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Invariant Metrics and Laplacians on Siegel-Jacobi Disk**

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Abstract Let $\mathbb{D}_n$ be the generalized unit disk of degree n. In this paper, Riemannian metrics on the Siegel-Jacobi disk $\mathbb{D}_n \times \mathbb{C}^{(m,n)}$ which are invariant under the natural action of the Jacobi group are found explicitly and the Laplacians of these invariant metrics are computed explicitly. These are expressed in terms of the trace form.

Keywords Invariant metrics, Siegel-Jacobi disk, Partial Cayley transform
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1 Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0\}$$

be the Siegel upper half plane of degree n and

$$\operatorname{Sp}(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that $\operatorname{Sp}(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

We define the semidirect product of $\operatorname{Sp}(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$ as

$$G^J := \operatorname{Sp}(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

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endowed with the following multiplication law

$$(M,(\lambda,\mu;\kappa))\cdot(M',(\lambda',\mu';\kappa')) = (MM',(\widetilde{\lambda}+\lambda',\widetilde{\mu}+\mu';\kappa+\kappa'+\widetilde{\lambda}\,{}^t\mu'-\widetilde{\mu}\,{}^t\lambda'))$$

with $M, M' \in \mathrm{Sp}(n,\mathbb{R})$, $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu}) = (\lambda,\mu)M'$. We call this group G^J the Jacobi group of degree n and index m. Then we get the natural action of G^J on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ (see [1, 2, 7–9, 15]) defined by

$$(M,(\lambda,\mu;\kappa))\cdot(\Omega,Z) = (M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$, $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We note that the action (1.2) is transitive.

For brevity, we write $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. For a coordinate $(\Omega,Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_n$ and $Z = (z_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
&\Omega = X + \mathrm{i}Y, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \text{ real},\\
&Z = U + \mathrm{i}V, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real},\\
&\mathrm{d}\Omega = (\mathrm{d}\omega_{\mu\nu}), \quad \mathrm{d}\overline{\Omega} = (\mathrm{d}\overline{\omega}_{\mu\nu}),\\
&\mathrm{d}Z = (\mathrm{d}z_{kl}), \quad \mathrm{d}\overline{Z} = (\mathrm{d}\overline{z}_{kl}),\\
&\frac{\partial}{\partial\Omega} = \Big(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\omega_{\mu\nu}}\Big), \quad \frac{\partial}{\partial\overline{\Omega}} = \Big(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\overline{\omega}_{\mu\nu}}\Big),\\
&\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial\overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix},
\end{aligned}$$

where δ_{ij} denotes the Kronecker delta symbol.

Siegel [6] introduced the symplectic metric $\mathrm{d}s_n^2$ on $\mathbb{H}_n$ invariant under the action (1.1) of $\mathrm{Sp}(n,\mathbb{R})$ given by

$$\mathrm{d}s_n^2 = \sigma(Y^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}\,), \tag{1.3}$$

and Maass [4] proved that the differential operator

$$\Delta_n = 4\sigma\Big(Y\ {}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big) \tag{1.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric $\mathrm{d}s_n^2$. Here $\sigma(A)$ denotes the trace of a square matrix A.

In [11], the author proved the following theorems.

Theorem 1.1 *For any two positive real numbers A and B, the following metric*

$$\begin{aligned}
\mathrm{d}s^2_{n,m;A,B} = {} & A\sigma(Y^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}\,) + B\{\sigma(Y^{-1}\,{}^tVVY^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}\,) + \sigma(Y^{-1}\,{}^t(\mathrm{d}Z)\mathrm{d}\overline{Z}\,)\\
& - \sigma(VY^{-1}\mathrm{d}\Omega Y^{-1}\,{}^t(\mathrm{d}\overline{Z}\,)) - \sigma(VY^{-1}\mathrm{d}\overline{\Omega}Y^{-1}\,{}^t(\mathrm{d}Z))\}
\end{aligned} \tag{1.5}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) *of the Jacobi group G^J.*

Theorem 1.2 *For any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m},\mathrm{d}s^2_{n,m;A,B})$ is given by*

$$\begin{aligned}
\Delta_{n,m;A,B} = {} & \frac{4}{A}\Big\{\sigma\Big(Y\ {}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big) + \sigma\Big(VY^{-1}\,{}^tV\ {}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial Z}\Big) + \sigma\Big(V\ {}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big)\\
& + \sigma\Big({}^tV\ {}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial\Omega}\Big)\Big\} + \frac{4}{B}\sigma\Big(Y\frac{\partial}{\partial Z}\ {}^t\Big(\frac{\partial}{\partial\overline{Z}}\Big)\Big).
\end{aligned} \tag{1.6}$$

Let

$$G_* = \mathrm{SU}(n,n) \cap \mathrm{Sp}(n,\mathbb{C})$$

be the symplectic group and

$$\mathbb{D}_n = \{ W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - \overline{W}W > 0 \}$$

be the generalized unit disk. Then G_* acts on $\mathbb{D}_n$ transitively by

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW + Q)(\overline{Q}W + \overline{P}\,)^{-1},$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$ and $W \in \mathbb{D}_n$. Using the Cayley transform of $\mathbb{D}_n$ onto $\mathbb{H}_n$, we can see that

$$\mathrm{d}s_*^2 = 4\sigma((I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,) \tag{1.7}$$

is a G_*-invariant Kähler metric on $\mathbb{D}_n$ (see [6]) and Maass [4] showed that its Laplacian is given by

$$\Delta_* = \sigma\Big((I_n - W\overline{W}\,)\ {}^t\Big((I_n - W\overline{W}\,)\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial W}\Big). \tag{1.8}$$

Let

$$G_*^J = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \,\middle|\, \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

be the Jacobi group with the following multiplication:

