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On the Hydrostatic and Darcy Limits of the Convective Navier-Stokes Equations**

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(*Dedicated to Professor Andrew Majda on the Occasion of his* 60*th Birthday*)

Abstract The author studies two singular limits of the convective Navier-Stokes equations. The hydrostatic limit is first studied: the author shows the existence of global solutions with a convex pressure field and derives them from the convective Navier-Stokes equations as long as the pressure field is smooth and strongly convex. The (friction dominated) Darcy limit is also considered, and a relaxed version is studied.

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1 Introduction

Convection theory is an essential part of geophysical fluid dynamics (see [17–19]). In this paper, we study (from a purely mathematical viewpoint) various models that can be formally derived from the convective Navier-Stokes equations, or more precisely what we call the generalized Navier-Stokes Boussinesq equations. The GNSB equations are just the Navier-Stokes equations with a forcing term advected by the fluid velocity with a possible source term, a typical example being the buoyancy force in classical convection theory. In the first section, we start our discussion with a one dimensional toy-model that describes, in a very crude way, a very fast convection process based on rearrangement theory. Then, using the concept of maps with convex potential and optimal transport theory, we introduce a multidimensional generalization of the model and provide a weak formulation for which global solutions are shown to exist for all suitable initial conditions. Next, we observe that our multidimensional toy-model can be interpreted as a singular, hydrostatic, limit of the GNSB equations. We provide a global existence theorem of suitable weak solutions (based on the analysis of the multidimensional toy-model), with a convex pressure field. As long as the pressure field is smooth and strongly convex, we show that these solutions are limits of the Leray solutions of the GNSB equations. Finally, we consider another singular (friction dominated) limit of the GNSB equations, what we call the Darcy Boussinesq equations, and study a relaxed version.

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2 A Toy Model for Ultra Fast Convection

2.1 Increasing rearrangements

Let us first recall a well-known result about rearrangements (see, for instance, [15]).

Proposition 2.1 *Let $p \in [1, +\infty]$ and $D = [0, 1]$. For every real valued function $z \in L^p(D)$, there is a unique non-decreasing function $Z = R[z] \in L^p(D)$ such that*

$$\int_D f(Z(x))\mathrm{d}x = \int_D f(z(x))\mathrm{d}x$$

holds for all continuous function f such that $|f(y)| \le 1 + |y|^p$. $R[z]$ is called the increasing rearrangement of z.

Notice that in the discrete case when

$$z(x) = z_j, \quad \frac{j}{N} < x < \frac{j+1}{N}, \ j = 0, \cdots, N-1,$$

then $R[z](x) = Z_j$, where $(Z_1, \cdots, Z_N)$ is just $(z_1, \cdots, z_N)$ sorted in increasing order.

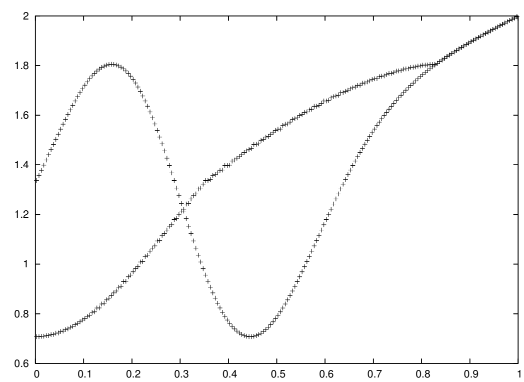

Figure 1 An example of function and its increasing rearrangement (discretization with $N = 200$ points)

2.2 Description of the model

We consider only the vertical coordinate $x = x_3 \in D = [0, 1]$. The temperature field is denoted by $y(t, x)$ and the heat source term by $G = G(t, x, y)$.

We use a time discrete scheme with a uniform time step $h > 0$. We denote by $y_n(x)$ the discrete approximation of $y(t, x)$ at time $t = nh$, for $n = 0, 1, 2, \cdots$. Each time step is made of a predictor step

$$\widetilde{y}_{n+1}(x) = y_n(x) + h\, G(nh, x, y_n(x))$$

describing the heating process due to the source term G, followed by a corrector step

$$y_{n+1} = R[\widetilde{y}_{n+1}]$$

describing the convection process in a very crude way: the temperature profile is just rearranged in increasing order. Overall, the algorithm describes a succession of stable equilibria modified by the source term at each time step.

