Chin. Ann. Math.
30B(6), 2009, 697–714
DOI: 10.1007/s11401-009-0205-3

Chinese Annals of Mathematics, Series B

Global Attractors and Determining Modes for the 3D Navier-Stokes-Voight Equations***

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(*Dedicated to Professor Andrew Majda on the Occasion of his* 60*th Birthday with Friendship and Admiration*)

Abstract The authors investigate the long-term dynamics of the three-dimensional Navier-Stokes-Voight model of viscoelastic incompressible fluid. Specifically, upper bounds for the number of determining modes are derived for the 3D Navier-Stokes-Voight equations and for the dimension of a global attractor of a semigroup generated by these equations. Viewed from the numerical analysis point of view the authors consider the Navier-Stokes-Voight model as a non-viscous (inviscid) regularization of the three-dimensional Navier-Stokes equations. Furthermore, it is also shown that the weak solutions of the Navier-Stokes-Voight equations converge, in the appropriate norm, to the weak solutions of the inviscid simplified Bardina model, as the viscosity coefficient $\nu \to 0$.

Keywords Navier-Stokes-Voight, Navier-Stokes-Voigt, Global attractor, Determining modes, Regularization of the Navier-Stokes, Turbulence models, Viscoelastic models

2000 MR Subject Classification 37L30, 35Q35, 35Q30, 35B40

1 Introduction

We consider the three-dimensional Navier-Stokes-Voight (or Navier-Stokes-Voigt) (NSV) model that governs the motion of a Kelvin-Voight linear viscoelastic incompressible fluid:

$$v_t - \nu\Delta v - \alpha^2\Delta v_t + (v\cdot\nabla)v + \nabla p = f(x), \quad x\in\Omega,\ t\in\mathbb{R}^+, \tag{1.1}$$

$$\begin{aligned} &\operatorname{div} v = 0, && x\in\Omega,\ t\in\mathbb{R}^+,\\ &v(x,t) = 0, && x\in\partial\Omega,\ t\in\mathbb{R}^+, \end{aligned} \tag{1.2}$$

$$v(x,0) = v_0(x), \quad x\in\Omega \tag{1.3}$$

in a bounded domain $\Omega\subset\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. Here $v = v(x,t)$ is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity, and f is a given force

Manuscript received June 12, 2009. Published online September 25, 2009.
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***Project supported by the Scientific and Research Council of Turkey (No. 106T337), the ISF Grant (No. 120/6), the BSF Grant (No. 2004271), and the National Science Foundation (Nos. DMS-0504619, DMS-0708832).

field. The length scale α is a characterizing parameter of the elasticity of the fluid in the sense that $\frac{\alpha^2}{\nu}$ is a characteristic relaxation time scale of the viscoelastic material.

The system (1.1)–(1.2) was introduced and studied by Oskolkov in [41]. It is interesting to observe that the inviscid version of the NSV equations (1.1)–(1.3), i.e., when $\nu = 0$, coincides with the simplified Bardina model of turbulent flows. The viscous simplified Bardina model was introduced, and studied, in [36] (see also [4]) as a simplified version of the Bardina subgrid scale model of turbulence (see [3]). In [5] the viscous and inviscid simplified Bardina model were shown to be globally well-posed. Viewed from the numerical analysis point of view the authors of [5] proposed the inviscid simplified Bardina model (or equivalently the inviscid NSV equations) as a non-viscous (inviscid) regularization of the 3D Euler equations, subject to periodic boundary conditions. Motivated by this observation, system (1.1)–(1.3) was also proposed in [5] as a regularization, for small values of α, of the 3D Navier-Stokes (NS) equations for the purpose of direct numerical simulations, for both the periodic and the no-slip Dirichlet boundary conditions.

It was shown in [41] that the initial boundary value problem (1.1)–(1.3) has a unique weak solution. Moreover, it was shown in [25] and [26] that the semigroup generated by the problem (1.1)–(1.3) has a finite dimensional global attractor.

In this paper, we give an estimate of the fractal and Hausdorff dimensions of the global attractor of a dynamical system generated by the problem (1.1)–(1.3), which is an improvement of the estimates done in [26]. Moreover, we derive estimates for the number of asymptotic determining modes of the solutions of the problem (1.1)–(1.3). We also show that there exists a number m such that each trajectory $v(t)$ on the global attractor of the dynamical system generated by this problem is uniquely determined by its projection $P_m v(t)$ onto the span$\{w_1, \cdots, w_m\}$ of the first m eigenfunctions of the Stokes operator. This observation is related to the notion of continuous data assimilations as it has been presented in [30, 39, 40].

It is worth stressing that by adding the regularizing term $(-\alpha\Delta v_t)$ to the NS equations, system (1.1)–(1.3) changes its parabolic character. In particular, the 3D system (1.1)–(1.3) is globally well-posed forward and backwards in time. The semigroup generated by problem (1.1)–(1.3) is only asymptotically compact. In this sense the system is similar to damped hyperbolic systems. We also remark that this type of inviscid regularization has been recently used for the two-dimensional surface quasi-geostropic model (see [29]). In particular, necessary and sufficient conditions for the formation of singularity were presented in terms of regularizing parameter. Similar criterion for the formation of singularity in the 3D Euler equations of inviscid incompressible flows is also reported in [35].

In addition, it was shown in [27] that the global attractor of the 3D NSV equations, driven by an analytic forcing, consists of analytic functions. As a consequence, the spectrum of the solutions of the 3D NSV system, lying on the global attractor, have exponentially decaying tail, despite the fact that the equations behave like a damped hyperbolic system, rather than the parabolic one. This result provides an additional evidence that the 3D NSV, with the small regularization parameter α, enjoys similar statistical properties as the 3D NS equations. These statistical properties were investigated further, both analytically and computationally, in [37]. In addition, the existence of probability invariant measures associated with the dynamics of the 3D NSV have been established in [44]. Moreover, the limiting behavior of these measures, as $\alpha \to 0$, and its relation to the notion of stationary statistical solutions of the 3D NS (see, e.g., [16] for details) have also been investigated in [44].

2 Preliminary

In this paper, we will use the following standard notations in the mathematical theory of NS equations:

$L^p(\Omega)$ $(1 \le p \le \infty)$ and $H^s(\Omega)$ are the usual Lebesgue and Sobolev spaces, respectively;

For $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$, we denote by

$$(u, v) = \sum_{j=1}^{3} (v_j, u_j)_{L^2(\Omega)}, \quad \|v\|^2 = \sum_{j=1}^{3} \|v_i\|^2_{L^2(\Omega)}, \quad \|\nabla v\|^2 := \sum_{j,i=1}^{3} \|\partial_i v_j\|^2_{L^2(\Omega)};$$

We set

$$\mathcal{V} := \{v \in (C_0^\infty(\Omega))^3 : \nabla \cdot v = 0\};$$

H is the closure of the set $\mathcal{V}$ in $(L_2(\Omega))^3$ topology;

P is the Helmholz-Leray orthogonal projection in $(L^2(\Omega))^3$ onto the space H, and $h := Pf$;

$A := -P\Delta$ is the Stokes operator subject to the no-slip homogeneous Dirichlet boundary condition with the domain $(H^2(\Omega))^3 \cap V$. The operator A is a self-adjoint positively definite operator in H, whose inverse A^{-1} is a compact operator from H into H. Thus it has an orthonormal system of eigenfunctions $\{w_j\}_{j=1}^\infty$ of A;

