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On Regularity and Singularity of Free Boundaries in Obstacle Problems**

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(*Dedicated to Professor Andrew Majda on the Occasion of his* 60*th Birthday*)

Abstract The author presents a simple approach to both regularity and singularity theorems for free boundaries in classical obstacle problems. This approach is based on the monotonicity of several variational integrals, the Federer-Almgren dimension reduction and stratification theorems, and some simple PDE arguments.

Keywords Free boundary, Monotonicity, Dimension reduction, Uniqueness of blow-ups
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1 Introduction

Let Ω be a bounded, smooth domain in $\mathbb{R}^n$, and consider a closed, convex subset K of $H^1(\Omega)$:

$$K = \{u \in H^1(\Omega) : u = \varphi \text{ on } \partial\Omega \text{ and } u \geq \psi \text{ in } \Omega\}.$$

Here φ is a smooth function on $\partial\Omega$, and ψ is smooth in $\overline{\Omega}$ with $\psi \leq \varphi$ on $\partial\Omega$. In K, there is a unique v that minimizes the Dirichlet integral $\int_\Omega |\nabla v|^2 \mathrm{d}x$. Such a v is called the solution of the obstacle problem. The classical obstacle problem is to study properties of such minimizers v. The obstacle problem, in fact, was one of the main motivations for the development of the theory of variational inequalities (see [8]), and it has many interesting applications (see [7]).

Suppose that the obstacle ψ is smooth (say $C^{2,\alpha}$ in Ω). Then the solution v of the obstacle problem is of class $C^{1,1}(\Omega)$, and that is the optimal regularity one can generally expect (see [6, 7]). Let

$$\begin{aligned}\wedge(v) &= \{x \in \Omega : v(x) = \psi(x)\},\\ N(v) &= \{x \in \Omega : v(x) > \psi(x)\},\\ \Gamma_v &= \partial N(v) \cap \Omega.\end{aligned}$$

Γ_v is called the free boundary of the solution v, and $\wedge(v)$ is called the set of coincidence. One of the most fascinating and challenging questions concerning the obstacle problem is the study of the properties of Γ_v. Without any further assumptions on ψ (besides the smoothness), one can easily construct examples to show that $\wedge(v)$ can be an arbitrary closed subset of Ω.

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In order to establish regularity of Γ_v, one of the natural assumptions would be that $\Delta\psi < 0$ in Ω. Under this assumption, with some usual normalizations, the problem is reduced to the study of Γ_u of the following normalized solutions $P_1(M)$ to the obstacle problems. Here $P_1(M)$ consists of such u:

(1) $u \geq 0$ in B_1, $\|u\|_{C^{1,1}(B_1)} \leq M$;

(2) $\Delta u = 1$ in $\{x \in B_1 : u(x) > 0\}$;

(3) $\underline{0} \in \Gamma_u = \partial\{x \in B_1 : u(x) > 0\} \cap B_1$.

Caffarelli [4] observed that $P_1(M)$ is compact. This is based on the following simple fact.

Lemma 1.1 (Nondegeneracy) *Let $u \in P_1(M)$, $x_0 \in B_{\frac{1}{2}}$ with $u(x_0) > 0$. Then*

$$\sup_{\partial B_r(x_0)} u(x) \geq \frac{1}{2n} r^2, \quad 0 < r \leq \frac{1}{2}.$$

Proof Consider

$$h(x) = u(x) - \frac{1}{2n}|x - x_0|^2, \quad \text{in } N(u) = \{x \in B_1 : u(x) > 0\}.$$

Then $\Delta h(x) = 0$ in $N(u)$ and $h(x_0) = u(x_0) > 0$. Thus $\sup\limits_{\partial(B_r(x_0)\cap N(u))} h(x) \geq u(x_0) > 0$. Since on Γ_u, $h \leq 0$, one easily deduces that $\sup\limits_{\partial B_r(x_0)\cap N(u)} h(x) \geq u(x_0) > 0$, and the conclusion of the lemma follows.