2.3 Numerics

Numerical experiments can be easily performed without using further space discretization, provided that the initial condition $y_0(x)$ is itself approximated by a piecewise constant profile. In Figures 2 and 3, with 500 grid points in $x \in [0,1]$, one can see the temperature profiles $y(t,x)$ at times $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$ and $t = 1$, in the case when the initial profile is $y_0(x) = x$ and the source term G is

$$G = G(x) = 1 + \exp(-25(x-0.2)^2) - \exp(-20(x-0.4)^2).$$

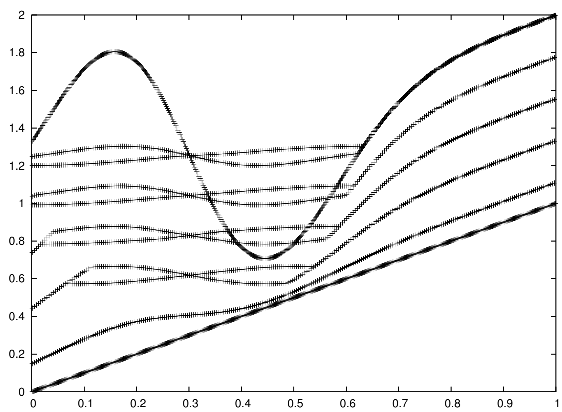

Figure 2 Profile $y = y(t,x)$ versus x at different t (predictor and corrector) with $h = 0.1$

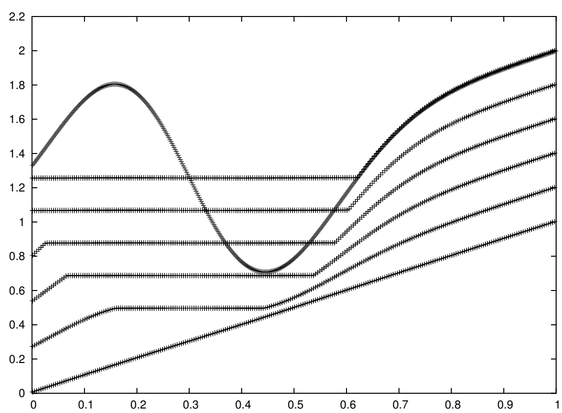

Figure 3 Profile $y = y(t,x)$ versus x at different t with $h = 0.005$ (fully converged)

In these pictures, we also draw the final temperature profile at time $t = 1$ that would have been obtained by inhibiting the rearrangement process (no corrector step in the scheme). The differences between Figure 2 and Figure 3 is entirely due to the size of h. In Figure 2, the time stepping $h = 0.1$ is crude and we easily see the discrepancy between the predictor and the corrector at each step. A very well converged solution is drawn in Figure 3, just by using the finer time step $h = 0.005$. We notice that the temperature profile gets flat as the rearrangement step starts acting on the discrete solution. In Figure 4, on can see how the area of rearrangement is spreading along the x axis as time goes on, which crudely describes the convection process and the mixing occurring in the fluid.

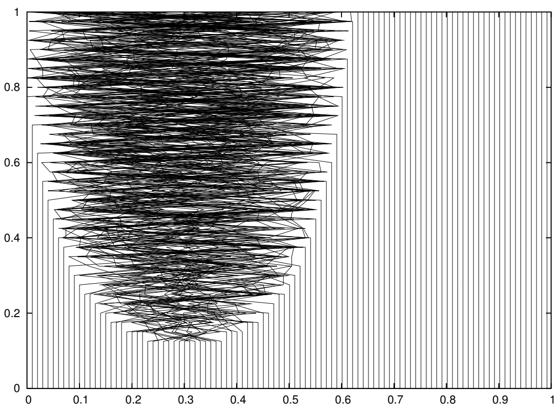

Figure 4 Extension of the rearrangement area where convection takes place (vertical t and horizontal x)

2.4 Convergence analysis

Theorem 2.1 *As $h \to 0$, the time-discrete scheme has a unique limit $y(t,x)$ in space $C^0(\mathbb{R}_+, L^2(D,\mathbb{R}^d))$ that satisfies the subdifferential inclusion:*

$$G(t,x,y) \in \partial_t y + \partial\Psi[y], \tag{2.1}$$

where $\Psi[y] = 0$ if y is non-decreasing as a function of $x \in D$, and $\Psi[y] = +\infty$ otherwise.

The proof follows from the analysis done in [5], where multidimensional scalar conservation laws (see [10, 20]) have been revisited in an L^2 setting rather than in the usual L^1, BV, setting of Kruzhkov and Volpert. As a matter of fact, as an output of this analysis, it turns out that the pseudo-inverse $x = u(t,y)$ of the temperature profile $y = y(t,x)$, precisely defined by

$$u(t,y') = \int_0^1 1\{y(t,x) < y'\}\mathrm{d}x, \quad y' \in \mathbb{R}, \tag{2.2}$$

is a Kruzhkov entropy solution to the scalar conservation law with source term

$$\partial_t u + \partial_y(G(t,u(t,y),y)) - (\partial_y G)(t,u(t,y),y) = 0. \tag{2.3}$$

2.5 Multidimensional rearrangements with convex potential

Proposition 2.1 admits a multidimensional generalization.

