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Stochastic Fractional Anderson Models with Fractional Noises***

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Abstract The authors are concerned with a class of one-dimensional stochastic Anderson models with double-parameter fractional noises, whose differential operators are fractional. A unique solution for the model in some appropriate Hilbert space is constructed. Moreover, the Lyapunov exponent of the solution is estimated, and its Hölder continuity is studied. On the other hand, the absolute continuity of the solution is also discussed.

Keywords Anderson models, Fractional noises, Lyapunov exponent, Hölder continuity, Absolute continuity

2000 MR Subject Classification 34A34, 49N60

1 Introduction

In this article, we are concerned with the following stochastic fractional Anderson model with a fractional noise potential:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\lambda u + W^H \diamond u, \quad \text{in } [0,T]\times\mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\Delta_\lambda = -(-\frac{1}{4\pi^2}\Delta)^{\frac{\lambda}{2}} = -(-\frac{1}{4\pi^2}\frac{\partial^2}{\partial x^2})^{\frac{\lambda}{2}}$ with $\lambda > 0$, $W^H(t,x) = \frac{\partial^2}{\partial t\partial x}B^H(t,x)$ with $H = (h_1,h_2) \in (\frac{1}{2},1)\times(\frac{1}{2},1)$ is the formal derivative of a double-parameter fractional field B^H (see Section 2), and "$\diamond$" denotes Skorokhod integral. The precise meaning of a solution of (1.1) will be stated in Section 2.

First, let us recall some related works on stochastic partial differential equations (SPDEs) as follows:

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x) + f(u(t,x)) + u(t,x)\dot{F}(t,x), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, \tag{1.2}$$

where $L = \frac{1}{2}\Delta$ or $L = -\Delta^2$, f is some specified function and $\dot{F}$ denotes a white noise or a fractional noise on some probability space $(\Omega,\mathcal{F},P)$. In [25] Uemura treated the 1-dimensional heat equation with $\dot{F}(t,x) = \dot{w}(x)$, where $\dot{w}(x)$ is a space noise, and studied the Hölder continuity of the solution on the model, whereas, in the case of $\dot{F}$ being a fractional noise, Nualart

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and Ouknine [19] discussed the existence and uniqueness of the solution under some restrictive conditions.

When the drift term $f = 0$, equation (1.2) is called an Anderson model. Hu got the Lyapunov exponent estimates on the solutions of the equations with fractional noise potentials under appropriate assumptions on Hurst parameter $H = (h_0, h_1, \cdots, h_d)$ in [10]. Furthermore, for the (deterministic) Cahn-Hilliard equation, the case of $L = -\Delta^2$ was first proposed in material science. A stochastic version of the equation was developed by Cardon and Weber in [6], who proposed stochastic Cahn-Hilliard equations with space-time white noises with space dimensions $d \leq 3$. Bo and Wang [5] considered stochastic Cahn-Hilliard equations with Lévy space-time white noises, and they established the local mild solution of the equation. Moreover, Bo et al. [2, 4] proposed a fourth-order stochastic Anderson model and Cahn-Hilliard equation with fractional noises, and discussed the existence, uniqueness of the solution.

On the other hand, stochastic fractional partial differential equations have been widely developed. For example, in [1, 8], the authors proved the existence, uniqueness and regularity of the solution for a stochastic fractional Laplacian operator partial differential equation driven by a space-time white noise in one dimension. Moreover, there are some papers discussing the stochastic fractional differential operator heat equations with Lévy noises and fractional noises (see, e.g., [12]). In particular, the main results of Bo et al. [2] can be covered by this paper as the case of $\lambda = 4$.

Motivated by these works, we now suggest a new "Anderson model" with a fractional Laplacian operator, and the noise term is a fractional noise, i.e., equation (1.1). In this article, we shall establish the existence and uniqueness of the solution of (1.1) on some Hilbert space. Then we estimate the Lyapunov exponent of the solution by a continuous embedding theorem and some estimates of the Green function. Moreover, we will prove the Hölder continuity of the solution and give the Hölder continuous order of the solution of (1.1). Another objective of this paper is to discuss the absolute continuity of the solution through Malliavin calculus.

The rest of this paper is organized as follows. In the coming section, we will give the definitions of multiple stochastic integrals with respect to double-parameter fractional noises and define a solution of (1.1), as well as the introduction of the Malliavin calculus. The existence and Lyapunov exponent estimate of the solution will be considered in Section 3. Section 4 is devoted to studying the Hölder continuity of the solution of (1.1). We will discuss the absolute continuity of the solution in the last section.

2 Preliminaries

In this section, we will define a multiple stochastic integral with respect to a double-parameter fractional noise (see also [2]), and then define a solution of (1.1) in S_ρ sense after proposing the Green function and inducting some properties. On the other hand, we will introduce the Malliavin calculus with respect to fractional noises.