The following fundamental result concerning free boundary regularity was first established by Caffarelli [3]. An alternate proof based on compactness arguments was later given in [4] (where the proofs are conceptually much more clear and relatively easier to follow than those in [3], but are nonetheless quite involved).

Theorem 1.1 (Caffarelli) *Let $u \in P_1(M)$ and $N(u)$ is not too thin at $\underline{0} \in \Gamma_u$. Then Γ_u is a C^1-hypersurface near $\underline{0}$.*

Here $\wedge(u)$ is not too thin means that there is a universal continuous monotone function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sigma(0^+) = 0$ and $\delta_{r_0}(\wedge(u)) \geq \sigma(r_0)$ for some small $r_0 > 0$.

$\delta_r(\wedge(u)) \geq \varepsilon$ means that $\wedge(u) \cap B_r$ cannot be put into a strip between two parallel hyperplanes with distance between these two planes $< \varepsilon r$.

It was a long outstanding problem to study the properties of Γ_u near a point $x_0 \in \Gamma_u$ where $\wedge(u)$ is, in fact, thin. We call this a singular point of Γ_u. In 1998, Caffarelli introduced a remarkable idea to tackle this difficulty and established the following result.

Theorem 1.2 (see [5]) *Let $x_0 \in \Gamma_u \cap B_{\frac{1}{2}}$ and suppose that $\wedge(u)$ is thin at x_0. Then*

(a) *There exists a unique non-negative quadratic polynomial*

$$Q_{x_0}(x) = \frac{1}{2}(x - x_0)^{\mathrm{T}} M_{x_0}(x - x_0) \quad \textit{with } \Delta Q_{x_0} = 1 \equiv \operatorname{trace} M,$$

such that $|u(x) - Q_{x_0}(x)| \leq |x|^2 \sigma(|x|)$.

(b) *M_{x_0} is continuous in x_0 because x_0 in the singular part of Γ_u.*

(c) *If $\dim\ker M_{x_0} = k$, then the singular set of Γ_u is contained in a C^1 k-dimensional submanifold near x_0.*

Remark 1.1 There are examples of solutions u of the obstacle problem such that $u \geq 0$ in B_1 and $\|u\|_{C^{1,1}(B_1)} \leq M_0$, $0 \in \Gamma_u$ and $\Delta u = 1 + h(x) > 0$ with h a smooth function in B_1 that vanishes at the infinite order at Γ_u. Moreover, singular points of Γ_u form a closed subset of a smooth hypersurface (for example, a hyperplane) of positive $\mathcal{H}^{n-1}$-measure. In other words, Caffarelli's theorem is, in general, the best possibility besides further smoothness of these k-dimensional submanifolds.

2 Weiss Montonicity and Its Consequences

For a harmonic function u in B_1, Almgren [1] showed that

$$N(r) = \frac{rD(r)}{H(r)}, \quad 0 < r < 1,$$

where

$$D(r) = \int_{B_r} |\nabla u|^2 \mathrm{d}x, \quad H(r) = \int_{\partial B_r} u^2,$$

is a monotone increasing function of r. In particular, $N(0^+) = \lim_{r \to 0^+} N(r)$ exists and it is the vanishing order of u at $\underline{0}$. Suppose that u vanishes at $\underline{0}$ with order $k \in \{1, 2, \cdots\}$. Then Almgren's monotonicity immediately implies that

$$\frac{D(r)}{r^{n-2+2k}} - k\frac{H(r)}{r^{n-1+2k}}, \quad 0 < r < 1$$

is a monotone increasing and non-negative function of $r \in (0, 1)$.

Weiss [11] proved a similar monotonicity formula for solutions $u \in P_1(M)$ of the obstacle problem (with a similar proof).