We denote by $\{\lambda_j\}_{j=1}^\infty$ $(0 < \lambda_1 \le \lambda_2 \le \cdots)$ the eigenvalues of the Stokes operator A corresponding to eigenfunctions $\{w_j\}_{j=1}^\infty$, repeated according to their multiplicities;

$V_s := D(A^{\frac{s}{2}})$, $\|v\|_s := \|A^{\frac{s}{2}} v\|$, $s \in \mathbb{R}$. $V := V_1 = (H_0^1(\Omega))^3 \cap H$ is the Hilbert space with the norm $\|v\|_1 = \|u\|_V = \|\nabla u\|$, thanks to the Poincaré inequality (2.3). Clearly $V_0 = H$;

For $u, v, w \in \mathcal{V}$, we define the following bilinear form

$$B(u, v) := P((u \cdot \nabla)v) \text{ and the trilinear form } b(u, v, w) = (B(u, v), w).$$

The bilinear form $B(\,\cdot\,, \cdot\,)$ can be extended as a continuous operator $B : V \times V \to V'$, where V' is the dual of V (see, e.g., [11]);

For each $u, v, w \in V$,

$$b(u, v, v) = 0 \quad \text{and} \quad b(u, v, w) = -b(u, w, v). \tag{2.1}$$

Next we formulate some well-known inequalities and a Gronwall type lemma that we will be using in what follows.

Young's inequality:

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{q\varepsilon^{\frac{1}{p-1}}} b^q \quad \text{for all } a, b, \varepsilon > 0, \text{ with } q = \frac{p}{p-1}, \ 1 < p < \infty. \tag{2.2}$$

Poincaré inequality:

$$\|u\| \le \lambda_1^{-\frac{1}{2}} \|u\|_1, \quad \forall\, u \in V, \tag{2.3}$$

where λ_1 is the first eigenvalue of the Stokes operator under the homogeneous Dirichlet boundary condition. Hereafter, C will denote a dimensionless scale invariant constant which might depend on the shape of the domain Ω.

Ladyzhenskaya inequalities (see [11, 32, 34]):

$$\|u\|_{L^3} \le C\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}, \quad \forall\, u \in V, \tag{2.4}$$

$$\|u\|_{L^4} \le C\|u\|^{\frac{1}{4}}\|u\|_1^{\frac{3}{4}}, \qquad \forall\, u \in V. \tag{2.5}$$

Sobolev inequality (see, e.g., [1]):

$$\|u\|_{L^6} \le C\|u\|_1, \quad \forall\, u \in V. \tag{2.6}$$

Gagliardo-Nirenberg inequalities (see, e.g., [2, 11, 34]):

$$\|u\|_{L^{\frac{6}{3-2\varepsilon}}} \le C\|u\|^{1-\varepsilon}\|u\|_1^{\varepsilon}, \quad 0 \le \varepsilon \le 1, \ \forall\, u \in V. \tag{2.7}$$

$$\|u\|_{L^p} \le C\|u\|^{\frac{2}{p}}\|u\|_{\frac{3}{2}}^{1-\frac{2}{p}}, \quad p \in [2,\infty), \ \forall\, u \in V_{\frac{3}{2}}. \tag{2.8}$$

Agmon inequality (see, e.g., [11]):

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_1^{\frac{1}{2}}\|Au\|^{\frac{1}{2}}, \quad \forall\, u \in V_2. \tag{2.9}$$

We will use also the following estimates of the trilinear form $b(u, v, w)$ which follow from (2.4)–(2.9) (see, e.g., [11]):

$$|b(u,v,w)| \le C\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}\|v\|_1\|w\|_1, \quad \forall\, u,v,w \in V, \tag{2.10}$$

$$|b(u,v,u)| \le C\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{3}{2}}\|v\|_1, \qquad \forall\, u,v \in V, \tag{2.11}$$

$$|b(u,v,w)| \le C\|u\|_1\|v\|_1\|w\|^{\frac{1}{2}}\|w\|_1^{\frac{1}{2}}, \quad \forall\, u,v,w \in V, \tag{2.12}$$

$$|b(u,v,w)| \le C\lambda_1^{\frac{1}{4}}\|u\|_1\|v\|_1\|w\|_1, \qquad \forall\, u,v,w \in V. \tag{2.13}$$

Lemma 2.1 (see [15, 23]) *Let $a(t)$ and $b(t)$ be locally integrable functions on $(0,\infty)$ which satisfy, for some $T > 0$, the conditions*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} a(\tau)\mathrm{d}\tau = \gamma, \quad \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} a^-(\tau)\mathrm{d}\tau = \Gamma, \quad \liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} b^+(\tau)\mathrm{d}\tau = 0,$$

where $\gamma > 0$, $\Gamma < \infty$, $a^- = \max\{-a, 0\}$ and $b^+ = \max\{b, 0\}$. If a non-negative, absolutely continuous function $\phi(t)$ satisfies

$$\phi'(t) + a(t)\phi(t) \le b(t), \quad t \in (0,\infty),$$

then $\phi(t) \to 0$ as $t \to \infty$.

Definition 2.1 (see, e.g., [15, 19, 33]) *A semigroup $S(t) : V \to V$, $t \ge 0$ is called asymptotically compact, if for any sequence of positive numbers $t_n \to \infty$ and any bounded sequence $\{v_n\} \subset V$ the sequence $\{S(t_n)v_n\}$ is precompact in V.*

Theorem 2.1 (see, e.g., [19, 33, 47]) *Assume that a semigroup $S(t) : V \to V$ for $t \ge t_0 > 0$ can be decomposed into the form*

$$S(t) = Y(t) + Z(t),$$

where $Z(t)$ is a compact operator in V for each $t \ge t_0 > 0$. Assume also that there is a continuous function $k : [t_0,\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $R > 0$, $k(t,R) \to 0$ as $t \to \infty$ and

$$\|Y(t)v\|_V \le k(t,R) \quad \text{for all } t \ge t_0 > 0 \text{ and all } \|v\|_V \le R.$$

Then $S(t) : V \to V$, $t \ge 0$ is asymptotically compact.

Next we state a result from [33] which will enable us to estimate the dimension of the global attractor for the system (1.1)–(1.3). This result is typically useful in the context of nonlinear damped hyperbolic systems, when the damping term is not strong enough to control the instabilities rising from the perturbed nonlinearity.

Theorem 2.2 (see [13,33]) *Let $S(t)$, $t \in \mathbb{R}^+$ be a semigroup generated by the problem*

$$v_t(t) = \Phi(v(t)), \quad v|_{t=0} = v_0$$

in the phase space H, and let $\mathcal{M} \subset H$ be a compact invariant subset with respect to $S(t)$. Let $S(t)$ and $\Phi(\,\cdot\,)$ be uniformly differentiable on $\mathcal{M}$ and let $L(t, v_0)$ be a differential of Φ at the point $S(t)v_0$, $v_0 \in \mathcal{M}$. Suppose that $L^c(t, v_0) := L(t, v_0) + L^*(t, v_0)$, $v_0 \in \mathcal{M}$ satisfies the inequality

$$(L^c(t)u, u) \le -h_0(t)\|u\|^2 + \sum_{k=1}^{m} h_{s_k}(t)\|u\|_{s_k}^2 \tag{2.14}$$

for some numbers $s_k < 0$ $(k = 1, \cdots, m)$ and some functions $h_0, h_{s_k} \in L_{1,\mathrm{loc}}(\mathbb{R})$, $h_{s_k}(t) \ge 0$, $h_0(t) \ge 0$ for all $t \in \mathbb{R}^+$. Then

$$\dim_{\mathcal{H}}(\mathcal{M}) \le \dim_f(\mathcal{M}) \le N,$$

where N is such that

$$-\overline{h}_0(T) + \sum_{k=0}^{m} \overline{h}_{s_k}(T) N^{s_k} < 0$$

for some $T > 0$. Here $\overline{h}_i(T) := \frac{1}{T}\int_0^T h_i(\tau)\mathrm{d}\tau$.