2.1 Skorokhod integral

Definition 2.1 *A double-parameter fractional Brownian field* $B^H = \{B^H(t, x) : (t, x) \in$

$[0,T]\times\mathbb{R}\}$ *with Hurst parameter* $H=(h_1,h_2)$ *for* $h_i\in(0,1)$ $(i=1,2)$ *is a centered Gaussian field with covariance*

$$\mathbf{E}[B^H(t,x)B^H(s,y)]=\frac{1}{4}[t^{2h_1}+s^{2h_1}-|t-s|^{2h_1}][|x|^{2h_2}+|y|^{2h_2}-|x-y|^{2h_2}] \tag{2.1}$$

for all $t,s\in[0,T]$ *and* $x,y\in\mathbb{R}$.

For $x,y\in\mathbb{R}$ and $h\in(0,1)$, we denote

$$\varphi_h(x-y):=h(2h-1)|x-y|^{2h-2}.$$

Let $n=1,2,\cdots$. Define $\mathcal{H}_n$ by

$$\begin{aligned}\mathcal{H}_n:=\Big\{&f:([0,T]\times\mathbb{R})^n\to\mathbb{R};\ f\text{ is measurable and}\\ &\int_{[0,T]^{2n}}\int_{\mathbb{R}^{2n}}\prod_{i=1}^n\varphi_{h_1}(s_i-r_i)\varphi_{h_2}(x_i-y_i)f(s,x)f(r,y)\\ &\times\mathrm{d}x_1\cdots\mathrm{d}x_n\mathrm{d}y_1\cdots\mathrm{d}y_n\mathrm{d}r_1\cdots\mathrm{d}r_n\mathrm{d}s_1\cdots\mathrm{d}s_n<\infty\Big\},\end{aligned}$$

which is a Hilbert space, and the inner product of $f,g\in\mathcal{H}_n$ is defined by

$$\begin{aligned}\langle f,g\rangle_{\mathcal{H}_n}:=&\int_{[0,T]^{2n}}\int_{\mathbb{R}^{2n}}\prod_{i=1}^n\varphi_{h_1}(s_i-r_i)\varphi_{h_2}(x_i-y_i)f(s,x)g(r,y)\\ &\times\mathrm{d}x_1\cdots\mathrm{d}x_n\mathrm{d}y_1\cdots\mathrm{d}y_n\mathrm{d}r_1\cdots\mathrm{d}r_n\mathrm{d}s_1\cdots\mathrm{d}s_n.\end{aligned} \tag{2.2}$$

Then we have a sequence of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$.

First we briefly introduce the following stochastic integral as follows:

$$\Big\{\int_0^t\int_{\mathbb{R}}f(s,x)B^H(\mathrm{d}s,\mathrm{d}x);t\in[0,T]\Big\},\quad f\in\mathcal{H}_1.$$

For the stochastic integral, it is easy to check the following properties.

Proposition 2.1 *For* $f,g\in\mathcal{H}_1$, *we have*

$$\begin{aligned}&\mathbf{E}\Big[\int_0^T\int_{\mathbb{R}}f(s,x)B^H(\mathrm{d}s,\mathrm{d}x)\Big]=0,\\ &\mathbf{E}\Big[\int_0^T\int_{\mathbb{R}}f(s,x)B^H(\mathrm{d}s,\mathrm{d}x)\int_0^T\int_{\mathbb{R}}g(s,x)B^H(\mathrm{d}s,\mathrm{d}x)\Big]=\langle f,g\rangle_{\mathcal{H}_1}.\end{aligned}$$

Proof See [10]. We omit the details.

Next, we will introduce the multiple integral for fractional noises. Suppose that $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_1$. Define

$$\begin{aligned}\widetilde{\mathcal{H}}_n:=\Big\{&f\in\mathcal{H}_n;\ f((s_1,u_1),\cdots,(s_n,u_n))=f((s_{\sigma(1)},u_{\sigma(1)}),\cdots,(s_{\sigma(n)},u_{\sigma(n)}))\\ &\text{for all permutations }\sigma=\{\sigma(1),\cdots,\sigma(n)\}\text{ of }\{1,2,\cdots,n\}\Big\}.\end{aligned}$$

If $f\in\widetilde{\mathcal{H}}_n$, then we say that f is “symmetric”. And define

$$\mathcal{C}_n:=\Big\{f\in\widetilde{\mathcal{H}}_n;\ f=\sum_{\text{finite sum}}a_{i_1,\cdots,i_n}e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_n},\ a_{i_1,\cdots,i_n}\in\mathbb{R}\Big\},$$

where $\otimes$ denotes the symmetric tensor product. Let $H_n(x)$ be the Hermite polynomial of degree $n \in \mathbb{N} \cup \{0\}$, i.e.,

$$H_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathrm{e}^{-\frac{x^2}{2}}], \quad x \in \mathbb{R}.$$

