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Self-similar Solutions for a Transport Equation with Non-local Flux**

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(*Dedicated to Professor Andrew Majda on the Occasion of his* 60*th Birthday*)

Abstract The authors construct self-similar solutions for an N-dimensional transport equation, where the velocity is given by the Riezs transform. These solutions imply non-uniqueness of weak solution. In addition, self-similar solution for a one-dimensional conservative equation involving the Hilbert transform is obtained.

Keywords Hilbert transform, Riesz transform, Transport equations, Self-similar solutions
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1 Introduction

In this paper, we shall construct self-similar solutions of the transport equation

$$\theta_t + R\theta \cdot \nabla\theta = 0, \quad \text{on } \mathbb{R}^N \times \mathbb{R}^+, \tag{1.1}$$

$$\theta(x,0) = \theta_0(x), \tag{1.2}$$

where $\theta : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$, $N \geq 2$, $R\theta = (R_1\theta, \cdots, R_N\theta)$ and $R_i\theta$ are the Riesz transform of θ in the i-th direction, i.e.,

$$R_i\theta(x) = \Gamma\Big(\frac{N+1}{2}\Big)\pi^{-\frac{N+1}{2}} P.V. \int_{\mathbb{R}^N} \frac{x_i - y_i}{|x-y|^{N+1}}\theta(y)\mathrm{d}y, \quad 1 \leq i \leq N. \tag{1.3}$$

Equation (1.1) was studied in [2], and the authors showed the blow-up in finite time for all positive initial data. For a simple proof of the formation of singularities with radial initial data see [10], and for the viscous case see [13].

The technique used in this paper to construct self-similar solutions of the form

$$\theta(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R}^N) \tag{1.4}$$

is based on a result of [11], where the author showed that the function $\theta(x,1)$ is such that $\Lambda\theta(x,1) = N$ in the unit ball (see Section 2). These are also self-similar solutions of the 1D

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transport equation

$$\theta_t + H\theta\theta_x = 0, \quad \text{on } \mathbb{R}\times\mathbb{R}^+, \tag{1.5}$$
$$\theta(x,0) = \theta_0(x), \tag{1.6}$$

where $H\theta$ is the Hilbert transform of θ, i.e.,

$$H\theta(x) = \frac{1}{\pi}P.V.\int \frac{\theta(y)}{x-y}\mathrm{d}y$$

(for more details on this equation, see [1, 7, 8, 14]).

In Section 3, we will see that this result can be used to show the existence of self-similar solutions of the equation

$$\theta_t + (\theta H\theta)_x = 0, \quad \text{in } \mathbb{R}^N\times\mathbb{R}^+, \tag{1.7}$$
$$\theta(x,0) = \theta_0(x), \tag{1.8}$$

which was studied from completely different contexts (vortex sheet, water wave, 1D model of the quasi-geostrophic equation, dislocations dynamics in solids and complex Burgers equation) in [1, 3–6, 9, 12] and references therein. Nevertheless we will follow the ideas of [4] to construct the self-similar solutions.

Next we shall comment briefly the notation: the spaces $W^{k,p}$ are the classical Sobolev spaces (k derivatives in L^p). The operator Λ^α is defined by the operator $(-\Delta)^{\frac{\alpha}{2}}$, i.e., in the Fourier space

$$\widehat{\Lambda^\alpha\theta}(\xi) = |\xi|^\alpha\widehat{\theta}(\xi),$$

and we recall the identity

$$\widehat{R_j\theta}(\xi) = -\mathrm{i}\frac{\xi_j}{|\xi|}\widehat{\theta}(\xi).$$

2 Riezs Transport Equation

2.1 Self-similar solutions

From the scaling invariance of equation (1.1), $\theta(x,t)\to\theta(\lambda x,\lambda t)$ with $\lambda>0$, we will consider a self-similar function with the following form:

$$\theta(x,t) = \Phi\Big(\frac{x}{t}\Big) = \Phi(\xi), \tag{2.1}$$

where $\xi = \frac{x}{t}$. The equalities

$$\partial_t\theta(x,t) = \partial_t\Phi\Big(\frac{x}{t}\Big) = -\frac{\xi}{t}\nabla\Phi(\xi),$$
$$R\theta(x,t) = R\Phi(\xi),$$
$$\nabla\theta(x,t) = \nabla\Big(\Phi\Big(\frac{x}{t}\Big)\Big) = \frac{1}{t}\nabla\Phi(\xi)$$

yield, from equation (1.1),

$$\nabla\Phi(\xi)\cdot(R\Phi(\xi)-\xi) = 0. \tag{2.2}$$

Now we shall show the existence of a solution to (2.2) by means of the following lemma.