3 Existence of Global Attractors

Applying the Helmholtz-Leray projector P to the system (1.1)–(1.2), we obtain the following equivalent functional differential equation

$$v_t + \nu A v + \alpha^2 A v_t + B(v, v) = h, \quad h = Pf, \tag{3.1}$$

$$v(0) = v_0. \tag{3.2}$$

The question of global existence and uniqueness of (3.1)–(3.2) first was studied in [41], where actually it was established that the problem (1.1)–(1.3) generates a continuous semigroup $S(t) : V \to V$, $t \in \mathbb{R}^+$. In [5] the authors proved also the global regularity for inviscid model of (3.1), i.e., when $\nu = 0$.

In this section we show that the semigroup $S(t)$ generated by the problem (1.1)–(1.3) has an absorbing ball in V and an absorbing ball in V_2. Then we show that $S(t) : V \to V$ for $t \in \mathbb{R}^+$ is an asymptotically compact semigroup, and deduce the existence of a global attractor in V.

Let us note that the formal estimates we provide below can be justified rigorously by using a Galerkin approximation procedure and passing to the limit, by using the relevant Aubin's compactness theorem as for the NS equations (see, for example, [11, 15, 45] or [47]).

Absorbing ball in *V* Taking the inner product of (3.1) with v, and noting that $(B(v, v), v) = 0$ due to (2.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2] + 2\nu\|v(t)\|_1^2 \le 2\|h\|_{-1}\|v(t)\|_1. \tag{3.3}$$

It is easy to see by Poincaré inequality (2.3) that

$$\nu\|v(t)\|_1^2 \geq \frac{\nu}{2}[\lambda_1\|v\|^2 + \|v(t)\|_1^2] \geq d_0[\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2],$$

where $d_0 := \frac{\nu}{2}\min\{\frac{1}{\alpha^2}, \lambda_1\} = \nu d_1$. Hence (3.3) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2] + d_0[\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2] \leq \frac{1}{\nu}\|h\|_{-1}^2.$$

By Gronwall's inequality, we have

$$\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2 \leq \mathrm{e}^{-d_0(t-s)}\Big[\|v(s)\|^2 + \alpha^2\|v(s)\|_1^2 - \frac{\|h\|_{-1}^2}{\nu d_0}\Big] + \frac{1}{\nu d_0}\|h\|_{-1}^2. \qquad (\mathrm{E}_1)$$

Therefore

$$\limsup_{t\to\infty}[\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2] \leq \frac{\|h\|_{-1}^2}{\nu d_0}. \qquad (\mathrm{E}_2)$$

The last inequality implies that the semigroup $S(t) : V \to V$, $t \in \mathbb{R}^+$ generated by the problem (1.1)–(1.3) (or equivalently (3.1)–(3.2)) has an absorbing ball

$$\mathcal{B}_1 := \Big\{v \in V : \|v\|_1 \leq \frac{2}{\sqrt{\nu\alpha^2 d_0}}\|h\|_{-1}\Big\}. \qquad (3.4)$$

Hence, the following uniform estimate is valid:

$$\|v(t)\|_1 \leq M_1, \qquad (3.5)$$

where $M_1 = \frac{2}{\nu\alpha\sqrt{d_1}}\|h\|_{-1}$ for t large enough ($t \gg 1$) depending on the initial data.

Asymptotic compactness By using the Galerkin procedure, it is not difficult to prove the following propositions.

Proposition 3.1 *Let $s \in \mathbb{R}$. If $w_0 \in V_s$, $g \in L^2([0,T); V_{s-2})$, then the linear problem*

$$z_t + \alpha^2 A z_t + \nu A z = g(t), \quad z(0) = 0 \qquad (3.6)$$

has a unique weak solution which belongs to $C([0,T); V_s)$ and the following inequality holds:

$$\sup_{t\in[0,T)}\|z(t)\|_s \leq C\|g\|_{L^2(0,T;V_{s-2})}, \quad s \in \mathbb{R}.$$

Proposition 3.2 *Let $h \in H$ be time independent. Then the semigroup $S(t)$, $t \geq 0$ is asymptotically compact semigroup in V.*

Proof Let $v_0 \in V$. First we observe that $S(t)$ has the representation

$$S(t)v_0 = Y(t)v_0 + Z(t)v_0, \qquad (3.7)$$

where $Y(t)$ is the semigroup generated by the linear problem

$$y_t + \nu A y + \alpha^2 A y_t = 0, \quad y(0) = v_0, \qquad (3.8)$$

and $z(t) = Z(t)(v_0)$ is the solution of the problem

$$z_t + \nu A z + \alpha^2 A z_t = h - B(v(t), v(t)), \quad z(0) = 0, \qquad (3.9)$$

where v is the solution of (1.1)–(1.3) (or equivalently (3.1)–(3.2)) with the initial data v_0. Taking the H inner product of (3.8) with y, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|y(t)\|^2 + \alpha^2\|y(t)\|_1^2] + d_0[\|y(t)\|^2 + \alpha^2\|y(t)\|_1^2] \le 0,$$

where we recall that $d_0 = \nu d_1 = \nu \frac{1}{2}\min\{\frac{1}{\alpha^2}, \lambda_1\}$. This inequality implies that

$$\|y(t)\|^2 + \alpha^2\|y(t)\|_1^2 \le \mathrm{e}^{-d_0 t}[\|v_0\|^2 + \alpha^2\|v_0\|_1^2] \quad \text{for all } t > 0. \tag{3.10}$$

So the semigroup $Y(t) : V \to V$ is exponentially contractive.

Due to Hölder's inequality and the Sobolev inequality (2.6), we have

$$\begin{aligned}\|B(v,v)\|_{-\frac{1}{2}} &= \sup_{\substack{\phi\in V\\ \|A^{\frac{1}{4}}\phi\|=1}} b(v,v,\phi) = \sup_{\substack{\phi\in V\\ \|A^{\frac{1}{4}}\phi\|=1}} \int_\Omega P((v\cdot\nabla)v)\cdot\phi\,\mathrm{d}x\\ &= \sup_{\substack{\phi\in V\\ \|A^{\frac{1}{4}}\phi\|=1}} \int_\Omega (v\cdot\nabla)v\cdot P\phi\,\mathrm{d}x\\ &= \sup_{\substack{\phi\in V\\ \|A^{\frac{1}{4}}\phi\|=1}} \int_\Omega (v\cdot\nabla)v\cdot\phi\,\mathrm{d}x \le C \sup_{\substack{\phi\in V\\ \|A^{\frac{1}{4}}\phi\|=1}} \|v\|_{L^6}\|v\|_1\|\phi\|_{L^3}.\end{aligned}$$