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ \overline{Q'} & \overline{P'} \end{pmatrix}, (\xi', \overline{\xi'}; \mathrm{i}\kappa') \right) \\ &= \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \begin{pmatrix} P' & Q' \\ \overline{Q'} & \overline{P'} \end{pmatrix}, (\widetilde{\xi} + \xi', \widetilde{\widetilde{\xi}} + \overline{\xi'}; \mathrm{i}\kappa + \mathrm{i}\kappa' + \widetilde{\xi}\,{}^t\overline{\xi'} - \widetilde{\widetilde{\xi}}\,{}^t\xi') \right), \end{aligned}$$

where $\widetilde{\xi} = \xi P' + \overline{\xi}\,\overline{Q'}$ and $\widetilde{\widetilde{\xi}} = \xi Q' + \overline{\xi}\,\overline{P'}$. Then we have the natural action of G_*^J on the Siegel-Jacobi disk $\mathbb{D}_n \times \mathbb{C}^{(m,n)}$ (see (2.6)) given by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \cdot (W, \eta) = ((PW + Q)(\overline{Q}W + \overline{P}\,)^{-1}, (\eta + \xi W + \overline{\xi}\,)(\overline{Q}W + \overline{P}\,)^{-1}), \tag{1.9}$$

where $W \in \mathbb{D}_n$ and $\eta \in \mathbb{C}^{(m,n)}$.

For brevity, we write $\mathbb{D}_{n,m} := \mathbb{D}_n \times \mathbb{C}^{(m,n)}$. For a coordinate $(W, \eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\mathrm{d}W = (\mathrm{d}w_{\mu\nu}), \quad \mathrm{d}\overline{W} = (\mathrm{d}\overline{w}_{\mu\nu}), \quad \mathrm{d}\eta = (\mathrm{d}\eta_{kl}), \quad \mathrm{d}\overline{\eta} = (\mathrm{d}\overline{\eta}_{kl})$$

and

$$\frac{\partial}{\partial W} = \Big(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial w_{\mu\nu}} \Big), \qquad \frac{\partial}{\partial \overline{W}} = \Big(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{w}_{\mu\nu}} \Big),$$

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{\eta}} = \begin{pmatrix} \frac{\partial}{\partial \overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{m1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial \overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{mn}} \end{pmatrix}.$$

In this paper, we find the G_*^J-invariant Riemannian metrics on $\mathbb{D}_{n,m}$ and their Laplacians. In fact, we prove the following theorems.

Theorem 1.3 *For any two positive real numbers A and B, the following metric $\mathrm{d}\widetilde{s}^2_{n,m;A,B}$ defined by*

$$\begin{aligned}\mathrm{d}\widetilde{s}^2_{n,m;A,B} &= 4A\sigma((I_n-W\overline{W})^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}) + 4B\{\sigma((I_n-W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta)\mathrm{d}\overline{\eta})\\
&+\sigma((\eta\overline{W}-\overline{\eta})(I_n-W\overline{W})^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta}))\\
&+\sigma((\overline{\eta}W-\eta)(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta))\\
&-\sigma((I_n-W\overline{W})^{-1}\,{}^t\eta\eta(I_n-\overline{W}W)^{-1}\overline{W}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W})\\
&-\sigma(W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n-W\overline{W})^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W})\\
&+\sigma((I_n-W\overline{W})^{-1}\,{}^t\eta\overline{\eta}(I_n-W\overline{W})^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W})\\
&+\sigma((I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n-W\overline{W})^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W})\\
&+\sigma((I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n-\overline{W}W)^{-1}(I_n-\overline{W})(I_n-W)^{-1}\\
&\times\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W})-\sigma((I_n-W\overline{W})^{-1}(I_n-W)(I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\eta\\
&\times(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W})\}\end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (1.9) of the Jacobi group G^J_*. Note that if $n=m=1$ and $A=B=1$, we get

$$\begin{aligned}\frac{1}{4}\mathrm{d}\widetilde{s}^2_{1,1;1,1} &= \frac{\mathrm{d}W\mathrm{d}\overline{W}}{(1-|W|^2)^2}+\frac{\mathrm{d}\eta\mathrm{d}\overline{\eta}}{1-|W|^2}+\frac{(1+|W|^2)|\eta|^2-\overline{W}\eta^2-W\overline{\eta}^2}{(1-|W|^2)^3}\mathrm{d}W\mathrm{d}\overline{W}\\
&+\frac{\eta\overline{W}-\overline{\eta}}{(1-|W|^2)^2}\mathrm{d}W\mathrm{d}\overline{\eta}+\frac{\overline{\eta}W-\eta}{(1-|W|^2)^2}\mathrm{d}\overline{W}\mathrm{d}\eta.\end{aligned}$$

Theorem 1.4 *For any two positive real numbers A and B, the Laplacian $\widetilde{\Delta}_{n,m;A,B}$ of $(\mathbb{D}_{n,m},\mathrm{d}\widetilde{s}^2_{n,m;A,B})$ is given by*

$$\begin{aligned}\widetilde{\Delta}_{n,m;A,B} &= \frac{1}{A}\Big\{\sigma\Big((I_n-W\overline{W})\ {}^t\Big((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial W}\Big)\\
&+\sigma\Big({}^t(\eta-\overline{\eta}W)\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big)+\sigma\Big((\overline{\eta}-\eta\overline{W})\ {}^t\Big((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big)\\
&-\sigma\Big(\eta\overline{W}(I_n-W\overline{W})^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\\
&-\sigma\Big(\overline{\eta}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\\
&+\sigma\Big(\overline{\eta}(I_n-W\overline{W})^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\\
&+\sigma\Big(\eta\overline{W}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\Big\}\\
&+\frac{1}{B}\sigma\Big((I_n-\overline{W}W)\frac{\partial}{\partial\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)\Big).\end{aligned}$$

Note that if $n=m=1$ and $A=B=1$, we get

$$\begin{aligned}\widetilde{\Delta}_{1,1;1,1} &= (1-|W|^2)^2\frac{\partial^2}{\partial W\partial\overline{W}}+(1-|W|^2)\frac{\partial^2}{\partial\eta\partial\overline{\eta}}+(1-|W|^2)(\eta-\overline{\eta}W)\frac{\partial^2}{\partial W\partial\overline{\eta}}\\
&+(1-|W|^2)(\overline{\eta}-\eta\overline{W})\frac{\partial^2}{\partial\overline{W}\partial\eta}-(\overline{W}\eta^2+W\overline{\eta}^2)\frac{\partial^2}{\partial\eta\partial\overline{\eta}}+(1+|W|^2)|\eta|^2\frac{\partial^2}{\partial\eta\partial\overline{\eta}}.\end{aligned}$$

The main ingredients for the proof of Theorems 1.3 and 1.4 are the partial Cayley transform, Theorems 1.1 and 1.2. The paper is organized as follows. In Section 2, we review the partial Cayley transform that was dealt with in [12]. In Section 3, we prove Theorem 1.3. In Section 4, we prove Theorem 1.4. In the final section, we briefly remark the theory of harmonic analysis on the Siegel-Jacobi disk.