Lemma 2.1 *The function*

$$v(\xi) = Nk(N)((1-|\xi|^2)_+)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R}^N), \tag{2.3}$$

where $k(N) = \Gamma(\frac{N}{2})(2^{\frac{1}{2}}\Gamma(\frac{3}{2})\Gamma(\frac{2N+1}{2}))^{-1}$ *and* f_+ *is the positive part of the function* f*, satisfies the equalities*

$$Rv(\xi) = \xi, \quad \textit{if } |\xi| < 1,$$

and

$$\nabla v(\xi) = 0, \quad \textit{if } |\xi| > 1.$$

Proof From [11], we know that $v(\xi)$ satisfies the following properties:
(1) $\Lambda v(\xi) = N$, if $|\xi| < 1$.
(2) $\Lambda v(\xi) \in L^1(R^N)$.
(3) Λv is radial.
Since

$$Rv = \nabla(\Lambda^{-1} v) \equiv \nabla \Psi, \tag{2.4}$$

$$\nabla \cdot Rv = \Lambda v, \tag{2.5}$$

we have $\Delta\Psi = \Lambda v$ and therefore Ψ is a radial function with $\Delta\Psi(\xi) = N$ if $|\xi| < 1$. This implies the following expression for Ψ:

$$\Psi(\xi) = \frac{|\xi|^2}{2} + a_0, \quad \text{if } |\xi| < 1,$$

where a_0 is constant. By using (2.4), we obtain

$$Rv(\xi) = \frac{\xi}{|\xi|}\frac{\partial}{\partial|\xi|}\Psi(\xi) = \xi, \quad \text{if } |\xi| < 1. \tag{2.6}$$

Thus, the function

$$\theta(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R}^N) \tag{2.7}$$

is a self-similar solution of equation (1.1) (almost everywhere).

Remark 2.1 We can check that the functions $\theta^T(x,t) = -\theta(x,(T-t))$, with $0 < T < \infty$, are solutions with an initial data $\theta^T(x,0) = -\theta(x,T)$, which collapse in a point in finite time T.

Remark 2.2 The previous ideas can be easily adapted to prove that the function,

$$\theta(x,t) = k(1)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R}), \tag{2.8}$$

is a self-similar solution to equation

$$\theta_t + H\theta\theta_x = 0, \quad \text{on } \mathbb{R} \times \mathbb{R}^+, \tag{2.9}$$

which is a one-dimensional version of equation (1.1).

2.2 Formal weak solutions and non-uniqueness

In this section, we shall check that the previous functions are solutions to (1.1) in the weak sense that we define below. In addition, we will be able to show the non-uniqueness.

Definition 2.1 *The function $\theta(x,t)$ is a weak solution to* (1.1), *if*

$$\theta \in C((0,T), L^q(\mathbb{R}^N)) \cap C((0,T), W^{1,p}(\mathbb{R}^N))$$

with $1 \le q < \infty$ and $1 \le p < 2$,

$$\partial_t \theta \in W^{1,p}(\mathbb{R}^N)$$

for all $t > 0$ with $1 \le p < 2$,

$$\int_{\mathbb{R}^N} (\theta(x,t)_t + R\theta(x,t) \cdot \nabla\theta(x,t))\phi(x,t)\mathrm{d}x = 0$$

for all $t \in (0,T)$ and all $\phi \in C_c^\infty((0,T) \times \mathbb{R}^N)$, and

$$\lim_{t \to 0^+} \theta(x,t) = \theta_0(x), \quad \text{in } L^q(\mathbb{R}^N).$$

Theorem 2.1 (Non-uniqueness) *The function*

$$\Phi(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}}$$

is a global weak solution to (1.1) *in the sense of Definition* 2.1 *with zero initial data.*

Proof Given a function $\phi(x,t) \in C_c^\infty((0,\infty) \times \mathbb{R}^N)$ and a fixed time $t > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\Phi(x,t)_t + R\Phi(x,t) \cdot \nabla\Phi(x,t))\phi(x,t)\mathrm{d}x \\
=& \int_{|x|<t} (\Phi(x,t)_t + R\Phi(x,t) \cdot \nabla\Phi(x,t))\phi(x,t)\mathrm{d}x \\
=& \int_{\varepsilon<|x|<t} (\Phi(x,t)_t + R\Phi(x,t) \cdot \nabla\Phi(x,t))\phi(x,t)\mathrm{d}x \\
&+ \int_{|x|<\varepsilon} (\Phi(x,t)_t + R\Phi(x,t) \cdot \nabla\Phi(x,t))\phi(x,t)\mathrm{d}x,
\end{aligned}$$