Hence due to the Sobolev inequality $\|\phi\|_{L^3} \le C\|A^{\frac{1}{4}}\phi\|$ and (2.6), we have

$$\|B(v,v)\|_{-\frac{1}{2}} \le C \sup_{\substack{\phi\in V\\ \|A^{\frac{1}{4}}\phi\|=1}} \|v\|_1^2\|A^{\frac{1}{4}}\phi\| \le C\|v\|_1^2, \tag{3.11}$$

and

$$B(v,v) \in L^\infty(\mathbb{R}^+; V_{-\frac{1}{2}}).$$

The function $v(t)$ as a solution of the problem (3.1)–(3.2) with $v_0 \in V$ belongs to $L^\infty(\mathbb{R}^+; V)$. Thus due to the inequality (3.11) and Proposition 3.1, the solution of the problem (3.9) belongs to $C(\mathbb{R}^+; V_{\frac{3}{2}})$, that is the operator $Z(t)$ maps V into $V_{\frac{3}{2}}$. Since the embedding $V_{\frac{3}{2}} \subset V$ is a compact embedding, the operator $Z(t)$ is a compact operator for each $t > 0$. Hence, the semigroup $S(t)$ satisfies the conditions of Theorem 2.1, and is an asymptotically compact semigroup. The proof is completed.

Since each bounded dissipative and asymptotically compact semigroup possesses a compact global attractor (see, e.g., [2, 19, 32, 47]), we have

Theorem 3.1 *If $h \in H$, then the semigroup $S(t) : V \to V$ has an absorbing ball $\mathcal{B}_1 = \{v \in V : \|v\|_1 \le M_1\}$ and a global attractor $\mathcal{A}_1 \subset V$. The attractor $\mathcal{A}_1$ is compact, connected and invariant.*

Next we show that the global attractor $\mathcal{A}_1$ is a bounded subset of V_2.

Taking the inner product in $V_{\frac{1}{2}}$ of the equation (3.9) with z, and remembering that $v(t) = y(t) + z(t) \in \mathcal{A}_1$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|z(t)\|_{\frac{1}{2}}^2 + \alpha^2\|z(t)\|_{\frac{3}{2}}^2] + 2\nu\|z(t)\|_{\frac{3}{2}}^2 = 2(h, z(t))_{\frac{1}{2}} - 2(B(v(t),v(t)), z(t))_{\frac{1}{2}}. \tag{3.12}$$

The first term on the right-hand side has the estimate

$$|2(h,z(t))_{\frac{1}{2}}| \le 2\|h\|_{-\frac{1}{2}}\|z(t)\|_{\frac{3}{2}} \le \frac{\nu}{2}\|z(t)\|_{\frac{3}{2}}^2 + \frac{2}{\nu}\|h\|_{-\frac{1}{2}}^2.$$

The second term, due to (3.11), has the following estimate

$$\begin{aligned}|2(B(v(t),v(t)),z(t))_{\frac{1}{2}}| &\le C\|B(v(t),v(t))\|_{-\frac{1}{2}}\|z(t)\|_{\frac{3}{2}}\\ &\le \frac{\nu}{2}\|z(t)\|_{\frac{3}{2}}^2 + \frac{C}{\nu}\|B(v(t),v(t))\|_{-\frac{1}{2}}^2 \le \frac{\nu}{2}\|z(t)\|_{\frac{3}{2}}^2 + \frac{C}{\nu}\|v\|_1^4.\end{aligned}$$

Taking into account the last two inequalities in (3.12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|z(t)\|_{\frac{1}{2}}^2 + \alpha^2\|z(t)\|_{\frac{3}{2}}^2] + 2d_0[\|z(t)\|_{\frac{1}{2}}^2 + \alpha^2\|z(t)\|_{\frac{3}{2}}^2] \le \frac{C}{\nu}(\|v(t)\|_1^4 + \|h\|_{-\frac{1}{2}}^2).$$

Integrating the last inequality, we obtain the estimate

$$\|z(t)\|_{\frac{3}{2}}^2 \le \frac{C}{d_0\alpha^2\nu}(M_1^4 + \|h\|_{-\frac{1}{2}}^2) = L_0. \tag{3.13}$$

Since the attractor $\mathcal{A}_1$ is invariant, $S(t)\mathcal{A}_1 = \mathcal{A}_1$, and due to (3.10) the inequality

$$\|v(t) - z(t)\|_1 = \|y(t)\|_1 \le C(\|y(0)\|_1)\mathrm{e}^{-d_0 t}$$

holds, we deduce that for each $u \in \mathcal{A}_1$ there exists a sequence $\{z(t_k)\}$, $t_k \to \infty$, corresponding to $v_k(0) \in \mathcal{A}_1$, such that

$$u = \lim_{k\to\infty} z(t_k), \quad v_k(0) \in \mathcal{A}_1. \tag{3.14}$$

Thanks to (3.13) the sequence $\{z(t_k)\}$ is belonging to a ball in $V_{\frac{3}{2}}$, whose radius L_0 depends only on M_1 and $\|h\|$. Hence, the sequence $\{z(t_k)\}$ is weakly compact in $V_{\frac{3}{2}}$. Thus, by using (3.14) and the inequality $\|u\|_{\frac{3}{2}} \le \liminf_{t_k\to\infty}\|z(t_k)\|_{\frac{3}{2}}$, we see that $\mathcal{A}_1$ is bounded in $V_{\frac{3}{2}}$. Knowing that $\mathcal{A}_1$ is bounded in $V_{\frac{3}{2}}$, we can use similar arguments to show that $\mathcal{A}_1$ is also bounded in $V_{\frac{5}{3}}$ and in V_2.

***V_2* absorbing ball** To show that the semigroup $S(t): V_2 \to V_2$ has an absorbing ball in the phase space $V_2 = D(A)$, we take H inner product of (3.1) with $Av(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|v(t)\|_1^2 + \alpha^2\|Av(t)\|^2] + 2\nu\|Av(t)\|^2 + 2(B(v(t),v(t)),Av(t)) = 2(h,Av(t)). \tag{3.15}$$

For the first term in the right-hand side of (3.15), we have

$$|2(h,Av(t))| \le \frac{1}{\nu}\|h\|^2 + \nu\|Av(t)\|^2. \tag{3.16}$$

By using the Agmon's inequality (2.9) and Young's inequality (2.2) with $p = \frac{4}{3}$, we can estimate the last term in the left-hand side of (3.15) as follows

$$2|(B(v,v),Av)| \le C\|v\|_{L^\infty(\Omega)}\|\|v\|_1\|\|Av\| \le C\|v\|_1^{\frac{3}{2}}\|\|Av\|^{\frac{3}{2}} \le \frac{3}{4}\epsilon\|Av\|^2 + \frac{C}{\epsilon^3}\|v\|_1^6.$$

Employing (3.16) and the last inequality with $\epsilon = \frac{2\nu}{3}$, from (3.15) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|v(t)\|_1^2 + \alpha^2\|Av(t)\|^2] + \nu\|Av(t)\|^2 \le \frac{1}{\nu}\|h\|^2 + \frac{C}{\nu^3}\|v(t)\|_1^6. \tag{3.17}$$

It follows from (3.17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|v(t)\|_1^2 + \alpha^2\|Av(t)\|^2] + d_0[\|v(t)\|_1^2 + \alpha^2\|Av(t)\|^2] \leq \frac{1}{\nu}\|h\|^2 + \frac{C}{\nu^3}\|v(t)\|_1^6.$$