We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real numbers and the field of complex numbers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $\Omega \in \mathbb{H}_g$, $\mathrm{Re}\,\Omega$ (resp. $\mathrm{Im}\,\Omega$) denotes the real (resp. imaginary) part of Ω. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M.

2 A Partial Cayley Transform

In this section, we review the partial Cayley transform (see [12]) of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ needed for the proof of Theorems 1.3 and 1.4.

We can identify an element $g = (M, (\lambda, \mu; \kappa))$ of G^J, where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$, with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $\mathrm{Sp}(m+n, \mathbb{R})$.

Set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ \mathrm{i}I_{m+n} & -\mathrm{i}I_{m+n} \end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* := T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{2.1}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\{Q\,{}^t(\lambda + \mathrm{i}\mu) - P\,{}^t(\lambda - \mathrm{i}\mu)\} \\ \frac{1}{2}(\lambda + \mathrm{i}\mu) & I_h + \mathrm{i}\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\{P\,{}^t(\lambda - \mathrm{i}\mu) - Q\,{}^t(\lambda + \mathrm{i}\mu)\} \\ \frac{1}{2}(\lambda - \mathrm{i}\mu) & -\mathrm{i}\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by the formulas

$$P = \frac{1}{2}\{(A + D) + \mathrm{i}(B - C)\}, \tag{2.2}$$

$$Q = \frac{1}{2}\{(A - D) - \mathrm{i}(B + C)\}. \tag{2.3}$$

From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \Big(\frac{1}{2}(\lambda+\mathrm{i}\mu), \frac{1}{2}(\lambda-\mathrm{i}\mu); \mathrm{i}\frac{\kappa}{2}\Big)\right) := \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa)\right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \Big(\frac{1}{2}(\lambda+\mathrm{i}\mu), \frac{1}{2}(\lambda-\mathrm{i}\mu); -\mathrm{i}\frac{\kappa}{2}\Big)\right).$$

Let

$$H_{\mathbb{C}}^{(n,m)} := \{(\xi,\eta;\zeta) \mid \xi,\eta\in\mathbb{C}^{(m,n)},\ \zeta\in\mathbb{C}^{(m,m)},\ \zeta+\eta\,{}^t\xi \text{ symmetric}\}$$

be the complex Heisenberg group endowed with the following multiplication:

$$(\xi,\eta;\zeta)\circ(\xi',\eta';\zeta') := (\xi+\xi',\eta+\eta';\zeta+\zeta'+\xi\,{}^t\eta'-\eta\,{}^t\xi').$$

We define the semidirect product

$$\mathrm{SL}(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication:

$$\begin{aligned}&\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi,\eta;\zeta)\right)\cdot\left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi',\eta';\zeta')\right)\\ &= \left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\widetilde{\xi}+\xi',\widetilde{\eta}+\eta';\zeta+\zeta'+\widetilde{\xi}\,{}^t\eta'-\widetilde{\eta}\,{}^t\xi')\right),\end{aligned}$$

where $\widetilde{\xi}=\xi P'+\eta R'$ and $\widetilde{\eta}=\xi Q'+\eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\{(\xi,\overline{\xi};\mathrm{i}\kappa) \mid \xi\in\mathbb{C}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)}\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion:

$$G_*^J \subset \mathrm{SU}(n,n)\ltimes H_{\mathbb{R}}^{(n,m)} \subset \mathrm{SL}(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta : G^J \to G_*^J$ by

$$\Theta\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa)\right) := \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \Big(\frac{1}{2}(\lambda+\mathrm{i}\mu), \frac{1}{2}(\lambda-\mathrm{i}\mu); -\mathrm{i}\frac{\kappa}{2}\Big)\right), \tag{2.4}$$

where P and Q are given by (2.2) and (2.3). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1g_2) = \Theta(g_1)\Theta(g_2)$.

According to [10, p. 250], G_*^J is of the Harish-Chandra type (see [5, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda,\mu;\kappa)\right)$$

be an element of G_*^J. Since the Harish-Chandra decomposition of an element $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ in $\mathrm{SU}(n,n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} P-QS^{-1}R & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},$$

the P_*^+-component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right), \quad W \in \mathbb{D}_n$$

of $\mathrm{SL}(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$ is given by

$$\left(\begin{pmatrix} I_n & (PW+Q)(\overline{Q}W+\overline{P}\,)^{-1} \\ 0 & I_n \end{pmatrix}, (0, (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P}\,)^{-1}; 0) \right). \tag{2.5}$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \,\middle|\, W \in \mathbb{D}_n,\ \eta \in \mathbb{C}^{(m,n)} \right\}$$

of the complexification of G_*^J. Indeed, $\mathbb{D}_{n,m}$ is embedded into P_*^+ given by

$$P_*^+ = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \,\middle|\, W = {}^tW \in \mathbb{C}^{(n,n)},\ \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (see [5, p. 119]). Then we get the natural transitive action of G_*^J on $\mathbb{D}_{n,m}$ defined by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \cdot (W, \eta) = ((PW+Q)(\overline{Q}W+\overline{P}\,)^{-1}, (\eta + \xi W + \overline{\xi}\,)(\overline{Q}W + \overline{P}\,)^{-1}), \tag{2.6}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$, $\xi \in \mathbb{C}^{(m,n)}$, $\kappa \in \mathbb{R}^{(m,m)}$ and $(W, \eta) \in \mathbb{D}_{n,m}$.

