Hence $C_G(x) = C_G(y)$, a contradiction.

Step 4 For any $1 \neq \bar{x} \in \bar{N}$, we have

- (iii) If $o(\bar{x})$ is a power of a prime q , then $|\bar{x}^{\bar{G}}|_{q'} = m_{q'}$.
- (iv) If $o(\bar{x})$ is not a prime power, then $|\bar{x}^{\bar{G}}| = m$.

Let $C_{\bar{G}}(\bar{x}) = \bar{H}$. For every element $\bar{y} \in \bar{H}$ of order of a power of a prime t where $t \in q'$, we may assume that y is a t -element and $y \in H$ by Lemma 2.3. Therefore we have that $\langle \bar{x}, \bar{y} \rangle$ is a cyclic group and hence $\langle x, y \rangle$ is Abelian. So $\bar{y} \in \overline{C_G(x)}$. It follows that $|\bar{H}|_t = |\overline{C_G(x)}|_t$. So $|\bar{H}|_{q'} = |C_G(x)|_{q'}$. Therefore

$$|\bar{x}^{\bar{G}}|_{q'} = \frac{|\bar{G}|_{q'}}{|\bar{H}|_{q'}} = \frac{|\bar{G}|_{q'}}{|C_G(x)|_{q'}} = \frac{|G|_{q'}/|Z(G)|_{q'}}{|C_G(x)|_{q'}/|Z(G)|_{q'}} = \frac{|G|_{q'}}{|C_G(x)|_{q'}} = |G : C_G(x)|_{q'} = m_{q'}.$$

Next, if \bar{x} is not a prime power order element, then $o(x)$ is also not a prime power. So we may assume that $x = x_1 x_2 \cdots x_s$, where the order of each x_i is a power of a prime p_i and x_i commute pairwise with $p_i \neq p_j$ ($i, j = 1, 2, \dots, s$ and $s \geq 2$). Since $o(\bar{x})$ is not a prime power, there at least exist two elements beyond $Z(G)$ among x_i ($1 \leq i \leq s$), say, x_1 and x_2 . So $\bar{x} = \bar{x}_1 \bar{x}_2 \bar{x}_3 \cdots \bar{x}_s$. Obviously, $C_{\bar{G}}(\bar{x}) = C_{\bar{G}}(\bar{x}_1) \cap C_{\bar{G}}(\bar{x}_2 \bar{x}_3 \cdots \bar{x}_s) = C_{\bar{G}}(\bar{x}_2) \cap C_{\bar{G}}(\bar{x}_1 \bar{x}_3 \cdots \bar{x}_s)$, and it is clear that $C_{\bar{G}}(\bar{x}) \leq C_{\bar{G}}(\bar{x}_1)$ and $C_{\bar{G}}(\bar{x}) \leq C_{\bar{G}}(\bar{x}_2)$. Hence $|\bar{x}_1^{\bar{G}}| \mid |\bar{x}^{\bar{G}}|$ and $|\bar{x}_2^{\bar{G}}| \mid |\bar{x}^{\bar{G}}|$. By (iii), we have $m_{p'_1} \mid |\bar{x}^{\bar{G}}|$ and $m_{p'_2} \mid |\bar{x}^{\bar{G}}|$. So $|\bar{x}^{\bar{G}}| = m$.

Step 5 If $1 \neq \bar{x} \in \bar{N}$ is not a prime power order element, then $C_{\bar{G}}(\bar{x}) = \overline{C_G(x)}$, particularly, $C_{\bar{N}}(\bar{x}) = \overline{C_N(x)}$.

Since

$$|\bar{x}^{\bar{G}}| = |\bar{G} : C_{\bar{G}}(\bar{x})| \leq |\bar{G} : \overline{C_G(x)}| = |G : C_G(x)| = m$$

while $|\bar{x}^{\bar{G}}| = m$ by (iv), thus we obtain $C_{\bar{G}}(\bar{x}) = \overline{C_G(x)}$. Particularly, $C_{\bar{N}}(\bar{x}) = C_{\bar{G}}(\bar{x}) \cap \bar{N} = \overline{C_G(x)} \cap \bar{N} = \overline{C_G(x) \cap N} = \overline{C_N(x)}$.