Lemma 2.1 (Weiss Monotonicity Formula) *Let $u \in P_1(M)$, $x_0 \in \Gamma_u$ such that $B_R(x_0) \subseteq B_1$. Then the function*

$$\Phi(x_0, r, u) \equiv \frac{1}{r^{n+2}} \int_{B_r(x_0)} [|\nabla u|^2(x) + 2u(x)] \mathrm{d}x - \frac{2\int_{\partial B_r(x_0)} u^2}{r^{n+3}}$$

is monotone increasing for $0 < r \leq R$. In fact,

$$\frac{\mathrm{d}}{\mathrm{d}r} \Phi(x_0, r, u) = \frac{2}{r^{n+2}} \int_{\partial B_r(x_0)} \Big(u_\rho - \frac{2u}{\rho}\Big)^2.$$

An easy consequence of this monotonicity formula is the following lemma concerning the existence of homogeneous degree 2 blow-ups for $u \in P_1(M)$ at a free boundary point.

Lemma 2.2 (Existence of Homogeneous Blow-Ups) *Let $u \in P_1(M)$. Then for any sequence $\{\lambda_i\}$, $\lambda_i \downarrow 0$, there is a subsequence $\{\lambda_i'\}$ such that $u^{\lambda_i'}(x) = \frac{u(\lambda_i' x)}{(\lambda_i')^2}$ converges uniformly to $u_0(x) \in P_1(M)$ such that $u_0(x) = |x|^2 u_0(\frac{x}{|x|})$.*

Proof We observe that, for any $0 < \lambda < 1$, $u^\lambda(x) \in P_1(M)$. We apply Lemma 2.1 to u^λ to obtain

$$\begin{aligned}\Phi(\underline{0}, 1, u^\lambda) - \Phi(\underline{0}, 0^+, u^\lambda) &= \int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r(\underline{0})} \Big(\frac{\partial u^\lambda}{\partial \rho} - \frac{2u^\lambda}{\rho}\Big)^2 \mathrm{d}r \\ &= \int_0^\lambda \frac{2}{r^{n+2}} \int_{\partial B_r(\underline{0})} \Big(\frac{\partial}{\partial \rho} u - \frac{2u}{\rho}\Big)^2 \mathrm{d}r \to 0, \quad \text{as } \lambda \to 0^+.\end{aligned}$$

Thus for a subsequence of $u^{\lambda_i'}$ such that $u^{\lambda_i'}(x) \to u_0(x)$ in $C^{1,\beta}$ (for any $0 < \beta < 1$, via the fact that $u^{\lambda_i} \in P_1(M)$) with $u_0 \in P_1(M)$, one has

$$\int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r} \Big(\frac{\partial u_0}{\partial \rho} - \frac{2u_0}{\rho}\Big)^2 \mathrm{d}r = \lim_{\lambda_i \to 0} \int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r} \Big(\frac{\partial u^{\lambda_i'}}{\partial \rho} - \frac{2u^{\lambda_i'}}{\rho}\Big)^2 \mathrm{d}r$$
$$= \lim_{\lambda_i'} \int_0^{\lambda_i'} \frac{2}{r^{n+2}} \int_{\partial B_r} \Big(\frac{\partial u}{\partial \rho} - \frac{2u}{\rho}\Big)^2 \mathrm{d}r = 0.$$

In other words, $u_0(x) = |x|^2 u_0(\frac{x}{|x|})$.

Now we let $\mathcal{F} = \{\Gamma_u : u \in P_1(M)\}$. Then it is easy to verify the following properties of $\mathcal{F}$:

(1) $\forall\, E \in \mathcal{F}$, $a \in E$, $E_{a,\lambda} \in \mathcal{F}$, where $E_{a,\lambda} = (\frac{E-a}{\lambda}) \cap B_1$, $0 < \lambda \le 1 - |a|$.

This is a direct consequence of the fact that if $u \in P_1(M)$, $a \in \Gamma_u$, then $\frac{u(\lambda(x-a))}{\lambda^2} \in P_1(M)$, for $0 < \lambda \le 1 - |a|$.