The author proved in [12] that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (2.6) of G_*^J on $\mathbb{D}_{n,m}$ through a partial Cayley transform $\Phi : \mathbb{D}_{n,m} \to \mathbb{H}_{n,m}$ defined by

$$\Phi(W, \eta) := (\mathrm{i}(I_n + W)(I_n - W)^{-1}, 2\mathrm{i}\eta(I_n - W)^{-1}). \tag{2.7}$$

In other words, if $g_0 \in G^J$ and $(W, \eta) \in \mathbb{D}_{n,m}$, we have

$$g_0 \cdot \Phi(W, \eta) = \Phi(g_* \cdot (W, \eta)), \tag{2.8}$$

where $g_* = T_*^{-1} g_0 T_*$. Φ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ which gives the partially bounded realization of $\mathbb{H}_{n,m}$ by $\mathbb{D}_{n,m}$. The inverse of Φ is

$$\Phi^{-1}(\Omega, Z) = ((\Omega - \mathrm{i}I_n)(\Omega + \mathrm{i}I_n)^{-1}, Z(\Omega + \mathrm{i}I_n)^{-1}).$$

3 Proof of Theorem 1.3

For $(W, \eta) \in \mathbb{D}_{n,m}$, we write

$$(\Omega, Z) := \Phi(W, \eta).$$

Thus

$$\Omega = \mathrm{i}(I_n + W)(I_n - W)^{-1}, \quad Z = 2\mathrm{i}\eta(I_n - W)^{-1}. \tag{3.1}$$

Since

$$\begin{aligned} &\mathrm{d}(I_n - W)^{-1} = (I_n - W)^{-1}\mathrm{d}W(I_n - W)^{-1}, \\ &I_n + (I_n + W)(I_n - W)^{-1} = 2(I_n - W)^{-1}, \end{aligned}$$

we get the following formulas from (3.1):

$$Y = \frac{1}{2\mathrm{i}}(\Omega - \overline{\Omega}\,) = (I_n - W)^{-1}(I_n - W\overline{W}\,)(I_n - \overline{W}\,)^{-1}, \tag{3.2}$$

$$V = \frac{1}{2\mathrm{i}}(Z - \overline{Z}\,) = \eta(I_n - W)^{-1} + \overline{\eta}(I_n - \overline{W}\,)^{-1}, \tag{3.3}$$

$$\mathrm{d}\Omega = 2\mathrm{i}(I_n - W)^{-1}\mathrm{d}W(I_n - W)^{-1}, \tag{3.4}$$

$$\mathrm{d}Z = 2\mathrm{i}\{\mathrm{d}\eta + \eta(I_n - W)^{-1}\mathrm{d}W\}(I_n - W)^{-1}. \tag{3.5}$$

According to (3.2) and (3.4), we obtain

$$\sigma(Y^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}\,) = 4\sigma((I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,). \tag{3.6}$$

From (3.2)–(3.4), we get

$$\sigma(Y^{-1}\,{}^t V V Y^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}\,) = \text{(a)} + \text{(b)} + \text{(c)} + \text{(d)}, \tag{3.7}$$

where

$$\begin{aligned}
\text{(a)} &:= 4\sigma((I_n - W\overline{W}\,)^{-1}\,{}^t\eta\eta(I_n - \overline{W}W)^{-1}(I_n - \overline{W}\,)(I_n - W)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,),\\
\text{(b)} &:= 4\sigma((I_n - W\overline{W}\,)^{-1}\,{}^t\eta\overline{\eta}(I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,),\\
\text{(c)} &:= 4\sigma((I_n - \overline{W}\,)^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n - \overline{W}W)^{-1}\\
&\quad\times (I_n - \overline{W}\,)(I_n - W)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,),\\
\text{(d)} &:= 4\sigma((I_n - \overline{W}\,)^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,).
\end{aligned}$$

According to (3.2) and (3.5), we get

$$\sigma(Y^{-1}\,{}^t(\mathrm{d}Z)\mathrm{d}\overline{Z}) = \text{(e)} + \text{(f)} + \text{(g)} = \text{(h)}, \tag{3.8}$$

where

$$\begin{aligned}
\text{(e)} &:= 4\sigma((I_n - W\overline{W}\,)^{-1}\,{}^t(\mathrm{d}\eta)\mathrm{d}\overline{\eta}),\\
\text{(f)} &:= 4\sigma((I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - W)^{-1}\,{}^t\eta\mathrm{d}\overline{\eta}),\\
\text{(g)} &:= 4\sigma((I_n - W\overline{W}\,)^{-1}\,{}^t(\mathrm{d}\eta)\overline{\eta}(I_n - \overline{W}\,)^{-1}\mathrm{d}\overline{W}\,),\\
\text{(h)} &:= 4\sigma((I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - W)^{-1}\,{}^t\eta\,\overline{\eta}(I_n - \overline{W}\,)^{-1}\mathrm{d}\overline{W}\,).
\end{aligned}$$

From (3.2)–(3.5), we get

$$-\sigma(VY^{-1}\mathrm{d}\Omega Y^{-1}\,{}^t(\mathrm{d}\overline{Z})) = \text{(i)} + \text{(j)} + \text{(k)} + \text{(l)}, \tag{3.9}$$

where

$$\begin{aligned}
\text{(i)} &:= -4\sigma(\eta(I_n - W)^{-1}(I_n - \overline{W}\,)(I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta})),\\
\text{(j)} &:= -4\sigma((I_n - \overline{W}\,)^{-1}\,{}^t\overline{\eta}\eta(I_n - W)^{-1}(I_n - \overline{W}\,)(I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,),\\
\text{(k)} &:= -4\sigma(\overline{\eta}(I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta})),\\
\text{(l)} &:= -4\sigma((I_n - \overline{W}\,)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W}\,)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}\,).
\end{aligned}$$