Theorem 2.2 *Let $p \in [1,+\infty]$ and D be a bounded domain in $\mathbb{R}^d$. For every $z \in L^p(D,\mathbb{R}^d)$, there is a unique $Z = R[z] \in L^p(D,\mathbb{R}^d)$ such that*

$$\int_D f(Z(x))\mathrm{d}x = \int_D f(z(x))\mathrm{d}x$$

holds for all continuous function f such that $|f(y)| \leq 1 + |y|^p$ and for which there is a convex lsc function $p : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ differentiable a.e. in D with $\nabla p(x) = Z(x)$. $R[z]$ is called the rearrangement of z as a map with convex potential.

This is a typical result in optimal transport theory (see [2, 7, 21]).

2.6 Multi-d generalization of the model

Thanks to the multidimensional rearrangement Theorem 2.2, it is straightforward to get a generalization of the model in higher dimension, just by keeping the same definitions and notations, where it is understood that

(1) D is now a smooth bounded domain in $\mathbb{R}^d$;

(2) $y(t,x) \in \mathbb{R}^d$ (generalized temperature) is now a vector field valued in $\mathbb{R}^d$;

(3) the source term $G = G(t,x,y)$ is also valued in $\mathbb{R}^d$ and has bounded derivatives.

We get the following time discrete scheme:

(1) time step $h > 0$, $y(t = nh, x) \sim y_n(x)$, $n = 0, 1, 2, \cdots$;

(2) predictor (heating): $\widetilde{y}_{n+1}(x) = y_n(x) + h\, G(nh, x, y_n(x))$;

(3) corrector (fast convection): $y_{n+1} = R[\widetilde{y}_{n+1}]$ is the unique rearrangement with convex potential $y_{n+1} = \nabla p_{n+1}$.

2.7 Analysis of the scheme

Take a smooth function f. Then

$$\int_D f(y_{n+1}(x))\mathrm{d}x = \int_D f(\widetilde{y}_{n+1}(x))\mathrm{d}x$$

$$= \int_D f(y_n(x) + h\, G(nh, x, y_n(x)))\mathrm{d}x \quad \text{(because } y_{n+1} \text{ is a rearrangement of } \widetilde{y}_{n+1})$$

$$= \int_D f(y_n(x))\mathrm{d}x + h \int_D (\nabla f)(y_n(x)) \cdot G(nh, x, y_n(x))\mathrm{d}x + o(h) \quad \text{(by definition of corrector } \widetilde{y}_{n+1}).$$

From this relation and optimal transport results, it is rather easy to deduce the following theorem.

Theorem 2.3 *As $h \to 0$, the time-discrete scheme has converging subsequences. Each limit y belongs to the space $C^0(\mathbb{R}_+, L^2(D, \mathbb{R}^d))$ and has a convex potential $p(t, \cdot)$ for each $t \geq 0$. In addition,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_D f(y(t,x))\mathrm{d}x = \int_D (\nabla f)(y(t,x)) \cdot G(t, x, y(t,x))\mathrm{d}x \tag{2.4}$$

for all smooth function f such that $|f(x)| \leq 1 + |x|^2$.

See [4] for a detailed proof (in the special case when G does not explicitly depend on t). Notice that formulation (2.4) is "self-consistent" thanks to the rearrangement theorem. Indeed, the knowledge of $f \to \int_D f(y(t,x))\mathrm{d}x$ for all suitable f is sufficient to recover $y(t,x)$ entirely, as a map with convex potential. Also notice that our global existence result does not imply uniqueness except for $d = 1$, where we can use the theory of scalar conservation laws as shown previously.

2.8 Remark on formulation (2.4)

Formulation (2.4) suggests the existence of a "gauge" field $v(t,x)$ such that

$$\partial_t y + (v\cdot\nabla)y = G(t,x,y), \quad \nabla\cdot v = 0, \quad v \parallel \partial D,$$

which, continuously in time, rearranges $y(t,x)$ so that y stays a map with a convex potential at any time. This intuition will be confirmed by the analysis of the generalized Navier-Stokes Boussinesq equations performed in the coming up section.

3 The GNSB Model and Its Singular Limits

3.1 The GNSB equations

Let D be a smooth bounded domain in $\mathbb{R}^3$ in which moves an incompressible fluid of velocity $v(t,x)$ at $x\in D$, $t\geq 0$, subject to the Navier-Stokes equations

$$\epsilon(\partial_t v + (v\cdot\nabla)v) - \nu\Delta v + \alpha v + \nabla p = y, \quad \nabla\cdot v = 0, \tag{3.1}$$

with $\epsilon,\nu>0$, $\alpha\geq 0$ and $v=0$ along ∂D. Physically speaking, ν is a viscosity parameter, α is a friction parameter and ϵ scales the inertia of the fluid. The force field y is subject to the advection equation

$$\partial_t y + (v\cdot\nabla)y = G(t,x,y), \tag{3.2}$$

where G is a given smooth function with bounded derivatives. This model, that we call generalized Navier-Stokes Boussinesq (GNSB) equations, includes the special case when