Let t_0 be so that (3.5) holds for all $t \geq t_0$. Then integrating the last inequality over the interval (t_0, t), we get

$$\|v(t)\|_1^2 + \alpha^2\|Av(t)\|^2 \leq [\|v(t_0)\|_1^2 + \alpha^2\|Av(t_0)\|^2]\mathrm{e}^{-d_0(t-t_0)} + \frac{R_2}{d_0}(1 - \mathrm{e}^{-d_0(t-t_0)}), \tag{3.18}$$

where $R_2 := \frac{1}{\nu}\|h\|^2 + \frac{C}{\nu^3}M_1^6$. The last inequality implies the existence of an absorbing ball

$$\mathcal{B}_2 := \{v \in V_2 : \|Av\| \leq M_2\}, \tag{3.19}$$

where $M_2^2 = \frac{2R_2}{(\alpha^2+\lambda_1^{-1})d_0}$. That is, for all $t \gg 1$, we have $\|Av(t)\| \leq M_2$.

Similarly, we can prove the following theorem.

Theorem 3.2 *If $h \in V_1$, then the semigroup $S(t) : V_2 \to V_2$ has a global attractor $\mathcal{A}_2 \subset V_2$. The attractor $\mathcal{A}_2$ is compact, connected and invariant. Moreover, $\mathcal{A}_2$ is a bounded set in V_3.*

Remark 3.1 Let us note that in case we assume in Theorem 3.1 that $h \in V_1$, instead of $h \in H$, then the attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide.

4 Estimates for the Number of Determining Modes

It is asserted, based on physical heuristic arguments, that the long-time behavior of turbulent flows is determined by a finite number degrees of freedom. This concept was formulated more rigorously for 2D NS equations by introducing the notion of determining modes in [17]. In [17] it was shown that there exists a number m such that if the first m Fourier modes of two different solutions of the NS equations have the same asymptotic behavior, as $t \to \infty$, then the remaining infinitely many number of modes have the same asymptotic behavior.

In [32] it was shown that the semigroup generated by the initial boundary value problem for the 2D NS equations with Dirichlet boundary condition has a global attractor which is compact, invariant and connected. It was also established in [32] that there exists a number m such that if projections of two different trajectories on the attractor on the m dimensional subspace of H, spanned on the first m eigenfunctions of the Stokes operator, coincide for each $t \in \mathbb{R}$, then these trajectories completely coincide for each $t \in \mathbb{R}$.

The results obtained in [17] and [32] were developed, generalized, and applied to various infinite dimensional dissipative problems (see, e.g., [7–9,15,16,18,20,22–24,33,39,40] and references therein).

In this section we are going to give estimates for the number of determining modes (both asymptotic and for trajectories on the attractor) for 3D NSV equations.

Asymptotic determining modes Let us denote by P_m the L^2-orthogonal projection from H onto the m-dimensional subspace $H_m = \mathrm{span}\{w_1, w_2, \cdots, w_m\}$. We set $Q_m = I - P_m$. Let v and u be two solutions of NSV equations

$$v_t + \nu Av + \alpha^2 Av_t + B(v, v) = h(t), \quad v(0) = v_0, \tag{4.1}$$

$$u_t + \nu Av + \alpha^2 Au_t + B(u, u) = g(t), \quad v(0) = v_0. \tag{4.2}$$

Definition 4.1 *A set of modes $\{w_1, \cdots, w_m\}$ is called asymptotically determining (see [15, 17]) if*

$$\lim_{t\to\infty} \|v(t) - u(t)\|_1 = 0$$

whenever

$$\lim_{t\to\infty} \|h(t) - g(t)\|_{-1} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|P_m(v(t) - u(t))\|_1 = 0.$$

Theorem 4.1 *Assume that the following conditions are satisfied:*

$$\|h(t)\|_{-1} \le \mathbf{h} < \infty, \quad \forall\, t \in \mathbb{R}, \tag{4.3}$$

$$\lim_{t\to\infty} \|h(t) - g(t)\|_{-1} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|P_m(v(t) - u(t))\| = 0. \tag{4.4}$$

Then the first m eigenfunctions of the Stokes operator are asymptotically determining for the NSV equations with homogeneous Dirichlet boundary conditions, provided that m is large enough such that

$$\lambda_{m+1} > C\frac{\mathbf{h}^4}{\alpha^4\nu^8 d_1^2}. \tag{4.5}$$

Proof It is clear that the function $w = v - u$ satisfies

$$w_t + \nu Aw + \alpha^2 Aw_t + B(v, w) + B(w, v) - B(w, w) = \theta(t), \quad v(0) = v_0, \tag{4.6}$$

where $\theta(t) = h(t) - g(t)$. It is clear from the proof of (E_2) that

$$\limsup_{t\to\infty} \|v(t)\|_1 \le \frac{\mathbf{h}}{\alpha\nu\sqrt{d_1}}. \tag{4.7}$$

Multiplying (4.6) by $q(t) = Q_m w(t)$ in H, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2 + \alpha^2\|q\|_1^2] + 2\nu\|q\|_1^2 + 2b(q, v, q)\\ &= 2(\theta, q) - 2b(v, p, q) - 2b(p, v, q) - 2b(p, p, q) + 2b(q, p, q),\end{aligned} \tag{4.8}$$

where $p = P_m w$. Before estimating the terms of (4.8) we observe that for each $\phi \in V$, we have

$$\|Q_m\phi\|_1 \ge \lambda_{m+1}\|Q_m\phi\| \quad \text{and} \quad \|P_m\phi\|_1 \le \lambda_m\|P_m\phi\|. \tag{4.9}$$

Due to the inequality (2.11) the term $b(q, v, q)$ has the estimate

$$2|b(q, v, q)| \le C\|q\|^{\frac{1}{2}}\|q\|_1^{\frac{3}{2}}\|v\|_1 \le \frac{C}{\lambda_{m+1}^{\frac{1}{4}}}\|q\|_1^2\|v\|_1. \tag{4.10}$$

The first term in the right-hand side of (4.8) has the estimate

$$2|(\theta, q)| \le \frac{2}{\nu}\|\theta\|_{-1}^2 + \frac{\nu}{2}\|q\|_1^2. \tag{4.11}$$

Employing the inequalities (2.12) and (4.9), we estimate the second term in the right-hand side of (4.8) as follows

$$2|b(v, p, q)| \le \|v\|_1\|p\|_1\|q\|^{\frac{1}{2}}\|q\|_1^{\frac{1}{2}} \le C\lambda_m\lambda_{m+1}^{-\frac{1}{4}}\|p\|_1(\|q\|_1^2 + \|v\|_1^2). \tag{4.12}$$

Other terms in the right-hand side of (4.8) can be estimated in a similar way to (4.12). Using estimates (4.10)–(4.12) and the estimates of other terms in the right-hand side of (4.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2+\alpha^2\|q\|_1^2]+\frac{\nu}{2}\|q\|_1^2+\|q\|_1^2\Big(\nu-\frac{C}{\lambda_{m+1}^{\frac{1}{4}}}\|v\|_1\Big)\le b(t), \tag{4.13}$$

where $b(t)$ satisfies the corresponding condition of Lemma 2.1.