Conjugating (3.9), we get

$$-\sigma(VY^{-1}\mathrm{d}\overline{\Omega}Y^{-1}\,{}^t(\mathrm{d}Z)) = \text{(m)} + \text{(n)} + \text{(o)} + \text{(p)}, \tag{3.10}$$

where

$$
\begin{aligned}
(\mathrm{m}) &:= -4\sigma(\overline{\eta}(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta)),\\
(\mathrm{n}) &:= -4\sigma((I_n-W)^{-1}\,{}^t\eta\overline{\eta}(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\mathrm{d}W),\\
(\mathrm{o}) &:= -4\sigma(\eta(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta)),\\
(\mathrm{p}) &:= -4\sigma((I_n-W)^{-1}\,{}^t\eta\eta(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\mathrm{d}W).
\end{aligned}
$$

If we add (f), (i) and (k), we get

$$
(\mathrm{f})+(\mathrm{i})+(\mathrm{k}) = 4\sigma((\eta\overline{W}-\overline{\eta})(I_n-W\overline{W})^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta})). \tag{3.11}
$$

Indeed, transposing the matrix inside (f), we get

$$
(\mathrm{f}) = 4\sigma(\eta(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta})).
$$

Adding (f) and (i) together with (k), we get (3.11) because

$$
\begin{aligned}
&(I_n-W)^{-1}-(I_n-W)^{-1}(I_n-\overline{W})(I_n-W\overline{W})^{-1}\\
&=(I_n-W)^{-1}\{(I_n-W\overline{W})-(I_n-\overline{W})\}(I_n-W\overline{W})^{-1}\\
&=(I_n-W)^{-1}(I_n-W)\overline{W}(I_n-W\overline{W})^{-1}=\overline{W}(I_n-W\overline{W})^{-1}.
\end{aligned}
$$

If we add formulas (g), (m) and (o), we get

$$
(\mathrm{g})+(\mathrm{m})+(\mathrm{o}) = 4\sigma((\overline{\eta}W-\eta)(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta)). \tag{3.12}
$$

Indeed, we can express (g) as

$$
(\mathrm{g}) = 4\sigma(\overline{\eta}(I_n-\overline{W})^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta)).
$$

Adding (g) and (m) together with (o), we get (3.12) because

$$
\begin{aligned}
&(I_n-\overline{W})^{-1}-(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\\
&=(I_n-\overline{W})^{-1}\{(I_n-\overline{W}W)-(I_n-W)\}(I_n-\overline{W}W)^{-1}\\
&=(I_n-\overline{W})^{-1}(I_n-\overline{W})W(I_n-\overline{W}W)^{-1}=W(I_n-\overline{W}W)^{-1}.
\end{aligned}
$$

If we add (a) and (p), we get

$$
(\mathrm{a})+(\mathrm{p}) = -4\sigma((I_n-W\overline{W})^{-1}\,{}^t\eta\eta(I_n-\overline{W}W)^{-1}\overline{W}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}). \tag{3.13}
$$

Indeed, transposing the matrix inside (p), we get

$$
(\mathrm{p}) = -4\sigma((I_n-W\overline{W})^{-1}\,{}^t\eta\eta(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}).
$$

Adding (a) and (p), we get (3.13) because

$$
\begin{aligned}
&(I_n-\overline{W}W)^{-1}(I_n-\overline{W})(I_n-W)^{-1}-(I_n-W)^{-1}\\
&=(I_n-\overline{W}W)^{-1}\{(I_n-\overline{W})-(I_n-\overline{W}W)\}(I_n-W)^{-1}\\
&=(I_n-\overline{W}W)^{-1}(-\overline{W})(I_n-W)(I_n-W)^{-1}=-(I_n-\overline{W}W)^{-1}\overline{W}.
\end{aligned}
$$

Adding (d) and (l), we get

$$(\mathrm{d})+(\mathrm{l}) = -4\sigma(W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n-W\overline{W}\,)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,) \tag{3.14}$$

because

$$\begin{aligned}&(I_n-\overline{W}\,)^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}-(I_n-\overline{W}\,)^{-1}\\ =\,&(I_n-\overline{W}\,)^{-1}\{(I_n-W)-(I_n-\overline{W}W)\}(I_n-\overline{W}W)^{-1}\\ =\,&(I_n-\overline{W}\,)^{-1}(I_n-\overline{W}\,)(-W)(I_n-\overline{W}W)^{-1}=-W(I_n-\overline{W}W)^{-1}.\end{aligned}$$

Adding (h) and (j), we get

$$(\mathrm{h})+(\mathrm{j}) = 4\sigma((I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n-W\overline{W}\,)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,). \tag{3.15}$$

Indeed, transposing the matrix inside (h), we get

$$(\mathrm{h}) = 4\sigma((I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\eta(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,).$$

Adding (h) and (j), we get (3.15) because

$$\begin{aligned}&(I_n-W)^{-1}-(I_n-W)^{-1}(I_n-\overline{W}\,)(I_n-W\overline{W}\,)^{-1}\\ =\,&(I_n-W)^{-1}\{(I_n-W\overline{W}\,)-(I_n-\overline{W}\,)\}(I_n-W\overline{W}\,)^{-1}\\ =\,&(I_n-W)^{-1}(I_n-W)\overline{W}(I_n-W\overline{W}\,)^{-1}=\overline{W}(I_n-W\overline{W}\,)^{-1}.\end{aligned}$$

Transposing the matrix inside (n), we get

$$(\mathrm{n}) = -4\sigma((I_n-W\overline{W}\,)^{-1}(I_n-W)(I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\eta(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,). \tag{3.16}$$