(2) $\forall\, E \in \mathcal{F}$, $a \in E$, $\{\lambda_i\} \downarrow 0$, there is a subsequence $\{\lambda_i'\}$ such that

$$E_{a,\lambda_i'} \rightharpoonup T \quad \text{with } T_{\underline{0},\lambda} \equiv T \;\text{ for } 0 < \lambda < 1.$$

In other words, there is a tangent cone of E at each point $a \in E$.

This property follows directly from Lemma 2.2 on the existence of homogeneous degree 2 blow-ups of u at points of Γ_u. Here we say $E_i \rightharpoonup F$ if for any $\varepsilon > 0$ and for all sufficiently large i, $i \ge i(\varepsilon)$, E_i is contained in the ε neighborhood of F.

The following result is simply a version of the dimension reduction principle of Federer [7] and Almgren's improvement thereof (stratification principle) [1, 10].

Consider $E \in \mathcal{F}$ and let

$$S_j = \{a \in E : \text{the invariant dimension of } T \le j, \text{ for all tangent cones } T \text{ of } E \text{ at } a\}$$

for $j = 0, 1, 2, \cdots, n$. Here, for a given tangent cone T of E at a (i.e., $T_{0,\lambda} \equiv T = \lim_{\lambda_i} E_{a,\lambda_i}$ for a sequence of $\lambda_i \downarrow 0$, $0 < \lambda \le 1$), a linear subspace V of $\mathbb{R}^n$ is called an invariant space of T if $(T + v) \cap B_1 \subset T$ for all $v \in V$. The maximum dimension of all such invariant spaces V is called the invariant dimension of T.

Theorem 2.1 (Reduction and Stratification Principle) (i) *For every $E \in \mathcal{F}$, $\dim_H E$ (the Hausdorff dimension of E) $\le n - 1$. Moreover, there is an $(n-1)$-dimensional hyperplane $T \in \mathcal{F}$.*

(ii) $\dim_H S_j \le j$, *for* $j = 0, 1, 2, \cdots, n-1$,

$$S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} = E.$$

Moreover, S_0 consists of isolated points.

Proof The proof of this theorem is now standard (see e.g., [9, 10]). On the other hand, an even simpler argument can be made after we establish the uniqueness of the homogeneous degree 2 blow-ups at singular points of Γ_u. We thus omit further details of the proof of this theorem.

3 Regularity Theorem

We now consider the top dimensional parts of the free boundaries.

Let $u \in P_1(M)$, $E = \Gamma_u \in \mathcal{F}$, $S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} \equiv E$. Let $F = S_{n-1} \mid S_{n-2}$. Thus for every point $a \in F$, there is a tangent cone of Γ_u at a which is an $(n-1)$-dimensional hyperplane.

Definition 3.1 *Let $a \in \Gamma_u$. We say that $\wedge(u)$ is not too thin at $\underline{a}$ if one of the following conditions is satisfied for $\wedge(u)$:*

(a) $\displaystyle \varliminf_{r\to 0} \frac{|B_r(a) \cap \wedge(u)|}{|B_r(a)|} > 0,$

(b) $\displaystyle \varliminf_{r\to 0} \delta_r(\wedge(u) \cap B_1(a)) > 0.$

Lemma 3.1 *Suppose that $a \in F$ and $\wedge(u)$ is not too thin at a. Then there is a homogeneous degree 2 blow-up u_0 of u at point a $\big($that is, a limit of a sequence of functions of the form $\frac{u(\lambda_i(x-a))}{\lambda_i^2}$, $\lambda_i \to 0^+\big)$ such that $u_0(x) = \frac{1}{2}(x_n^+)^2$ in a suitable coordinate system of $\mathbb{R}^n$.*

Proof Since $a \in F$, there is a sequence of $\{\lambda_i\}$, $\lambda_i \downarrow 0$, such that $u_i(x) = \frac{u(\lambda_i(x-a))}{\lambda_i^2} \to u_0(x)$ in $C^{1,\alpha}(B_1)$ such that

(1) $u_0(x) = |x|^2 u_0(\frac{x}{|x|})$,

(2) $\Gamma_{u_i} \to \Gamma_{u_0}$, $\Gamma_{u_0} = \{x_n = 0\}$ for a suitable choice of coordinate system.