For $e \in \mathcal{H}_1$ and $\|e\|_{\mathcal{H}_1} = 1$, define a multiple integral of Itô-type of the function $e^{\otimes n}$ by

$$\begin{aligned}&\int_{[0,T]^n}\int_{\mathbb{R}^n} e^{\otimes n}(s_1, \cdots, s_n, u_1, \cdots, u_n) B^H(\mathrm{d}s_1, \mathrm{d}u_1) \cdots B^H(\mathrm{d}s_n, \mathrm{d}u_n)\\ &= H_n\Big(\int_{[0,T]}\int_{\mathbb{R}} e(r, z) B^H(\mathrm{d}r, \mathrm{d}z)\Big).\end{aligned} \tag{2.3}$$

Then

$$\begin{aligned}&I_n(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n})\\ &= \int_{[0,T]^n}\int_{\mathbb{R}^n} e_{i_1} \otimes \cdots \otimes e_{i_n}(s_1, \cdots, s_n, u_1, \cdots, u_n) B^H(\mathrm{d}s_1, \mathrm{d}u_1) \cdots B^H(\mathrm{d}s_n, \mathrm{d}u_n)\end{aligned}$$

by the polarization argument (see, e.g., [7, 10]).

For each $f \in \mathcal{C}_n$, we have

$$I_n(f) = \sum_{\text{finite sum}} a_{i_1, \cdots, i_n} I_n(e_{i_1} \otimes \cdots \otimes e_{i_n}). \tag{2.4}$$

Then the following isometry holds:

$$\mathbf{E}|I_n(f)|^2 = n! \|f\|^2_{\mathcal{H}_n}. \tag{2.5}$$

Note that for $f \in \widetilde{\mathcal{H}}_n$ there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{C}_n$ such that $f_k \to f$ in $\mathcal{C}_n$. It follows from (2.5) that $\{I_n(f_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, and the limit point of $I_n(f_k)$ (as $k \to \infty$) is independent of the choice of $\{f_k\}_{k\in\mathbb{N}}$. We call the limit point the multiple integral of Itô-type and denote it by

$$\begin{aligned}I_n(f) &:= \int_{[0,T]^n}\int_{\mathbb{R}^n} f(s, x) B^H(\mathrm{d}s_1, \mathrm{d}x_1) \cdots B^H(\mathrm{d}s_n, \mathrm{d}x_n)\\ &= \lim_{k\to\infty} I_n(f_k), \quad \text{in } L^2(\Omega) \text{ sense.}\end{aligned} \tag{2.6}$$

It is easy to check that for $f, g \in \widetilde{\mathcal{H}}_n$ there holds

$$\mathbf{E}[I_n(f)I_n(g)] = n!\langle f, g\rangle_{\mathcal{H}_n}. \tag{2.7}$$

Let $F = \bigoplus_{n=0}^{\infty} F_n$, where F_n is the nth chaos of F (see, e.g., [11]). For each $\rho \in \mathbb{R}$, we introduce a Hilbert space

$$S_\rho = \Big\{F = \bigoplus_{n=0}^{\infty} F_n;\ \sum_{n=0}^{\infty} [n!]^\rho \mathbf{E}|F_n|^2 < \infty\Big\},$$

and define

$$\|F\|_\rho := \sqrt{\sum_{n=0}^{\infty} [n!]^\rho \mathbf{E}|F_n|^2}\,.$$

In particular, if $\rho = 0$, then $S_\rho = L^2(\Omega)$.

2.2 Definition of the solution

In order to define the solution of (1.1), we first introduce the fractional Laplacian Δ_λ with $\lambda > 0$ and the symmetric fractional derivative of order λ on $\mathbb{R}$. This is a non-local operator defined via the Fourier transform $\mathcal{F}$:

$$\mathcal{F}(\Delta_\lambda \upsilon)(x) = -|x|^\lambda \mathcal{F}(\upsilon)(x).$$

The Green function $G_\lambda(t,x)$ associated to equation (1.1) on $[0,T]\times\mathbb{R}$ is the fundamental solution of the Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t} G(t,x) = \Delta_\lambda G(t,x), \quad \text{in } [0,T]\times\mathbb{R}, \\ G(0,x) = \delta_0(x), \end{cases} \tag{2.8}$$

where δ_0 is the Dirac distribution. Using Fourier transform, we see that $G_\lambda(t,x)$ is given by

$$G_\lambda(t,x) = \mathcal{F}^{-1}(\mathrm{e}^{-t|\,\cdot\,|^\lambda})(x) = \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}\pi x\xi}\mathrm{e}^{-t|\xi|^\lambda}\mathrm{d}\xi = \mathcal{F}(\mathrm{e}^{-t|\,\cdot\,|^\lambda})(x).$$

The function $G_\lambda(t,x)$ has the following properties (see, e.g., [1, 8]), which will be used later on.