From (3.7)–(3.16), we obtain

$$\begin{aligned}&\sigma(Y^{-1}\,{}^tVVY^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}\,)+\sigma(Y^{-1}\,{}^t(\mathrm{d}Z)\mathrm{d}\overline{Z}\,)\\ &-\sigma(VY^{-1}\mathrm{d}\Omega Y^{-1}\,{}^t(\mathrm{d}\overline{Z}\,))-\sigma(VY^{-1}\mathrm{d}\overline{\Omega}Y^{-1}\,{}^t(\mathrm{d}Z))\\ =\,&(\mathrm{a})+(\mathrm{b})+(\mathrm{c})+(\mathrm{d})+\cdots+(\mathrm{m})+(\mathrm{n})+(\mathrm{o})+(\mathrm{p})\\ =\,&4\sigma((I_n-W\overline{W}\,)^{-1}\,{}^t(\mathrm{d}\eta)\mathrm{d}\overline{\eta}\,)+4\sigma((\eta\overline{W}-\overline{\eta}\,)(I_n-W\overline{W}\,)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta}\,))\\ &+4\sigma((\overline{\eta}W-\eta)(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n-W\overline{W}\,)^{-1}\,{}^t(\mathrm{d}\eta))\\ &-4\sigma((I_n-W\overline{W}\,)^{-1}\,{}^t\eta\eta(I_n-\overline{W}W)^{-1}\overline{W}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,)\\ &-4\sigma(W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n-W\overline{W}\,)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,)\\ &+4\sigma((I_n-W\overline{W}\,)^{-1}\,{}^t\eta\overline{\eta}(I_n-W\overline{W}\,)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,)\\ &+4\sigma((I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n-W\overline{W}\,)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,)\\ &+4\sigma((I_n-\overline{W}\,)^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n-\overline{W}W)^{-1}\\ &\times(I_n-\overline{W}\,)(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,)\\ &-4\sigma((I_n-W\overline{W}\,)^{-1}(I_n-W)(I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\eta(I_n-W)^{-1}\mathrm{d}W(I_n-\overline{W}W)^{-1}\mathrm{d}\overline{W}\,).\end{aligned}$$

Consequently, the complete proof follows from the above formula, (3.6), Theorem 1.1 and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (2.6) of G^J_* on $\mathbb{D}_{n,m}$ through the partial Cayley transform.

4 Proof of Theorem 1.4

From (3.1), (3.4) and (3.5), we get

$$\frac{\partial}{\partial \Omega} = \frac{1}{2\mathrm{i}}(I_n - W)\Big[\ {}^t\Big\{(I_n - W)\frac{\partial}{\partial W}\Big\} - \ {}^t\Big\{{}^t\eta\ {}^t\Big(\frac{\partial}{\partial \eta}\Big)\Big\}\Big], \tag{4.1}$$

$$\frac{\partial}{\partial Z} = \frac{1}{2\mathrm{i}}(I_n - W)\frac{\partial}{\partial \eta}. \tag{4.2}$$

We need the following lemma for the proof of Theorem 1.4. Maass [3] observed the following useful fact.

Lemma 4.1 (a) *Let A be an $m\times n$ matrix and B an $n\times l$ matrix. Assume that the entries of A commute with the entries of B. Then ${}^t(AB) = {}^tB\,{}^tA$.*

(b) *Let A, B and C be a $k\times l$, an $n\times m$ and an $m\times l$ matrix respectively. Assume that the entries of A commute with the entries of B. Then*

$${}^t(A\,{}^t(BC)) = B\,{}^t(A\,{}^tC).$$

Proof The proof follows immediately from a direct computation.

From (3.2), (4.1) and Lemma 4.1, we get the following formula:

$$4\sigma\Big(Y\ {}^t\Big(Y\frac{\partial}{\partial \overline{\Omega}}\Big)\frac{\partial}{\partial \Omega}\Big) = (\alpha) + (\beta) + (\gamma) + (\delta), \tag{4.3}$$

where

$$(\alpha) := \sigma\Big((I_n - W\overline{W}\,)\ {}^t\Big((I_n - W\overline{W}\,)\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial W}\Big),$$

$$(\beta) := -\sigma\Big(\eta(I_n - W)^{-1}(I_n - W\overline{W}\,)\ {}^t\Big((I_n - W\overline{W}\,)\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial \eta}\Big),$$

$$(\gamma) := -\sigma\Big((I_n - W\overline{W}\,)(I_n - \overline{W}\,)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial W}\Big),$$

$$(\delta) := \sigma\Big(\eta(I_n - W)^{-1}(I_n - W\overline{W}\,)(I_n - \overline{W}\,)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big).$$

According to (3.2) and (4.2), we get

$$4\sigma\Big(Y\frac{\partial}{\partial Z}\ {}^t\Big(\frac{\partial}{\partial \overline{Z}}\Big)\Big) = \sigma\Big((I_n - \overline{W}W)\frac{\partial}{\partial \eta}\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)\Big). \tag{4.4}$$

From (3.2), (3.3) and (4.2), we get

$$4\sigma\Big(VY^{-1}\,{}^tV\ {}^t\Big(Y\frac{\partial}{\partial \overline{Z}}\Big)\frac{\partial}{\partial Z}\Big) = (\epsilon) + (\zeta) + (\eta) + (\theta), \tag{4.5}$$

where

$$(\epsilon) := \sigma\Big(\eta(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big),$$

$$(\zeta) := \sigma\Big(\overline{\eta}(I_n - W\overline{W}\,)^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big),$$

$$(\eta) := \sigma\Big(\eta(I_n - W)^{-1}(I_n - \overline{W}\,)(I_n - W\overline{W}\,)^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big),$$

$$(\theta) := \sigma\Big(\overline{\eta}(I_n - \overline{W}\,)^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big).$$

Using (3.2), (3.3), (4.1), (4.2) and Lemma 4.1, we get

$$4\sigma\Big(V\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big) = (\iota)+(\kappa)+(\lambda)+(\mu), \tag{4.6}$$

where

$$(\iota) := \sigma\Big(\overline{\eta}\,{}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big),$$
$$(\kappa) := \sigma\Big(\eta(I_n-W)^{-1}(I_n-\overline{W})\,{}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big),$$
$$(\lambda) := -\sigma\Big(\eta(I_n-W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big),$$
$$(\mu) := -\sigma\Big(\overline{\eta}(I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big).$$

Similarly, we get

$$4\sigma\Big({}^tV\,{}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial\Omega}\Big) = (\nu)+(\xi)+(o)+(\pi), \tag{4.7}$$

where

$$(\nu) := \sigma\Big({}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big),$$
$$(\xi) := \sigma\Big((I_n-W)(I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big),$$
$$(o) := -\sigma\Big(\eta(I_n-W)^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big),$$
$$(\pi) := -\sigma\Big(\eta(I_n-\overline{W}\,)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big).$$