where $0 < \varepsilon < t$. The second term on the right-hand side of the last expression is equal to zero. In addition, we have the following identities:

$$\nabla\Phi(x,t) = \begin{cases} 0, & |x| > t, \\ Nk(N)\dfrac{\frac{x}{t^2}}{\big(1 - \frac{|x|^2}{t^2}\big)^{\frac{1}{2}}}, & |x| < t, \end{cases} \tag{2.10}$$

$$\partial_t\Phi(x,t) = \begin{cases} 0, & |x| > t, \\ Nk(N)\dfrac{-\frac{|x|^2}{t^3}}{\big(1 - \frac{|x|^2}{t^2}\big)^{\frac{1}{2}}}, & |x| < t. \end{cases} \tag{2.11}$$

Thus, if $p < 2$, we obtain

$$\begin{aligned}\|\nabla\Phi(\,\cdot\,,t)\|_{L^p(\mathbb{R}^N)} &= Nk(N)\Big(\int_{|x|<t}\frac{\frac{|x|^p}{t^{2p}}}{\big(1-\frac{|x|^2}{t^2}\big)^{\frac{p}{2}}}\mathrm{d}x\Big)^{\frac{1}{p}}\\ &= Nk(N)t^{\frac{N}{p}-p}\Big(\int_{|x|<t}\frac{|x|^p}{(1-|x|^2)^{\frac{p}{2}}}\mathrm{d}x\Big)^{\frac{1}{p}}\\ &= C(N)t^{\frac{N}{p}-p}\Big(\int_0^1\frac{r^{N-1+p}}{(1-r^2)^{\frac{p}{2}}}\mathrm{d}r\Big)^{\frac{1}{p}} = C(N,p)t^{\frac{N}{p}-p},\\ \|\partial_t\Phi(\,\cdot\,,t)\|_{L^1(\mathbb{R}^N)} &= Nk(N)\Big(\int_{|x|<t}\frac{\frac{|x|^{2p}}{t^{3p}}}{\big(1-\frac{|x|^2}{t^2}\big)^{\frac{p}{2}}}\mathrm{d}x\Big)^{\frac{1}{p}}\\ &= Nk(N)t^{\frac{N}{p}-p}\Big(\int_{|x|<1}\frac{|x|^{2p}}{(1-|x|^2)^{\frac{p}{2}}}\mathrm{d}x\Big)^{\frac{1}{p}}\\ &= C(N)t^{\frac{N}{p}-p}\Big(\int_0^1\frac{r^{2p+N-1}}{(1-r^2)^{\frac{p}{2}}}\mathrm{d}r\Big)^{\frac{1}{p}} = C(N,p)t^{\frac{N}{p}-p}.\end{aligned} \tag{2.12}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_t\Phi(x,t)\phi(x,t)\mathrm{d}x &\le \|\partial_t\Phi(\,\cdot\,,t)\|_{L^1(\mathbb{R}^N)}\|\phi(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}\\ &= C(N,1)t^{N-1}\|\phi(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}R\Phi(x,t)\cdot\nabla\Phi(x,t)\phi(x,t)\mathrm{d}x &\le \|R\Phi(\,\cdot\,,t)\|_{L^q(\mathbb{R}^N)}\|\nabla\Phi(\,\cdot\,,t)\|_{L^p(\mathbb{R}^N)}\|\phi(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}\\ &\le C(N,q,p)t^{N-p}\|\phi(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)},\end{aligned}$$

where $1 < p < 2$, $\frac{1}{p}+\frac{1}{q} = 1$ and $t > 0$. Then, we can conclude

$$\lim_{\varepsilon\to t}\int_{\varepsilon<|x|<t}(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t))\phi(x,t)\mathrm{d}x = 0, \quad \forall\, t > 0,$$

and

$$\int_{\mathbb{R}^N}(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t))\phi(x,t)\mathrm{d}x = 0, \quad \forall\, t > 0,\ \forall\,\phi\in C_c^\infty((0,\infty)\times\mathbb{R}^N).$$

In addition, it is easy to check that

$$\lim_{t\to 0^+}\Phi(x,t) = 0, \quad \text{in } L^p(\mathbb{R}^N),\ \text{with } 1\le p<\infty.$$

3 One Dimensional Conservative Equation

In this section, we will construct self-similar solutions for the equation

$$\theta_t + (\theta H\theta)_x = 0, \quad \text{in } \mathbb{R}^N\times\mathbb{R}^+, \tag{3.1}$$

$$\theta(x,0) = \theta_0(x), \tag{3.2}$$

where $\theta : \mathbb{R}\to\mathbb{R}$ and $H\theta$ is the Hilbert transform of the function θ.