Step 6 Let $g_0 \in N - Z(G)$ and \bar{g}_0 be not a prime power order element. Consider the conjugacy class $\bar{g}_0^{\bar{N}}$ of \bar{g}_0 in \bar{N} . Then there exists some non-central element x of N such that $\bar{g}_0^{\bar{N}} \cap \overline{C_N(x)} = \emptyset$.

Consider the conjugacy class g_0^N of g_0 in N . Noting that N operates transitively on the set g_0^N with $|g_0^N| > 1$, we have that there exists an element x of N such that x operates without fixed point on g_0^N by Lemma 2.4. It follows that $g_0^N \cap C_N(x) = \emptyset$. So $\bar{g}_0^{\bar{N}} \cap \overline{C_N(x)} = \emptyset$.

Step 7 There exists a $\{p, r\}$ -element g of N such that \bar{g} is a $\{p, r\}$ -element of \bar{N} for any prime divisor $p(\neq r)$ of $|N|$.

According to Step 1, for any prime divisor $p(\neq r)$ of $|N|$, there exists a non-central p -element in N , say, x . By (i), we have $C_N(x) = C_N(x)_p \times C_N(x)_{p'}$. We claim $C_N(x)_r \not\leq Z(G)$. Otherwise, $C_N(x)_r \leq Z(G)$. As R is non-central, there exists a non-central r -element z such that $z \in N \setminus C_N(x)_r$. So $C_N(x)_r < \langle C_N(x)_r, z \rangle \leq C_G(z)$, in contradiction to $|z^G| = |x^G| = m$. Take $z \in C_N(x)_r \setminus Z(G)$ and let $g = xz$. Thus $\bar{g} = \bar{x}\bar{z}$. It is clear that g and \bar{g} satisfy the requirement of Step 7.

Step 8 If there exist an r -element x of N and a prime divisor $p(\neq r)$ of $|N|$ such that $p \nmid |\overline{C_N(x)}|$, then $|\bar{N}|_p \mid m$.

As

$$|x^N| = |N : C_N(x)| = |\bar{N} : \overline{C_N(x)}|,$$

it leads to $|\bar{N}|_p \mid |x^N|$ since p does not divide $|\overline{C_N(x)}|$. It follows that $|\bar{N}|_p \mid |x^G|$ by Lemma 2.1. The hypotheses of the theorem imply $|\bar{N}|_p \mid m$.

Step 9 $|\overline{C_N(y)}|$ and $|\bar{N}|$ have the same prime divisors for any r -element y of N .

If it is not true, there exist an r -element x_0 of N and a prime divisor p of N such that p does not divide $|\overline{C_N(x_0)}|$. Obviously, $p \neq r$. By Step 8, we have $|\bar{N}|_p \mid m$.

By Step 7, we may take a $\{p, r\}$ -element $g \in N$ such that \bar{g} is a $\{p, r\}$ -element. Applying Step 6, we have that there exists a non-central element x of N such that $\bar{g}^{\bar{N}} \cap \overline{C_N(x)} = \emptyset$. Consider that $\overline{C_N(x)}$ operates on $\bar{g}^{\bar{N}}$ by conjugation. Notice that no element in $\overline{C_N(x)}$ distinct from 1 centralizes any element in $\bar{g}^{\bar{N}}$ by Steps 3 and 5. So all orbits of $\overline{C_N(x)}$ on $\bar{g}^{\bar{N}}$ have the same length $|\overline{C_N(x)}|$, which implies $|\overline{C_N(x)}| \mid |\bar{g}^{\bar{N}}|$. Also $|\bar{g}^{\bar{N}}| \mid |\bar{g}^{\bar{G}}|$ by Lemma 2.1, so

$$|\overline{C_N(x)}| \mid |\bar{g}^{\bar{G}}|. \quad (3.1)$$

On the other hand, it is obvious that $\overline{C_N(g)}$ operates without fixed points on $\bar{g}^{\bar{G}} - \bar{g}^{\bar{G}} \cap \overline{C_N(g)}$. By Steps 3 and 5 once again, we have

$$|\overline{C_N(g)}| \mid (|\bar{g}^{\bar{G}}| - |\bar{g}^{\bar{G}} \cap \overline{C_N(g)}|). \quad (3.2)$$