Adding (γ), (ν) and (ξ), we get

$$(\gamma)+(\nu)+(\xi) = \sigma\Big({}^t(\eta-\overline{\eta}W)\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big) \tag{4.8}$$

because

$$-(I_n-W\overline{W}\,)(I_n-\overline{W}\,)^{-1}+(I_n-W)(I_n-\overline{W}\,)^{-1} = -W(I_n-\overline{W}\,)(I_n-\overline{W}\,)^{-1} = -W.$$

Adding (β), (ι) and (κ), we get

$$(\beta)+(\iota)+(\kappa) = \sigma\Big((\overline{\eta}-\eta\overline{W}\,)\,{}^t\Big((I_n-W\overline{W}\,)\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big) \tag{4.9}$$

because

$$-(I_n-W)^{-1}(I_n-W\overline{W}\,)+(I_n-W)^{-1}(I_n-\overline{W}\,) = -(I_n-W)^{-1}(I_n-W)\overline{W} = -\overline{W}.$$

If we add (η) and (o), we get

$$(\eta)+(o) = -\sigma\Big(\eta\overline{W}(I_n-W\overline{W}\,)^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big) \tag{4.10}$$

because

$$\begin{aligned}
&(I_n - W)^{-1}(I_n - \overline{W})(I_n - W\overline{W})^{-1} - (I_n - W)^{-1}\\
=&(I_n - W)^{-1}\{I_n - \overline{W} - (I_n - W\overline{W})\}(I_n - W\overline{W})^{-1}\\
=&(I_n - W)^{-1}(I_n - W)(-\overline{W})(I_n - W\overline{W})^{-1} = -\overline{W}(I_n - W\overline{W})^{-1}.
\end{aligned}$$

If we add (θ) and (μ), we get

$$(\theta) + (\mu) = -\sigma\Big(\overline{\eta}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big) \tag{4.11}$$

because

$$\begin{aligned}
&(I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1} - (I_n - \overline{W})^{-1}\\
=&(I_n - \overline{W})^{-1}\{I_n - W - (I_n - \overline{W}W)\}(I_n - \overline{W}W)^{-1}\\
=&(I_n - \overline{W})^{-1}(I_n - \overline{W})(-W)(I_n - \overline{W}W)^{-1} = -W(I_n - \overline{W}W)^{-1}.
\end{aligned}$$

If we add (δ), (ϵ), (λ) and (π), we get

$$(\delta) + (\epsilon) + (\lambda) + (\pi) = \sigma\Big(\eta\overline{W}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big) \tag{4.12}$$

because

$$\begin{aligned}
&(I_n - W)^{-1}(I_n - W\overline{W})(I_n - \overline{W})^{-1} + (I_n - \overline{W}W)^{-1} - (I_n - W)^{-1} - (I_n - \overline{W})^{-1}\\
=&(I_n - W)^{-1}\{(I_n - W\overline{W}) - (I_n - \overline{W})\}(I_n - \overline{W})^{-1} + (I_n - \overline{W}W)^{-1} - (I_n - \overline{W})^{-1}\\
=&\overline{W}(I_n - \overline{W})^{-1} + (I_n - \overline{W}W)^{-1} - (I_n - \overline{W})^{-1} = -(I_n - \overline{W})(I_n - \overline{W})^{-1} + (I_n - \overline{W}W)^{-1}\\
=& - I_n + (I_n - \overline{W}W)^{-1} = \{-(I_n - \overline{W}W) + I_n\}(I_n - \overline{W}W)^{-1} = \overline{W}W(I_n - \overline{W}W)^{-1}.
\end{aligned}$$

From (4.3) and (4.5)–(4.12), we obtain

$$\begin{aligned}
&\sigma\Big(Y\ {}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big) + \sigma\Big(VY^{-1}\,{}^tV\ {}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial Z}\Big)\\
&+\sigma\Big(V\ {}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big) + \sigma\Big({}^tV\ {}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial\Omega}\Big)\\
=&\ (\alpha) + (\beta) + (\gamma) + (\delta) + \cdots + (\nu) + (\xi) + (o) + (\pi)\\
=&\ \sigma\Big((I_n - W\overline{W})\ {}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial W}\Big) + \sigma\Big({}^t(\eta - \overline{\eta}W)\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial W}\Big)\\
&+\sigma\Big((\overline{\eta} - \eta\overline{W})\ {}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big) - \sigma\Big(\eta\overline{W}(I_n - W\overline{W})^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big)\\
&-\sigma\Big(\overline{\eta}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big)\\
&+\sigma\Big(\overline{\eta}(I_n - W\overline{W})^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big)\\
&+\sigma\Big(\eta\overline{W}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big).
\end{aligned}$$

Consequently, the complete proof follows from (4.4), the above formula, Theorem 1.2 and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (2.6) of G^J_* on $\mathbb{D}_{n,m}$ through the partial Cayley transform.

Remark 4.1 We proved in [11] that the following two differential operators D and $L := \frac{1}{4}\Delta_{n,m;1,1} - D$ on $\mathbb{H}_{n,m}$ defined by

$$D = \sigma\Big(Y \frac{\partial}{\partial Z}\,{}^t\Big(\frac{\partial}{\partial \overline{Z}}\Big)\Big)$$

and

$$L = \sigma\Big(Y\,{}^t\Big(Y\frac{\partial}{\partial \overline{\Omega}}\Big)\frac{\partial}{\partial \Omega}\Big) + \sigma\Big(VY^{-1}\,{}^tV\,{}^t\Big(Y\frac{\partial}{\partial \overline{Z}}\Big)\frac{\partial}{\partial Z}\Big)$$
$$+\,\sigma\Big(V\,{}^t\Big(Y\frac{\partial}{\partial \overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big) + \sigma\Big({}^tV\,{}^t\Big(Y\frac{\partial}{\partial \overline{Z}}\Big)\frac{\partial}{\partial \Omega}\Big)$$

are invariant under the action (1.2) of G^J. By (4.4) and the proof of Theorem 1.4, we see that the following differential operators $\widetilde{D}$ and $\widetilde{L} := \widetilde{\Delta}_{n,m;1,1} - \widetilde{D}$ on $\mathbb{D}_{n,m}$ defined by