$$\alpha = 0, \quad G = 0, \quad y \parallel e_3, \quad y = \eta e_3, \quad \eta = \eta(t,x)\in\mathbb{R},$$

namely

$$\begin{aligned}&\epsilon(\partial_t v + (v\cdot\nabla)v) - \Delta v + \nabla p = \eta e_3, \quad \nabla\cdot v = 0,\\ &\partial_t\eta + (v\cdot\nabla)\eta = 0,\end{aligned}$$

which is a classical model for convection theory. Mathematically speaking, global existence of weak solutions in 3D follows from Leray theory (using Diperna-Lions theory in [11] for the advection equation), while global existence of smooth solutions in 2D follows from Hou-Li [13] and Chae [8].

3.2 Three singular limits of the GNSB model

While keeping unchanged the advection equation with source term

$$\partial_t y + (v\cdot\nabla)y = G(t,x,y), \quad \nabla\cdot v = 0$$

and dropping inertia terms (i.e., setting $\epsilon = 0$), we may consider three possible limits:

$$\begin{aligned}\text{Stokes SB}: \epsilon = 0,\ \nu = 1,\ \alpha = 0\ &\Rightarrow\ -\Delta v + \nabla p = y,\\ \text{Darcy DB}: \epsilon = 0,\ \nu = 0,\ \alpha = 1\ &\Rightarrow\ v + \nabla p = y,\\ \text{Hydrostatic HB}: \epsilon = \nu = 0\ &\Rightarrow\ \nabla p = y.\end{aligned}$$

We call these limits "Stokes-Boussinesq" (SB), "Darcy Boussinesq" (DB) and "Hydrostatic Boussinesq" (HB) equations, respectively. We will concentrate our analysis on the last two limits.

3.3 A remarkable identity

A remarkable feature of all these singular limits is the following identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D f(y(t,x))\mathrm{d}x = \int_D (\nabla f)(y(t,x)) \cdot G(t,x,y(t,x))\mathrm{d}x \tag{3.3}$$

for all suitable smooth function f. Indeed, this just follows from equation (3.2), using that the velocity field v is divergence free and parallel to the boundary ∂D. This property can be rigorously derived for the GNSB model and does not depend on the parameters ϵ, α and ν. Of course (3.3) is not sufficient to recover the solutions of the different models, with the remarkable exception of the last "hydrostatic" model (HB), as will be discussed in Section 4. Also notice that (3.3) exactly coincides with the formulation (2.4) obtained in Section 2 for the multidimensional version of our "toy-model".

4 Analysis of the HB System

We are going to show that the convexity of the pressure field $p(t,x)$ with respect to the space variable x is a key condition for the HB system. Under this condition, global (weak) solutions will be shown to exist for all suitable initial conditions, and strong solutions (with strongly convex pressure) of the HB system will be shown to be limits of the Leray solutions of the GNSB equations as the parameters (ϵ,α,ν) vanish.

4.1 Remark on the solvability of the HB system

As a matter of fact, the convexity of p, which a priori looks as an artificial condition, is a very natural solvability condition for the HB system:

$$\partial_t y + (v\cdot\nabla)y = G(t,x,y), \quad \nabla\cdot v = 0, \quad \nabla p = y.$$

This system looks strange since there is no direct equation for v. Let us consider, for simplicity, the case of 2 space variables $x=(x_1,x_2)$ and write $v=(-\partial_1\psi,\partial_2\psi)$, where $\psi(t,x_1,x_2)\in\mathbb{R}$ is the so-called "stream" function. Now, taking the (2D) curl of the evolution equation in $y=(\partial_1 p,\partial_2 p)$, we get

$$\partial_{11}p\,\partial_{22}\psi + \partial_{22}p\,\partial_{11}\psi - 2\partial_{12}p\,\partial_{12}\psi = \partial_1(G_2) - \partial_2(G_1),$$

which is a well-posed linear elliptic equation in ψ whenever $D_x^2 p(t,x)$ is a field of positive matrices uniformly bounded away from 0 and $+\infty$. Thus, a natural solvability condition for the HB system is the uniform strict convexity of the pressure field $p(t,x)$ with respect to x. This analysis is still correct in higher dimension $d>2$. (In higher dimension, v should be viewed as a $d-1$ form and p as a zero form. The divergence free condition (locally) means that

$v = \mathrm{d}A$, where A is a $d-2$ form. Then, curling again the evolution equation for y, we get a linear elliptic system for A: $d(M(t,x) * \mathrm{d}A) = d(G(x, \mathrm{d}p))$, where $*$ denotes Hodge duality and $M = D_x^2 p(t,x)$.)

We conclude this subsection by saying that requiring p to be convex is a natural solvability condition for the HB system.