Let us choose $t_1>0$ so large that we have $\|v(t)\|_1\le M_1$ for all $t\ge t_1$ and m so that $\mu(m):=\lambda_{m+1}-(\frac{CM_1}{\nu})^4>0$. Then it follows from the last inequality the following relation

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2+\alpha^2\|q\|_1^2]+\frac{\nu}{2}\|q\|_1^2\le b(t)\quad\text{for all } t\ge t_1,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2+\alpha^2\|q\|_1^2]+d_m[\|q\|^2+\alpha^2\|q\|_1^2]\le b(t)\quad\text{for all } t\ge t_1, \tag{4.14}$$

where $d_m=\frac{\nu}{4}\min\{\frac{1}{\alpha^2},\lambda_{m+1}\}$. Thus, due to Lemma 2.1 the statement of the theorem follows.

Remark 4.1 Let us observe that the number m, for which $\lambda_{m+1}>\frac{C\mathbf{h}^4}{\nu^8\lambda_1^2}$ holds, is an upper bound for the minimal number of asymptotically determining modes for weak solutions (i.e., solutions belonging to $L^\infty(\mathbb{R}^+;H)\cap L_{\mathrm{loc}}(\mathbb{R}^+;V)$) of the initial boundary value problem for the 3D Navier Stokes equations. In fact, for weak solutions of NS equations instead of (4.13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2+\lambda_{m+1}^{\frac{3}{4}}(\nu\lambda_{m+1}^{\frac{1}{4}}-C\|v\|_1)\|q\|^2\le b(t),$$

and instead of (4.7) we have for weak solutions of NS equations (see, e.g., [10, 11, 20, 47])

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|v(\tau)\|_1^2\,\mathrm{d}\tau\le\frac{\mathbf{h}^2}{T\nu^3\lambda_1^2}+\frac{\mathbf{h}^2}{\nu^2\lambda_1}.$$

Hence

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|v(\tau)\|_1\,\mathrm{d}\tau\le\frac{\mathbf{h}}{\sqrt{T}\,\nu^{\frac{3}{2}}\lambda_1}+\frac{\mathbf{h}}{\nu\sqrt{\lambda_1}}.$$

Thus, the function $a(t):=\lambda_{m+1}^{\frac{3}{4}}(\nu\lambda_{m+1}^{\frac{1}{4}}-C\|v\|_1)$ satisfies conditions of Lemma 2.1 provided that T is large enough and

$$\lambda_{m+1}>C\frac{\mathbf{h}^4}{\nu^8\lambda_1^2}.$$

Different estimates of asymptotic determining modes for weak solutions of 3D NS equations are obtained in [12] (see also [14, 15] and references therein). The estimate obtained in [12] involves generalization of the so called mean rate dissipation of energy, per mass and time, i.e., it involves

$$\varepsilon=\nu\limsup_{t\to\infty}\frac{1}{t}\int_0^t\sup_{x\in\Omega}\|\nabla v(x,\tau)\|^2\mathrm{d}\tau.$$

For other related results concerning estimates of the number of asymptotic determining degrees of freedom for weak solutions of the 3D NS equations, see, e.g., [10, 20] and references therein.

Determining modes on the attractor Next we give an estimate of determining modes for trajectories on the attractor.

Definition 4.2 *A set of modes* $\{w_1, \cdots, w_m\}$ *is called determining on the attractor* (*in the sense of* [32]) *if for each two trajectories* $v(t)$ *and* $u(t)$ *on the attractor* $\mathcal{A}_1$*, the equality*

$$\|P_m(v(t) - u(t))\|_1 = 0 \quad \textit{for all } t \in \mathbb{R}$$

implies

$$v(t) = u(t), \quad \forall\, t \in \mathbb{R}.$$

Let v and u be arbitrary two trajectories in the attractor $\mathcal{A}_1$ of (3.1). Then $w = v - u$ satisfies

$$w_t + \alpha^2 A w_t + \nu A w + B(w, v) + B(u, w) = 0. \tag{4.15}$$

Taking the inner product of (4.15) with $q = Q_m w$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2 + \|q\|_1^2] + 2\nu\|q\|_1^2 = -2b(w, v, q) - 2b(u, w, q). \tag{4.16}$$

Assume that $P_m w(t) = 0$ for all $t \in \mathbb{R}$. Then $Q_m w = q$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2 + \|q\|_1^2] + 2\nu\|q\|_1^2 = 2b(q, v, q). \tag{4.17}$$

Due to (2.11) we have

$$|2b(q, v, q)| \le C\|q\|_1^{\frac{1}{2}}\|q\|_1^{\frac{3}{2}}\|v\|_1.$$

Noting that on the attractor $\mathcal{A}_1$ we have $\|v\|_1 \le M_1$, we employ the last inequality and inequality (4.9) to obtain from (4.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2 + \alpha^2\|q\|_1^2] + \nu\|q\|_1^2 + \|q\|^{\frac{1}{2}}\|q\|_1^{\frac{3}{2}}(\nu\lambda_{m+1}^{\frac{1}{4}} - CM_1) \le 0. \tag{4.18}$$

Let us choose m, large enough, so that $\lambda_{m+1} \ge (\frac{M_1 C}{\nu})^4$. Then (4.18) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|q\|^2 + \alpha^2\|q\|_1^2] + l_m[\|q\|^2 + \alpha^2\|q\|_1^2] \le 0,$$

where $l_m = \frac{\nu}{2}\min\{\lambda_{m+1}, \frac{1}{\alpha^2}\}$.

Finally, we integrate the last inequality and get

$$\|q(t)\|^2 + \alpha^2\|q(t)\|_1^2 \le \exp[-l_m(t - s)][\|q(s)\|^2 + \alpha^2\|q(s)\|_1^2]. \tag{4.19}$$

Passing to the limit as $s \to -\infty$, we obtain

$$\|q(t)\|^2 + \alpha^2\|q(t)\|_1^2 = 0 \quad \text{for all } t \in \mathbb{R}.$$

Thus, the following theorem is true.

Theorem 4.2 *Let* v *and* u *be two solutions of the problem* (1.1)–(1.3) *from the attractor* $\mathcal{A}_1$*. Assume that* $P_m(u(t)) = P_m(v(t))$, $\forall\, t \in \mathbb{R}$, *where* m *is so that*

$$\lambda_{m+1} \ge C\frac{\|h\|_{-1}^4}{\alpha^4\nu^8 d_1^2}. \tag{4.20}$$

Then $v(t) = u(t)$ *for all* $t \in \mathbb{R}$.

5 Estimates of Dimensions of the Global Attractor

In this section we show the differentiability of the semigroup with respect to the initial data. This is to prepare for implementing Theorem 2.2 in order to estimate the dimension of the global attractor.

Theorem 5.1 *Let u_0 and v_0 be two elements of V. Then there is a constant $K = K(\|u_0\|_1, \|v_0\|_1)$ such that*

$$\|S(t)v_0 - S(t)u_0 - \Lambda(t)(v_0 - u_0)\|_1 \le K\|v_0 - u_0\|_1^2, \tag{5.1}$$

where the linear operator $\Lambda(t) : V \to V$, for $t > 0$, is the solution operator of the problem

$$\xi_t + \alpha^2 A\xi_t + A\xi + B(\xi, v) + B(v, \xi) = 0, \quad \xi(0) = v_0 - u_0, \tag{5.2}$$

and $v(t) = S(t)v_0$. That is, for every $t > 0$, the map $S(t)v_0$, as a map $S(t) : V \to V$ is Fréchet differentiable with respect to the initial data, and its Fréchet derivative $D_{v_0}(S(t)v_0)w_0 = \Lambda(t)w_0$.