$$\widetilde{D} = \sigma\Big((I_n - \overline{W}W)\frac{\partial}{\partial \eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)\Big)$$

and

$$\widetilde{L} = \sigma\Big((I_n - W\overline{W}\,)\,{}^t\Big((I_n - W\overline{W}\,)\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial W}\Big) + \sigma\Big({}^t(\eta - \overline{\eta}W)\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial W}\Big)$$
$$+\,\sigma\Big((\overline{\eta} - \eta\overline{W}\,)\,{}^t\Big((I_n - W\overline{W}\,)\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial \eta}\Big) - \sigma\Big(\eta\overline{W}(I_n - W\overline{W}\,)^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big)$$
$$-\,\sigma\Big(\overline{\eta}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big)$$
$$+\,\sigma\Big(\overline{\eta}(I_n - W\overline{W}\,)^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big)$$
$$+\,\sigma\Big(\eta\overline{W}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big)$$

are invariant under the action (2.6) of G^J_*. Indeed, it is very complicated and difficult at this moment to express the generators of the algebra of all G^J_*-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly. We propose an open problem to find other explicit G^J_*-invariant differential operators on $\mathbb{D}_{n,m}$.

5 Remark on Harmonic Analysis on Siegel-Jacobi Disk

It might be interesting to develop the theory of harmonic analysis on the Siegel-Jacobi disk $\mathbb{D}_{n,m}$. The theory of harmonic analysis on the generalized unit disk $\mathbb{D}_n$ can be done explicitly by the work of Harish-Chandra because $\mathbb{D}_n$ is a symmetric space. However, the Siegel-Jacobi disk $\mathbb{D}_{n,m}$ is not a symmetric space. The work for developing the theory of harmonic analysis on $\mathbb{D}_{n,m}$ explicitly is complicated and difficult at this moment. We observe that this work on $\mathbb{D}_{n,m}$ generalizes the work on the generalized unit disk $\mathbb{D}_n$.

More precisely, if we put $G_* = \mathrm{SU}(n,n) \cap \mathrm{Sp}(n,\mathbb{C})$, then the Jacobi group

$$G^J_* = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \,\middle|\, \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

acts on the Siegel-Jacobi disk $\mathbb{D}_{n,m}$ transitively via the transformation behavior (1.9). It is easily seen that the stabilizer K^J_* of the action (1.9) at the base point $(0,0)$ is given by

$$K^J_* = \left\{ \left(\begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}, (0,0; \mathrm{i}\kappa) \right) \,\middle|\, P \in U(n),\ \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

Therefore, G_*^J/K_*^J is biholomorphic to $\mathbb{D}_{n,m}$ via the correspondence

$$gK_*^J \mapsto g\cdot(0,0), \quad g\in G_*^J.$$

We observe that the Siegel-Jacobi disk $\mathbb{D}_{n,m}$ is not a reductive symmetric space.

Let

$$\Gamma_{n,m} := \mathrm{Sp}(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)},$$

where $\mathrm{Sp}(n,\mathbb{Z})$ is the Siegel modular group of degree n and

$$H_{\mathbb{Z}}^{(n,m)} = \{(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu,\kappa \text{ are integral}\}.$$

We set

$$\Gamma_{n,m}^* := T_*^{-1}\Gamma_{n,m}T_*,$$

where T_* was already defined in Section 2. Clearly, the arithmetic subgroup $\Gamma_{n,m}^*$ acts on $\mathbb{D}_{n,m}$ properly continuously. We can describe a fundamental domain $\mathcal{F}_{n,m}^*$ for $\Gamma_{n,m}^*\setminus\mathbb{D}_{n,m}$ explicitly using a partial Cayley transform and a fundamental domain $\mathcal{F}_{n,m}$ for $\Gamma_{n,m}\setminus\mathbb{H}_{n,m}$ which is described explicitly in [13]. The G_*^J-invariant metric $d\widetilde{s}_{n,m;A,B}$ on $\mathbb{D}_{n,m}$ induces a metric on $\mathcal{F}_{n,m}^*$ naturally. It may be interesting to investigate the spectral theory of the Laplacian $\widetilde{\Delta}_{n,m;A,B}$ on a fundamental domain $\mathcal{F}_{n,m}^*$. But this work is very complicated and difficult at this moment.

For instance, we consider the case $n=m=1$ and $A=B=1$. In this case,

$$G_*^J = \left\{ \left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi,\overline{\xi};\mathrm{i}\kappa) \right) \,\middle|\, p,q,\xi\in\mathbb{C},\ |p|^2-|q|^2=1,\ \kappa\in\mathbb{R} \right\}$$

and

$$K_*^J = \left\{ \left(\begin{pmatrix} p & 0 \\ 0 & \overline{p} \end{pmatrix}, (0,0;\mathrm{i}\kappa) \right) \,\middle|\, p\in\mathbb{C},\ |p|=1,\ \kappa\in\mathbb{R} \right\}.$$

$d\widetilde{s}_{1,1;1,1}$ is a G_*^J-invariant Riemannian metric on $\mathbb{D}_{1,1}=\mathbb{D}_1\times\mathbb{C}$ (see Theorem 1.3) and $\widetilde{\Delta}_{1,1;1,1}$ is its Laplacian. It is well-known that the theory of harmonic analysis on the unit disk $\mathbb{D}_1$ has been well developed explicitly (see [3, pp. 29–72]). I think that so far nobody has investigated the theory of harmonic analysis on $\mathbb{D}_{1,1}$ explicitly. For example, inversion formula, Plancherel formula, Paley-Wiener theorem on $\mathbb{D}_{1,1}$ have not been described explicitly until now. It seems that it is interesting to develop the theory of harmonic analysis on the Siegel-Jacobi disk $\mathbb{D}_{1,1}$ explicitly.

Finally, we mention that it may be interesting to investigate differential operators on $\mathbb{D}_{n,m}$ which are invariant under the natural action (1.9) of the Jacobi group G_*^J in detail (see [14]).

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