4.2 Global solutions to the HB system

In the HB model, the field y is required to be a gradient $y = \nabla p$. If we a priori assume that $p(t,x)$ is the restriction of a convex function of $x \in \mathbb{R}^d$, then the field y is completely determined by the knowledge of all observables

$$t \to \rho_f(t) = \int_D f(y(t,x))\mathrm{d}x$$

because of the multidimensional rearrangement Theorem 2.2. Then (3.3) is just enough to determine a solution to the HB system under the convexity assumption. With this formulation, the HB system coincides with the multi-d toy model discussed in Section 2, since (2.4) and (3.3) are identical. So we immediately get the following global existence result from Theorem 2.3.

Theorem 4.1 *Let G be a smooth function with bounded first derivatives, and C the convex cone of all maps $y \in L^2(D, \mathbb{R}^3)$ such that $y(x) = \nabla p(x)$ a.e. in D for some convex lsc $p : \mathbb{R}^d \to \mathbb{R} \cup +\infty$. We say that $(t \to y(t,\cdot)) \in C^0(\mathbb{R}_+, L^2(D, \mathbb{R}^3))$ valued in the cone C is a solution to the HB system, if*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D f(y(t,x))\mathrm{d}x = \int_D (\nabla f)(y(t,x)) \cdot G(t,x,y(t,x))\mathrm{d}x$$

for all smooth function f such that $|f(x)| \leq 1 + |x|^2$. Then, for each $y_0 \in C$, there is always a global solution y such that $y(t=0,\cdot) = y_0$.

4.3 Rigorous derivation of the HB model

As long as the HB system admits solutions with strictly uniform pressure, these solutions are indeed limits of the GNSB equations as the parameters ϵ, α and ν vanish.

Theorem 4.2 *Let $(y = \nabla p, v)$ be a smooth solution of the HB hydrostatic Boussinesq model with $p(t,x)$ strongly convex in $x \in D$. Then y is obtained as the limit of the Leray solutions (y', v') to the full GNSB Navier-Stokes Boussinesq equations with same initial conditions as the parameters (ϵ, ν, α) vanish.*

The proof is based on a "relative entropy" type method (see [10]) quite similar to the analysis performed for the hydrostatic limit of the Euler equations by the author in [3]. More precisely, we obtain a Gronwall type estimates for the following functional:

$$\int_D \Big\{K(t, y'(t,x), y(t,x)) + \frac{\epsilon}{2}|v'|^2\Big\}\mathrm{d}x,$$

where

$$K(t,y',y) = p^*(t,y') - p^*(t,y) - \nabla p^*(t,y)\cdot(y'-y) \sim |y-y'|^2$$

and

$$p^*(t,z) = \sup_{x\in D}(x\cdot z - p(t,x))$$

is the Legendre-Fenchel transform of p.

The details can be found in [6] (with application to the rigorous derivation of Hoskins' "x-z" semi-geostrophic equations). Notice that the convergence is proven under a strong convexity assumption which is not realistic in the large, for global solutions of the HB system, once singularities have developed. The analysis beyond singularities is widely open.

5 Analysis of the DB Equations

In this section, we consider the DB model, where the friction term is still acting. The source term G does not play a crucial role in the DB equations, in opposition with the HB model. Let us ignore it for the sake of simplicity and consider the homogeneous DB model:

$$v + \nabla p = y, \quad \nabla\cdot v = 0, \quad v \parallel \partial D, \tag{5.1}$$

$$\partial_t y + (v\cdot\nabla)y = 0. \tag{5.2}$$

This system can be written in short:

$$\partial_t y + (\Pi[y]\cdot\nabla)y = 0, \tag{5.3}$$

where Π denotes the Helmholtz-Leray projector on divergence free fields, parallel to ∂D. Under this form, the DB system can be seen as a pseudo-differential version of the inviscid Burgers equation (see [9]) and has been discussed in different settings. It corresponds in particular to the Angenent-Haker-Tannenbaum model for optimal transportation in [1] (see [4] for a discussion). Existence of local smooth solutions does not make difficulties. However, the existence of global weak solutions is a challenging issue. In this paper, we are going to introduce a relaxed version of the DB equations which seems to us well suited for a future global analysis.