Proof It is easy to see that the function $\eta(t) := v(t) - u(t) - \xi(t) = S(t)(v_0 - u_0) - \xi(t)$ satisfies

$$\eta_t + \alpha^2 A\eta_t + \nu A\eta + B(\eta, v) + B(v, \eta) - B(w, w) = 0,$$

where $w = v - u$. Taking the inner product of the last equation with η, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\eta\|^2 + \alpha^2\|\eta\|_1^2] + 2\nu\|\eta\|_1^2 = -2b(\eta, v, \eta) - 2b(w, w, \eta). \tag{5.3}$$

By using inequalities (3.5) and (2.5) and Young's inequality we can estimate the terms in the right-hand side of (5.3) as follows.

By (2.11), we have

$$|2b(\eta, v, \eta)| \le C\|v\|_1\|\eta\|^{\frac{1}{2}}\|\eta\|_1^{\frac{3}{2}} \le CM_1\|\eta\|^{\frac{1}{2}}\|\eta\|_1^{\frac{3}{2}} \le \frac{CM_1}{4}(\|\eta\|^2 + 3\|\eta\|_1^2).$$

By (2.10),

$$|2b(w, w, \eta)| = |2b(w, \eta, w)| \le C\lambda_1^{-\frac{1}{2}}\|\eta\|_1\|w\|_1^2 \le \nu\|\eta\|_1^2 + \frac{C}{4\nu\lambda_1}\|w\|_1^4.$$

Hence, from (5.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\eta\|^2 + \alpha^2\|\eta\|_1^2] \le \frac{CM_1}{4}(\|\eta\|^2 + 3\|\eta\|_1^2) + \frac{C}{4\nu\lambda_1}\|w\|_1^4. \tag{5.4}$$

The function $w(t) = v(t) - u(t) = S(t)v_0 - S(t)u_0$ satisfies

$$w_t + \alpha^2 Aw_t + \nu Aw + B(w, v) + B(v, w) - B(w, w) = 0, \quad w(0) = v_0 - u_0 := w_0.$$

Taking the inner product of the last equation with w, and using (2.13) and (E_1) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\|w\|^2 + \alpha^2\|w\|_1^2] + 2\nu\|w\|_1^2 &= 2b(w, v, w) \le 2C\lambda_1^{\frac{1}{4}}\|v\|_1\|w\|_1^2 \\ &\le \kappa_1\|v\|_1[\|w\|^2 + \alpha^2\|w\|_1^2],\end{aligned}$$

where $k_1 = 2C\lambda_1^{\frac{1}{4}}\alpha^{-4}[\|v(0)\|^2 + \alpha^2\|v(0)\|_1^2 + \frac{1}{\nu d_0}\|h\|_{-1}^2]^{\frac{1}{2}}$. Integrating the last inequality we get

$$\|w(t)\|_1^2 \le \Big(1 + \frac{1}{\lambda_1\alpha^2}\Big)\|w(0)\|_1^2 \exp(\kappa_1 t). \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\eta\|^2 + \alpha^2\|\eta\|_1^2] \le A_1[\|\eta\|^2 + \alpha^2\|\eta\|_1^2] + A_2\|w(0)\|_1^4 \exp(2\kappa_1 t).$$

Integrating and using Gronwall's inequality, one has

$$\|\eta(t)\|_1^2 \le A(t)\|w(0)\|_1^4,$$

where $A(t) := \frac{A_2}{2\kappa_1\alpha^2}\exp[(2\kappa_1 + A_1)t]$. So we have

$$\frac{\|v(t) - u(t) - \xi(t)\|_1}{\|v_0 - u_0\|_1} \le \sqrt{A(t)}\,\|v_0 - u_0\|_1.$$

Thus the differentiability of $S(t)$ with respect to the initial data follows.

We rewrite (3.1) in the following form:

$$\widehat{v}_t = -\frac{\nu}{\alpha^2}\widehat{v} + \frac{\nu}{\alpha^2}G^{-2}\widehat{v} - G^{-1}B(G^{-1}\widehat{v}, G^{-1}\widehat{v}) + G^{-1}h, \tag{5.6}$$

where $G^2 = I + \alpha^2 A$, and $\widehat{v} = Gv$. The equation of linear variations corresponding to (5.6) has the form

$$w_t = L(t)w, \tag{5.7}$$

where

$$L(t)w := -\frac{\nu}{\alpha^2}w + \frac{\nu}{\alpha^2}G^{-2}w - G^{-1}B(G^{-1}w, G^{-1}\widehat{v}) - G^{-1}B(G^{-1}\widehat{v}, G^{-1}w).$$

Now we consider the quadratic form

$$(L(t)w, w) = -\frac{\nu}{\alpha^2}\|w\|^2 + \frac{\nu}{\alpha^2}\|G^{-1}w\|^2 - b(G^{-1}w, G^{-1}\widehat{v}, G^{-1}w).$$

By using inequality (2.11) and the inequality $\|G^{-1}u\|_1 \le \frac{1}{\alpha}\|u\|$, we get

$$|b(G^{-1}w, G^{-1}\widehat{v}, G^{-1}w)| \le \frac{1}{\alpha^{\frac{5}{2}}}\|G^{-1}w\|^{\frac{1}{2}}\|w\|^{\frac{3}{2}}\|\widehat{v}\|.$$

Employing Young's inequality with $p = \frac{4}{3}$, $\epsilon = \frac{2\nu}{3\alpha^2}$ and the fact that on the global attractor $\mathcal{A}_1$ the estimate $\|\widehat{v}\| \le (\lambda_1 + \alpha^2)^{\frac{1}{2}}M_1$ holds, we obtain

$$|b(G^{-1}w, G^{-1}\widehat{v}, G^{-1}w)| \le \frac{\nu}{2\alpha^2}\|w\|^2 + \frac{C(\lambda_1 + \alpha^2)^2M_1^4}{\nu^3\alpha^4}\|G^{-1}w\|^2.$$

Due to the last inequality the quadratic form $(L(t)w, w)$ has the following estimate

$$(L(t)w, w) \le -\frac{\nu}{2\alpha^2}\|w\|^2 + \Big(\frac{\nu}{\alpha^2} + \frac{C(\lambda_1 + \alpha^2)^2M_1^4}{\nu^3\alpha^4}\Big)\|G^{-1}w\|^2. \tag{5.8}$$

Thus, we can use Theorem 2.2 to get the desired estimate for the fractal dimension of the attractor $\mathcal{A}_1$,

$$d_f(\mathcal{A}_1) \le C\frac{(\lambda_1 + \alpha^2)^2M_1^4}{\nu^4\alpha^2} + 2 \le C\frac{(\lambda_1 + \alpha^2)^2\|h\|_{-1}^4}{\nu^8\alpha^6 d_1^2} + 2. \tag{5.9}$$

We recall that $M_1 = \frac{2}{\nu\alpha\sqrt{d_1}}\|h\|_{-1}$, $d_1 = \frac{1}{2}\min\{\alpha^{-2}, \lambda_1\}$. Let us note that in our situation

$$\overline{h}_0(t) = \frac{\nu}{2\alpha^2}, \quad s_0 = 0, \quad s_1 = -1, \quad \overline{h}_{s_1}(t) = \frac{\nu}{\alpha^2} + \frac{C(\lambda_1 + \alpha^2)^2M_1^4}{\nu^3\alpha^4}$$

and $\overline{h}_{s_k}(t) = 0$, $k \ge 2$.