Since $\wedge(u)$ is not thin at u, this implies either $| \wedge (u_0)| > 0$ or $\delta_1(\wedge(u_0)) > 0$. Thus $\wedge(u_0)$ has to be a half-space bounded by $\{x_n = 0\}$. Therefore $u_0(x) = \frac{1}{2}(x_n^+)^2$ follows by a suitable choice of coordinate system.

Remark 3.1 The hypothesis that $\wedge(u)$ is not too thin at $a \in F$ can be replaced by one of the following two conditions: either $\varlimsup_i | \wedge (u_i) \cap B_1| \geq \varepsilon > 0$ or $\varlimsup_i \delta_1(\wedge(u_i)) \geq \varepsilon > 0$ in the preceding proof.

Theorem 3.1 (Regularity of Free Boundary) *Suppose that $a \in F$ and $\wedge(u)$ is not too thin at a. Then Γ_u is a C^1 hypersurface near a.*

Proof From the conclusion of Lemma 3.1, we see that there is an $r_0 > 0$ such that

$$\left\| \frac{u(r_0(x-a))}{r_0^2} - \frac{1}{2}(x_n^+)^2 \right\|_{C^{1,\alpha}(B_1)} \leq \frac{1}{100n}.$$

Let $v(x) = \frac{u(r_0(x-a))}{r_0^2}$. We need to show that $\Gamma_v \cap B_{\frac{1}{2}}$ is a C^1 hypersurface passing the origin $\underline{0}$. In order to do so, we first consider the following auxiliary functions $h(x)$ parameterized by $x_0 \in N(v) \cap B_{\frac{2}{3}}$,

$$h(x) \equiv \vec{e} \cdot \nabla v(x) - v(x) + \frac{1}{2n}|x - x_0|^2, \quad \text{on } N(v) \cap B_1.$$

Here $\vec{e}$ is a unit vector in $\mathbb{R}^n$. It is clear that $\Delta h(x) = 0$ in $N(v) \cap B_1$ and that $h(x) > 0$ on Γ_v. On $\partial B_1 \cap N(v)$, we have

$$h(x) \geq \vec{e} \cdot \vec{e}_n x_n^+ - \frac{1}{2}(x_n^+)^2 + \frac{1}{2n}|x - x_0|^2 - \frac{1}{50n} \geq \Big(\frac{1}{20} - \frac{1}{50}\Big)\frac{1}{n} + x_n^+\Big(\vec{e} \cdot \vec{e}_n - \frac{1}{2}x_n^+\Big) > 0$$

whenever $\vec{e} \cdot \vec{e}_n \geq \frac{1}{2}$. Therefore $h(x) > 0$ in $N(v) \cap B_1$. In particular, we have

$$h(x_0) = \vec{e} \cdot \nabla v(x_0) - v(x_0) > 0 \quad \text{for all } x_0 \in N(v) \cap B_{\frac{2}{3}} \text{ and } \vec{e} \cdot \vec{e}_n \geq \frac{1}{2}.$$

A direct consequence of the above monotone property of $v(x)$ in the direction $\vec{e}$ is that $\Gamma_v \cap B_{\frac{2}{3}}$ is a Lipschitz graph $x_n = g(x_1, \cdots, x_{n-1})$ with $\operatorname{Lip} g \leq \frac{\sqrt{3}}{2}$.

One is now in a position to apply the result in [2] to conclude that $\Gamma_v \cap B_{\frac{1}{2}}$ is a $C^{1,\alpha}$ graph. We should also note that, with a slightly more expanded argument following the above idea, one can show that $\Gamma_v \cap B_{\frac{1}{2}}$ is a C^1 graph (by improving the sizes of cones of monotonicity for v when points go to free boundary Γ_v) without using the result in [2].