5.1 A relaxed version of the DB equations

For notational simplicity, we normalize the volume of D (i.e., its d-dimensional Lebesgue measure) to be 1. To any sufficiently smooth solution of the DB equations, we associate a time-dependent probability measure $c(t,\mathrm{d}x,\mathrm{d}a)$ defined on the product space $D^2 = D\times D$ by

$$\int_{D^2} f(x,a)c(t,\mathrm{d}x,\mathrm{d}a) = \int_D f(X(t,a),a)\mathrm{d}a \tag{5.4}$$

for all continuous functions f on $\mathbb{R}^{2d}$, where $X(t,a)$ denotes the position at time t of a fluid parcel initially located at point $a\in D$. By definition,

$$\int_{D^2} f(a)c(t,\mathrm{d}x,\mathrm{d}a) = \int_D f(a)\mathrm{d}a \tag{5.5}$$

for all continuous functions f. Since $X(t,\cdot)$ is a time-dependent volume preserving map of D, we also have

$$\int_{D^2} f(x)c(t,\mathrm{d}x,\mathrm{d}a) = \int_D f(X(t,a))\mathrm{d}a = \int_D f(x)\mathrm{d}x \tag{5.6}$$

for all continuous function f. Thus, $c(t,\cdot,\cdot)$ admits the Lebesgue measure as its projection ("margin") on each factor of D^2, which can be written in short:

$$\int_D c(t,\mathrm{d}x,a) = 1, \tag{5.7}$$

$$\int_D c(t,x,\mathrm{d}a) = 1. \tag{5.8}$$

By differentiating (5.4) with respect to t, we get, for any smooth function f,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{D^2} f(x,a)c(t,\mathrm{d}x,\mathrm{d}a)\\
&= \int_D (\nabla_x f)(X(t,a),a)\cdot\partial_t X(t,a)\mathrm{d}a\\
&= \int_D (\nabla_x f)(X(t,a),a)\cdot v(t,X(t,a))\mathrm{d}a\\
&= \int_{D^2} \nabla_x f(x,a)\cdot v(t,x)c(t,\mathrm{d}x,\mathrm{d}a) \quad \text{(using } \partial_t X(t,a) = v(t,X(t,a)) \text{ and definition (5.4) of } c)\\
&= \int_{D^2} \nabla_x f(x,a)\cdot (y(t,x) - \nabla_x p(t,x))c(t,\mathrm{d}x,\mathrm{d}a) \quad \text{(because of the first (5.1) DB equation)}\\
&= \int_{D^2} \nabla_x f(x,a)\cdot (y(t,X(t,a)) - \nabla_x p(t,x))c(t,\mathrm{d}x,\mathrm{d}a) \quad \text{(by definition (5.4) of } c)\\
&= \int_{D^2} \nabla_x f(x,a)\cdot (y(0,a) - \nabla_x p(t,x))c(t,\mathrm{d}x,\mathrm{d}a)
\end{aligned}$$

(because of the second DB equation (5.2) which asserts that y is advected by v and implies $y(t,X(t,a)) = y(0,a)$ for all $a\in D$).

This holds true for all smooth functions f, which means, in weak sense,

$$\partial_t c + \nabla_x\cdot((y(0,a) - \nabla_x p(t,x))c) = 0. \tag{5.9}$$

Notice that, in the special case when f depends only on x, we get

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{D^2} f(x)c(t,\mathrm{d}x,\mathrm{d}a) \quad \left(\text{since} \int c(t,x,\mathrm{d}a) = 1\right)\\
&= \int_{D^2} \nabla f(x)\cdot(y(0,a) - \nabla_x p(t,x))c(t,\mathrm{d}x,\mathrm{d}a)\\
&= \int_{D^2} \nabla f(x)\cdot y(0,a)c(t,\mathrm{d}x,\mathrm{d}a) - \int_D \nabla f(x)\cdot\nabla_x p(t,x)\mathrm{d}x \quad \text{(using constraint (5.8)).}
\end{aligned}$$

This just means that $\nabla_x p(t,\cdot)$ is the L^2 projection of the field $\int y(0,a)c(t,x,\mathrm{d}a)$ onto the subspace of all gradient fields, or, in PDE terms, that p solves the elliptic problem

$$-\Delta_x p(t,x) = \nabla_x\cdot\int_D y(0,a)c(t,x,\mathrm{d}a), \tag{5.10}$$

with inhomogeneous Neumann boundary condition along D:

$$\partial_n p(t,x) = n \cdot \int_D y(0,a)c(t,x,\mathrm{d}a), \tag{5.11}$$

where n denotes the outward normal vector along ∂D. From now on, let us call relaxed DB (RDB) equations the coupled equations (5.9)–(5.10). The RDB equations are of interest because they make sense in a much wider framework than the original DB (or NSB) equations. Indeed, in formulations (5.9)–(5.10), the time-dependent probability measure c does not need to be as singular as the one given by definition (5.4). In particular, initial conditions $c(t=0,\,\cdot\,,\,\cdot\,)$ that are absolutely continuous with respect to the Lebesgue measure on D^2 are perfectly suitable and their density can be assumed to be smooth as a function of $(x,a) \in D^2$. Also notice that the pairing between $\nabla_x p$ and c in the RDB equation (5.9) is well-defined as well as $\nabla_x p(t,\,\cdot\,)$ is an L^1 function. (Indeed, thanks to constraint (5.8), this is enough to make $\nabla_x p(t,x)c(t,\mathrm{d}x,\mathrm{d}a)$ a well-defined vector-valued measure on D^2.) This is a mild condition, automatically satisfied if we assume, for instance, that $y(0,\,\cdot\,)$ is an L^∞ function. Indeed,

$$\Big| \int_D y(0,a)c(t,x,\mathrm{d}a) \Big| \le \|y(0,\,\cdot\,)\|_{L^\infty}$$

follows from constraint (5.8), for Lebesgue a.e. in $x \in D$ and all t. Therefore, the vector field $\int y(0,a)c(t,x,\mathrm{d}a)$ is automatically L^2, as well as its L^2 projection $\nabla_x p(t,\,\cdot\,)$.