6 The Inviscid Limit

Here we show that when $\nu \to 0$ the weak solution of the initial boundary value problem for the NSV system, i.e., of the problem (1.1)–(1.3), is tending to the weak solution of the initial boundary value problem for the inviscid simplified Bardina model

$$\begin{aligned} &u_t - \alpha^2 \Delta u_t + (u \cdot \nabla) u + \nabla p = f, \quad && x \in \Omega,\ t > 0, \\ &\nabla \cdot u = 0, && x \in \Omega,\ t > 0, \end{aligned} \tag{6.1}$$

$$\begin{aligned} &u(x,0) = v_0(x), \quad && x \in \Omega, \\ &u(x,t) = 0, && x \in \partial\Omega,\ t > 0. \end{aligned} \tag{6.2}$$

The problem of existence and uniqueness of solutions of the initial boundary value problem, with periodic boundary conditions, for the 3D viscous and inviscid simplified Bardina models was studied in [5]. In particular, it was shown in [5] that the problem (6.1)–(6.2) has a unique solution $u \in C^1(\mathbb{R}; V)$, for initial value $u_0 \in V$.

Applying to (6.1) the Helmholtz-Leray operator P we obtain the equivalent functional differential equation

$$u_t + \alpha^2 A u_t + B(u,u) = h, \tag{6.3}$$

$$u(0) = v_0. \tag{6.4}$$

Let $v(t)$ be the solution of (6.1) with initial $v(0) = v_0 \in V$. Denote $w = v - u$. Then w satisfies the relation

$$w_t + \alpha^2 A w_t + B(w,v) + B(u,w) = -\nu A v, \tag{6.5}$$

$$w(0) = 0, \tag{6.6}$$

which holds in the space V'. Taking the action of (6.5) on w, which belongs to V, and using a lemma of Lions-Magenes concerning the derivative of functions with values in Banach space (see [48, Lemma 1.2, Chapter III, p.169]), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|w\|^2 + \alpha^2 \|w\|_1^2] = -2\nu(\nabla v, \nabla w) - 2b(w,v,w). \tag{6.7}$$

For the first term in the right-hand side we have

$$|2\nu(\nabla v, \nabla w)| \le \nu^2 \|v\|_1^2 + \|w\|_1^2.$$

We estimate the second term by using the inequality (2.13),

$$|2b(w,v,w)| \le C\lambda_1^{\frac{1}{4}} \|v\|_1 \|w\|_1^2.$$

Utilizing the last two inequalities in (6.7), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[\|w\|^2 + \alpha^2\|w\|_1^2] &\le \nu^2 \|v\|_1^2 + (1 + C\lambda_1^{\frac{1}{4}} \|v\|_1)\|w\|_1^2 \\ &\le \nu^2 \|v\|_1^2 + \alpha^{-2}(1 + 2C\lambda_1^{\frac{1}{4}}\|v\|_1)[\|w\|^2 + \alpha^2\|w\|_1^2]. \end{aligned}$$

Integrating the last inequality and using the standard Gronwall's lemma, we get the estimate

$$\|w(t)\|^2 + \alpha^2 \|w(t)\|_1^2 \le \nu^2 \int_0^t \|v(\tau)\|_1^2 \,\mathrm{d}\tau \exp\Big(\frac{t}{\alpha^2} + \frac{2C\lambda_1^{\frac{1}{4}}}{\alpha^2} \int_0^t \|v(\tau)\|_1 \,\mathrm{d}\tau\Big). \tag{6.8}$$

Next we show that on each finite interval $[0,T]$, we can estimate $\|v\|_1$ by a constant depending only on $\|v_0\|, \|v_0\|_1$ and the parameter α. Indeed, (3.3) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2] \leq \alpha^{-2}\|h\|_{-1}^2 + \alpha^2\|v(t)\|_1^2.$$

Integrating the last inequality over $(0,t)$ with respect to time variable, we obtain

$$\|v(t)\|^2 + \alpha^2\|v(t)\|_1^2 \leq \|v_0\|^2 + \alpha^2\|v_0\|_1^2 + t\alpha^{-2}\|h\|_{-1}^2 + \int_0^t [\|v(\tau)\|^2 + \alpha^2\|v(\tau)\|_1^2]\mathrm{d}\tau.$$

By using the Gronwall inequality we get

$$\|v(t)\|_1^2 \leq D_T\mathrm{e}^T \quad \text{for all } t \in [0,T].$$

Here $D_T := \alpha^{-2}[\|v_0\|^2 + \alpha^2\|v_0\|_1^2 + T\alpha^{-2}\|h\|_{-1}^2]$. Hence (6.8) implies

$$\|w(t)\|^2 + \alpha^2\|w(t)\|_1^2 \leq \nu^2 T D_T \mathrm{e}^T \exp(\alpha^{-2}T + 2C\alpha^{-2}\lambda^{\frac{1}{4}} T D_T^{\frac{1}{2}} \mathrm{e}^{\frac{T}{2}}). \tag{6.9}$$

Remark 6.1 The problem of convergence of solutions of the NSV equations to solutions of NS equations as $\alpha \to 0$ was studied in [42]. It was shown in [42] that strong solutions of the NSV equations converge to strong solutions of the NS equations as $\alpha \to 0$, under specified smallness conditions on the initial data of the problem.

Remark 6.2 The results obtained in this paper are valid also for the solutions of the initial boundary value problem for the 3D NSV equations with periodic boundary conditions.

Finally we would like to notice that the results reported here can be extended to other similar equations, a subject of future work. For instance, for the 3D equations of motion of Kelvin-Voight fluids of order $L \geq 1$,

$$\begin{aligned} &v_t + (v\cdot\nabla)v - \mu_0\Delta v_t - \mu_1\Delta v - \sum_{l=1}^{L}\beta_l\Delta u_l + \nabla p = f, \\ &\partial_t u_l + \alpha_l u_l - v = 0, \quad l = 1, \cdots, L, \end{aligned}$$

where $\mu_0, \mu_1, \beta_l, \alpha_l > 0$, $l = 1, \cdots, L$. Also for the generalized Benajamin-Bona-Mahony (GBBM) equation

$$u_t - \alpha^2\Delta u_t + \nu\Delta u + \nabla\cdot\vec{F}(u) = h, \tag{6.10}$$

where a smooth vector field $\vec{F}(u)$ satisfies the growth condition

$$|\vec{F}(u)| \leq C(1 + |u|^2).$$

The problem of existence of a finite dimensional global attractor and estimates for the number of determining modes on the global attractor of Kelvin-Voight fluids of order $L \geq 1$ was established in [28]. In [49] the existence of a finite dimensional global attractor was established for 1D GBBM equation under periodic boundary conditions. The existence of a finite dimensional global attractor for 3D GBBM under periodic boundary conditions was proved in [6]. In [46] it was shown the existence of the global attractor for GBBM equation in $H^1(\mathbb{R}^3)$. Moreover, the existence of a global attractor for a similar two-dimensional model describing the motion of a second-grade fluid was established in [38].

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