4 Singularity Theorem

From discussions in the preceding section, we conclude that Γ_u, for $u \in P_1(M)$, can be decomposed into two points: $\Gamma_u = R_u + S_u$.

(1) R_u consists of those points a on the free boundary Γ_u such that Γ_u has a tangent cone at a which is an $(n-1)$-dimensional hyperplane, say $\{x_n = 0\}$, and that u has, at a, a degree 2 blow-up of the form $\frac{1}{2}(x_n^+)^2$.

We have shown that Γ_u is a C^1 hypersurface near any point of a of R_u. In particular, R_u is an open subset consisting of regular points of Γ_u.

(2) If $a \in S_u$, then either $a \in S_{n-2} \subset \Gamma_u$ (note that the Hausdorff dimension of $S_{n-2} \leq n-2$), or at $\underline{a}$, u has a homogeneous degree 2 blow-up $u_0(x)$ with $\{u_0(x) = 0\} = \{x_n = 0\}$ for a suitable choice of coordinate system of $\mathbb{R}^n$.

In this last case, $|\wedge(u_0)| = 0$. Since $\Delta u_0 = 1$ whenever $u_0 > 0$, we conclude $\Delta u_0 = 1$ everywhere on $\mathbb{R}^n$. Since $u_0(x) = 0$ on $\{x_n = 0\}$, the only solution is given by $\frac{x_n^2}{2}$.

Remark 4.1 Points a of Γ_u with blow-ups of u at a given by $\frac{x_n^2}{2}$ (for a suitable choice of coordinate system) may be of positive $\mathcal{H}^{n-1}$-dimensional measure.

We shall call S_u the singular set of the free boundary Γ_u. The important fact from the discussion above is that for all $a \in S_u$, u has a homogeneous degree 2 blow-up v at a such that $v \geq 0$ and $\Delta v = 1$ in $\mathbb{R}^n$. Hence v is a non-negative quadratic polynomial in $\mathbb{R}^n$.

Write $v(x)$ as $\frac{1}{2}x^{\mathrm{T}}Mx$. Then M is non-negative and trace $M = 1$. Moreover, if $a \in S_j$, then there is a quadratic blow-up v of u at a such that $D^2v(0) = M$ has rank $\geq n-j$, $j = 0, 1, 2, \cdots, n-2$. One can also use this to verify that $\dim_H S_j \leq j$, which would be even simpler than the proof, say, in [10].

Our main result of this section is the following theorem.

Theorem 4.1 *For $x_0 \in S_u \cap B_{\frac{1}{2}} \subset \Gamma_u$, there is a unique non-negative quadratic polynomial $Q_{x_0}(x) = \frac{1}{2}(x - x_0)^{\mathrm{T}}M_{x_0}(x - x_0)$ with $\Delta Q_{x_0} = 1 = \operatorname{trace} M_{x_0}$, such that $|u(x) - Q_{x_0}(x)| \leq |x|^2\varepsilon(|x|)$, where $\varepsilon(r)$ is a monotone, continuous function on $\mathbb{R}_+$ with $\varepsilon(0^+) = 0$. Moreover, M_{x_0} is continuous in x_0 for $x_0 \in S_u$.*

As a consequence of this theorem, one has the following corollary.

Corollary 4.1 *If $x_0 \in S_u$ and $\dim\ker M_{x_0} = j$, then S_u near x_0 is contained in a C^1 j-dimensional submanifold for $j = 0, 1, 2, \cdots, n-1$. In particular, S_j is contained in a union* *of j-dimensional C^1 submanifolds for $j = 0, 1, \cdots, n-2$.*

The above result is exactly parallel to the singularity theorem of Caffarelli [3]. The only difference is that our proof will be based upon a new monotonicity formula.