Let us summarize this discussion with a proposition.

Proposition 5.1 *The relaxed version RDB* (5.9)–(5.11) *of the Darcy-Boussinesq equations makes full sense when, y is in L^∞ and $c(t,\,\cdot\,,\,\cdot\,)$ is, for each t, a probability measure on D^2 with the Lebesgue measure as projection on each factor of D^2.*

However, the global existence of weak solutions for the RDB equations is not clear and is an interesting issue.

5.2 Formal conservation of the Boltzmann entropy

In the relaxed regime of the RDB equations, let us assume that c is absolutely continuous with respect to the Lebesgue measure and can be thought as a function $c(t,x,a)$. Let us consider the “Boltzmann entropy”

$$\eta[c(t,x,\,\cdot\,)] = \int_D (\ln c(t,x,a) - 1)c(t,x,a)\mathrm{d}a \in [0,+\infty],$$

and try to get an evolution equation for it. Formally, we get

$$\begin{aligned}
&\partial_t \eta[c(t,x,\,\cdot\,)] \\
&= \int_D \ln c(t,x,a)\partial_c(t,x,a)\mathrm{d}a \\
&= -\int_D \ln c(t,x,a)\nabla_x \cdot [(y(0,a) - \nabla_x p(t,x))c(t,x,a)]\mathrm{d}a
\end{aligned}$$

$$= -\nabla_x \cdot \int_D \ln c(t,x,a)(y(0,a) - \nabla_x p(t,x))c(t,x,a)\mathrm{d}a$$
$$+ \int_D \nabla_x c(t,x,a) \cdot (y(0,a) - \nabla_x p(t,x))\mathrm{d}a$$
$$= -\nabla_x \cdot \int_D [\ln c(t,x,a)(y(0,a) - \nabla_x p(t,x)) - y(0,a)]c(t,x,a)\mathrm{d}a \quad \text{(using constraint (5.8))},$$

Thus, we conclude that the following proposition holds.

Proposition 5.2 *Let* (c,p) *be a sufficiently smooth solution of the relaxed version* RDB (5.9)–(5.11). *Then there is an additional conservation law for the Boltzmann entropy, namely*

$$\partial_t \int_D (\ln c - 1)c\,\mathrm{d}a + \nabla_x \cdot \int_D [(y(0,a) - \nabla_x p(t,x)) \ln c - y(0,a)]c\,\mathrm{d}a = 0. \tag{5.12}$$

5.3 One-dimensional and discrete versions of the RDB equations

Remarkably, the relaxed version (5.9)–(5.10) of the Darcy-Boussinesq is not trivial in the one-dimensional case $d = 1$. Then the pressure gradient is directly given by

$$\partial_x p(t,x) = \int_D y(0,a)c(t,x,\mathrm{d}a),$$

and the RDB equations can be compactly written as

$$\partial_t c(t,x,a) + \partial_x \int_D (y(0,a) - y(0,b))c(t,x,a)c(t,x,\mathrm{d}b) = 0. \tag{5.13}$$

Notice that constraint (5.8) is automatically propagated and does not need to be enforced, but at time $t = 0$. Furthermore, the RDB equations still make sense when the variable a is discrete, ranging from 1 to N, with discrete sums substituting for Lebesgue integrals, since no differential operator is acting on the variable a. Then a can be interpreted as a phase index, and the resulting RDB equations are similar to a multiphase Darcy model, quite similar to the one used, for instance, in oil recovery models. Combining one space dimension for x and discreteness for a, we reduce the RDB equations to an $N \times N$ system of first order conservation laws, namely

$$\partial_t c_a(t,x) + \partial_x \sum_{a=1}^{N} (y_a - y_b)c_a(t,x)c_b(t,x) = 0, \quad a = 1, \cdots, N, \tag{5.14}$$

with obvious change of notations. This system is automatically hyperbolic in the range $c_a(t,x) > 0$. Indeed, from Proposition 5.2, we immediately get an additional conservation law involving a strictly convex function of c, namely the Boltzmann entropy

$$\eta[c(t,x,\cdot)] = \sum_{a=1}^{N} (\ln c_a(t,x) - 1)c_a(t,x),$$

which is enough to enforce hyperbolicity (and therefore, local well-posedness for smooth initial conditions $c > 0$ (see [10, 16, 20])). The global existence of weak solutions in this much simpler situation seems a much more tractable problem than in the general case. As a matter of fact, very similar systems have been analyzed in the framework of chromatography or electrophoresis, successfully thanks to compensated compactness and kinetic tools, in particular by François James et al [14].