Since N contains a Sylow r -subgroup R of G , $|\overline{C_N(g)}|_r = |\overline{C_N(x)}|_r$, from which the relationship between (3.1) and (3.2) becomes

$$|\overline{C_N(g)}|_r \mid |\bar{g}^{\bar{G}} \cap \overline{C_N(g)}|. \quad (3.3)$$

Notice $|\bar{N}|_p \mid m$. Also, Step 4 implies that $|\bar{N}|_p \mid |\bar{g}^{\bar{G}}|$. Obviously, $|\overline{C_N(g)}|_p \leq |\bar{N}|_p$. So $|\overline{C_N(g)}|_p \mid |\bar{g}^{\bar{G}}|$. Noticing (3.2), we have

$$|\overline{C_N(g)}|_p \mid |\bar{g}^{\bar{G}} \cap \overline{C_N(g)}|. \quad (3.4)$$

By (3.3) and (3.4), we have

$$|\overline{C_N(g)}|_{\{p,r\}} \mid |\overline{g}^G \cap \overline{C_N(g)}|. \quad (3.5)$$

Noticing that $\overline{C_N(g)}$ is Abelian by (ii), we get

$$|\overline{g}^G \cap \overline{C_N(g)}| = |\overline{C_N(g)}|_{\{p,r\}}, \quad (3.6)$$

a contradiction.

Step 10 The final contradiction.

Let p be a prime divisor of $|\overline{N}|$ with $p \neq r$. Choose a non-central r -element x_0 such that $|\overline{C_N(x_0)}|_p$ is as small as possible. By Step 9 we have $|\overline{C_N(x_0)}|_p > 1$. So, it is clear that we may take a $\{p, r\}$ -element $g \in C_N(x_0)$ such that \overline{g} is also a $\{p, r\}$ -element. Arguing as in Step 9, we may see that (3.1) and (3.2) still hold, and therefore (3.3) also holds.

Noting $C_N(g) = C_N(x_0)$ by Step 3, we have $|\overline{C_N(g)}|_p = |\overline{C_N(x_0)}|_p$. Noticing $|\overline{C_N(x_0)}|_p \leq |\overline{C_N(x)}|_p$ by the choice of x_0 , we have $|\overline{C_N(g)}|_p \leq |\overline{C_N(x)}|_p$. Consequently, $|\overline{C_N(g)}|_p \mid |\overline{g}^G|$ by (3.1). By using a similar argument as in Step 9, (3.4) is obtained. Arguing as in Step 9 once again, we have (3.5), and thus equation (3.6) holds, a contradiction.

Corollary 3.1 *Let G be a finite group with a non-central Sylow r -subgroup R and N a normal subgroup of G containing R . If $|x^G| = 1$ or m for every element x of N , then either N is Abelian or $N_{r'} \leq Z(G)$.*

Proof By Theorem 1.1, we have that N is nilpotent. So, for any $p(\neq r)$ -element x of N , we have $r \nmid |G : C_G(x)|$. If $N_{r'} \not\leq Z(G)$, then $r \nmid m$. So $R \leq Z(N)$. However, since R is non-central, there exists a non-central r -element x_0 such that $N \leq C_N(x_0)$. Thus we have $N_{r'} \leq Z(C_G(x_0))$ by (i) in the proof of Theorem 1.1. So N is Abelian.

Corollary 3.2 (see [8, Theorem 1]) *Let G be a group. If $|x^G| = 1$ or m for every element x in G , then G is nilpotent.*

Proof Let $N = G$. Obviously, N satisfies the hypotheses of Theorem 1.1. So $N = G$ is nilpotent by Theorem 1.1.

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