Let $u \in P_1(M)$ and v be a non-negative quadratic polynomial such that $\Delta v = 1$. Without loss of generality, we assume that v is a homogeneous degree 2 blow-up of u at $\underline{0} \in S_u$, and that u is already close to v on B_1 (say in $C^{1,\alpha}$ norm). Let

$$D(w,r) = \int_{B_r} |\nabla w|^2(x)\,\mathrm{d}x,$$
$$H(w,r) = \int_{\partial B_r} w^2,$$

where $w = u - v$.

Lemma 4.1 (Generalized Almgren-Weiss Monotonicity)

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\Big] = 2\int_{\partial B_r} \frac{|\rho\frac{\partial w}{\partial \rho} - 2w|^2}{r^{n+4}} \geq 0 \quad \textit{for } 0 < r < 1.$$

Proof A direct calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\Big] = \frac{2}{r^{n+4}}\int_{\partial B_r} \Big|\rho\frac{\partial w}{\partial \rho} - 2w\Big|^2 + \varepsilon(r),$$

where

$$\varepsilon(r) = \frac{2}{r^{n+3}}\int_{B_r} (2w - \rho w_\rho)\Delta w\,\mathrm{d}x.$$

Since v is a homogeneous function of degree 2, we have $2w - \rho w_\rho = 2u - \rho u_\rho$. If $x \in \wedge(u)$, i.e., $u(x) = 0$, then $\nabla u(x) = 0$ ($u \geq 0$ and $C^{1,1}$), and hence $2u - \rho u_\rho = 0$. If $x \in N(u)$, then $\Delta w = \Delta u - \Delta v = \Delta u - 1 = 0$ on the set $\{u(x) > 0\}$. Hence $\varepsilon(r) \equiv 0$.

From our assumption, we have that $D(w,1) - 2H(w,1)$ is small since v is close to u on B_1 in the $C^{1,\alpha}$ norm. On the other hand, since v is a blow-up of u at $\underline{0}$, that means that there is a sequence of $\lambda_i \downarrow 0$ such that $u^{\lambda_i}(x) \to v(x) \in C^{1,\alpha}$ norm as $i \to +\infty$. Because

$$\frac{D(w,\lambda_i)}{\lambda_i^{n+2}} - 2\frac{H(w,\lambda_i)}{\lambda_i^{n+3}} = D(w_i,1) - 2H(w_i,1) \to 0, \quad \text{as } i \to +\infty,$$

where $w_i = u^{\lambda_i}(x) - v(x)$, we conclude

$$0 \leq \frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}} \leq D(w,1) - 2H(w,1).$$

Lemma 4.2 (Convexity)

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\int_{\partial B_r} w^2}{r^{n+3}}\Big) \geq \frac{2}{r}\Big[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\Big] \geq 0.$$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\int_{\partial B_r} w^2}{r^{n+3}} = \frac{2}{r}\Big[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\Big] + \frac{2}{r^{n+3}}\int_{B_r} w\Delta w\,\mathrm{d}x.$$

Notice that one has $w \in C^{1,1}(B_1)$ and, when $u > 0$, $\Delta w = \Delta u - \Delta v = 0$. On the other hand, if $u = 0$, $\Delta u = 0$ a.e. on $\{u = 0\}$, then $w\Delta w = v\Delta v = v$ a.e. on $\{u = 0\}$. Hence $w\Delta w \geq 0$ a.e. in B_1. The conclusion of the convexity lemma follows.

The reason that we call Lemma 4.2 the convexity lemma is that if we let $f(t) = \frac{\int_{\partial B_r} w^2}{r^{n+3}}$, $r = \mathrm{e}^t$, $-\infty < t < 0$, then $f_{tt}(t) \geq 0$ by Lemmas 4.2 and 4.1. An easy consequence of either one of these two lemmas is the uniqueness of homogeneous degree 2 blow-up at any point a of S_u. The rest of the proof of the main theorem follows.

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