5.4 Inviscid Burgers and Le Roux equations

Not so surprisingly, in the case of two "phases", namely $N = 2$, the discrete RDB equations reduce to the inviscid Burgers equation. Indeed, denoting $c(t,x) = c_1(t,x)$ and $q = y_2 - y_1$, we immediately get from (5.14) that

$$\partial_t c(t,x) + q\partial_x(c(1-c)) = 0. \tag{5.15}$$

In this ultimate model, we see how the classical "entropy condition" (which goes back to Oleinik, Kruzhkov, etc. see [10, 16, 20]) is precisely the right selection principle to get convectly stable solutions! Notice that it is not surprising to get this equation. As a matter of fact, it occurs in the most classical models for two-phase flows in porous media, for use, in particular, in oil recovery engineering. An interesting case is the three phase model when $y_1 = 0$, $y_2 = q$ and $y_3 = -q$, where q is a given constant. Then, we get, for $c = c_1$ and $w = (c_3 - c_2)$, the following system

$$\partial_t c + \partial_x(cw) = 0, \quad \partial_t w + \partial_x\Big(\frac{w^2}{2} + cq^2\Big) = 0, \tag{5.16}$$

which is known in the hyperbolic conservation law literature as the Le Roux system (see [20]) and has been rigorously derived from the lattice fluid models (see [12]).

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References

[1] Angenent, S., Haker, S. and Tannenbaum, A., Minimizing flows for the Monge-Kantorovich problem, *SIAM J. Math. Anal.*, **35**, 2003, 61–97.

[2] Brenier, Y., Polar factorization and monotone rearrangement of vector-valued functions, *Comm. Pure Appl. Math.*, **44**, 1991, 375–417.

[3] Brenier, Y., Remarks on the derivation of the hydrostatic Euler equations, *Bull. Sci. Math.*, **127**, 2003, 585–595.

[4] Brenier, Y., Optimal transport, convection, magnetic relaxation and generalized Boussinesq equations, *J. Nonlinear Sci.*, to appear.

[5] Brenier, Y., L2 formulation of multidimensional scalar conservation laws, *Arch. Rational Mech. Anal.*, to appear. arXiv:math/0609761v1

[6] Brenier, Y. and Cullen, M., Rigorous derivation of the *x-z* semigeostrophic equations, *CMS*, to appear.

[7] Caffarelli, L. A., Boundary regularity of maps with convex potentials, *Comm. Pure Appl. Math.*, **45**, 1992, 1141–1151.

[8] Chae, D., Global regularity for the 2D Boussinesq equations with partial viscosity terms, *Adv. Math.*, **203**, 2006, 497–513.

[9] Constantin, P., Lax, P. D. and Majda, A., A simple one-dimensional model for the three-dimensional vorticity equation, *Comm. Pure Appl. Math.*, **38**, 1985, 715–724.

[10] Dafermos, C. M., Hyperbolic Conservation Laws in Continuum Physics, Grundlehren der Mathematischen Wissenschaften, **325**, Springer-Verlag, Berlin, 2000.

[11] DiPerna, R. J. and Lions, P. L., Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.*, **98**, 1989, 511–547.

[12] Fritz, J. and Tóth, B., Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas, *Commun. Math. Phys.*, **249**, 2004, 1–27.

[13] Hou, T. Y. and Li, C. M., Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dyn. Sys.*, **12**, 2005, 1–12.

[14] James, F., Peng, Y.-J. and Perthame, B., Kinetic formulation for chromatography and some other hyperbolic systems, *J. Math. Pures Appl.*, **74**, 1995, 367–385.

[15] Lieb, E. and Loss, M., Analysis, Graduate Studies in Mathematics, **14**, A. M. S., Providence, RI, 2001.

[16] Majda, A., Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Applied Mathematical Sciences, **53**, Springer-Verlag, New York, 1984.

[17] Majda, A., Introduction to PDEs and Waves for the Atmosphere and Ocean, Courant Lecture Notes, **9**, A. M. S., Providence, RI, 2003.

[18] Majda, A. J. and Wang, X. M., Non-linear Dynamics and Statistical Theories for Basic Geophysical Flows, Cambridge University Press, Cambridge, 2006.

[19] Pedlosky, J., Geophysical Fluid Dynamics, Springer-Verlag, New York, 1979.

[20] Serre, D., Systems of Conservation Laws, Vol. 1–2, Cambridge University Press, Cambridge, 2000.

[21] Villani, C., Topics in Optimal Transportation, Graduate Studies in Mathematics, **58**, A. M. S., Providence, RI, 2003.