We will use the techniques developed in [4] to obtain formally a self-similar solution.

We sketch the mean features of equation (3.1) in the following lemma.

Lemma 3.1 *Let $Z(w,t)$ be a complex function $Z: M \to \mathbb{C}$, where $M = \{w = x + \mathrm{i}\, y : y > 0\}$, such that*

$$Z_t + ZZ_w = 0, \quad \text{on } M, \tag{3.3}$$

$$Z(w,0) = R\theta_0(x,y) - \mathrm{i}\, P\theta_0(x,y). \tag{3.4}$$

$P\theta(x,y)$ is the convolution with the Poisson kernel and $R\theta(x,y)$ is the convolution with the harmonic conjugate Poisson kernel, i.e.,

$$P\theta(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{y^2+(x-s)^2}\theta(s)\mathrm{d}s, \quad R\theta(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-s}{y^2+(x-s)^2}\theta(s)\mathrm{d}s. \tag{3.5}$$

Then, if $Z(w,t)$ is analytic on M and vanishes at infinity,

$$\theta(x,t) = -\Im(Z(w,t)|_{y=0}) \tag{3.6}$$

is a solution to (3.1), with $\theta(x,0) = \theta_0(x)$ on the points where θ and $H\theta$ are differentiable.

Proof If $Z(w,t)$ satisfies the statements of Lemma 3.1, we can write it in the following way:

$$Z(w,t) = R\theta(x,y;t) - \mathrm{i}\, P\theta(x,y;t), \tag{3.7}$$

where $\theta(x,t) = -\Im(Z(w,t)|_{y=0})$. In addition, we know

$$Z_t + ZZ_x = 0, \quad \text{on } M,$$

and from (3.7) it follows $Z(w,t)|_{y=0} = H\theta(x,t) - \mathrm{i}\,\theta(x,t)$. By taking the limit $y \to 0^+$ in equation (3.7), we have the desired result.

Next we shall use the previous lemma to prove the following theorem.

Theorem 3.1 *The function*

$$\theta(x,t) = \frac{1}{\sqrt{t\pi}}\left(\left(1 - \frac{\pi x^2}{4t}\right)_+\right)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R})$$

is a self-similar solution (at least in a weak sense) to (3.1) with the initial data $\theta_0 = \delta_0$, where δ_0 is the Dirac Delta.

Proof By Lemma 3.1, we have to study the solutions of the equation

$$Z_t + ZZ_w = 0, \quad \text{on } M, \tag{3.8}$$

$$Z(w,0) = \frac{1}{\pi}\frac{x}{x^2+y^2} - \mathrm{i}\frac{1}{\pi}\frac{y}{x^2+y^2}. \tag{3.9}$$

A standard argument yields that the solution is constant along the following complex trajectories

$$X^1(x,y,t) = \frac{1}{\pi}\frac{x}{x^2+y^2}t + x, \tag{3.10}$$

$$X^2(x,y,t) = -\frac{1}{\pi}\frac{y}{x^2+y^2}t + y. \tag{3.11}$$

Thus

$$Z(X^1(x,y,t), X^2(x,y,t), t) = Z_0(x,y),$$

and one can check that the solution, $Z(w,t)$, satisfies the requirements of Lemma 3.1. In addition,

$$\theta(X^1,t) = -\Im(Z(X^1,X^2,t)|_{X^2=0}) = P\theta_0(x,y,t)|_{X^2=0} = \frac{y}{\pi t}\Big|_{X^2=0}.$$

The function

$$y = \sqrt{\pi t}\left(\left(1-\frac{\pi x^2}{t}\right)_+\right)^{\frac{1}{2}}$$

satisfies equation (3.11) with $X^2 = 0$, and by equation (3.10) we have

$$X^1 = \begin{cases} 2x, & |x| < \sqrt{\dfrac{t}{\pi}}, \\ \dfrac{t}{\pi x} + x, & |x| > \sqrt{\dfrac{t}{\pi}}. \end{cases}$$

Furthermore, we can conclude that

$$\theta(x,t) = \frac{1}{\sqrt{t\pi}}\left(\left(1-\frac{\pi x^2}{4t}\right)_+\right)^{\frac{1}{2}}.$$

Remark 3.1 This solution was obtained in [3] by using the techniques in Section 2. In fact, they constructed self-similar solutions to the equation

$$u_t + \Lambda^\alpha u u_x = 0, \quad \text{on } \mathbb{R}\times\mathbb{R}^+, \tag{3.12}$$

$$u(x,0) = H(x), \tag{3.13}$$

where $H(x)$ is the Heaviside function and $0 < \alpha \le 2$.

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