Lemma 2.1 *For $\lambda \in (0,2]$, we have the following cases.*

(1) *For any $t \in (0,+\infty)$ and $x \in \mathbb{R}$, there hold*

$$G_\lambda(t,x) > 0 \quad \textit{and} \quad \int_{\mathbb{R}} G_\lambda(t,x)\mathrm{d}x = 1.$$

(2) $\frac{\partial^n G_\lambda}{\partial x^n}(t,x) = t^{-\frac{n+1}{\lambda}}\frac{\partial^n G_\lambda}{\partial y^n}(1,y)\big|_{y=t^{-\frac{1}{\lambda}}x}$ *for all $n \geq 0$. In particular, when $n = 0$, it is called the scaling property. That is*

$$G_\lambda(t,x) = t^{-\frac{1}{\lambda}}G_\lambda(1,t^{-\frac{1}{\lambda}}x).$$

(3) *G_λ is C^∞ on $(0,\infty)\times\mathbb{R}$, and for $n \geq 0$, there exists a $C_n > 0$ such that for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$,*

$$\Big|\frac{\partial^n G_\lambda}{\partial x^n}(t,x)\Big| \leq \frac{1}{t^{\frac{1+n}{\lambda}}}\frac{C_n}{1+t^{-\frac{2}{\lambda}}|x|^2}.$$

(4) *For any $(s,t) \in \mathbb{R}_+\times\mathbb{R}_+$,*

$$G_\lambda(s,\,\cdot\,) * G_\lambda(t,\,\cdot\,) = G_\lambda(s+t,\,\cdot\,).$$

(5) $\int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x G_\lambda^\alpha(t,x) < \infty$ *if and only if $\frac{1}{2} < \alpha < 1+\lambda$.*

Remark 2.1 Throughout the paper, we restrict $\lambda \in (1,2]$.

For $k \in \mathbb{N}$, set

$$f_k^{(\lambda)}(t,x;s_1,z_1,\cdots,s_k,z_k) := \int_{\mathbb{R}} G_\lambda(t-s_k,x-z_k)\cdots G_\lambda(s_2-s_1,z_2-z_1)G_\lambda(s_1,z_1-y)u_0(y)\mathrm{d}y$$

and

$$\widetilde{f}_k^{(\lambda)}(t,x;s_1,z_1,\cdots,s_k,z_k) = \mathrm{Sym}[f_k^{(\lambda)}(t,x;s_1,z_1,\cdots,s_k,z_k)],$$

where "Sym" denotes the symmetrization with respect to k variables $(s_1,z_1),\cdots,(s_k,z_k)$.

Now, we define a solution of (1.1) in S_ρ sense.

Definition 2.2 *We say that a stochastic field* $\{u(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ *is a solution of* (1.1) *in* S_ρ *sense, if* $u(t,x)=\lim_{n\to\infty}u_n(t,x)$ *on* S_ρ *(i.e.,* $\|u(t,x)-u_n(t,x)\|_\rho\to 0$ *for each* $(t,x)\in[0,T]\times\mathbb{R}$*), where*

$$\begin{cases} u_0(t,x)=G_\lambda(t)*u_0(x), \\ u_n(t,x)=\displaystyle\sum_{k=0}^{n} I_k(\widetilde{f}_k^{(\lambda)})(t,x), \quad n=1,2,\cdots, \end{cases}$$

and $I_k(\widetilde{f}_k^{(\lambda)})(t,x)$ *is defined by* (2.6)*,* $I_0(\widetilde{f}_0^{(\lambda)})(t,x)=u_0(t,x)$*.*

The following lemma gives an embedding from $L^{\frac{1}{h}}$ to $\mathcal{H}_1$ (see [15]), which is useful for our derivations below.

Lemma 2.2 *If* $h\in(\frac{1}{2},1)$ *and* $f,g\in L^{\frac{1}{h}}([a,b])$*, then*

$$\int_a^b\int_a^b f(u)g(v)|u-v|^{2h-2}\mathrm{d}u\mathrm{d}v\le C(h)\|f\|_{L^{\frac{1}{h}}([a,b])}\|g\|_{L^{\frac{1}{h}}([a,b])},$$

where $C(h)>0$ *is a constant depending only on* h*.*

In fact, if $a=-\infty$ or $b=+\infty$, the above inequality also holds.

2.3 Malliavin calculus with respect to fractional noises

Note that $\{B^H(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ is Gaussian, so we will develop the Malliavin calculus for fractional noises (see, e.g., [22]).

Let $B^H(h)=\int_0^T\int_{\mathbb{R}}h(t,x)B^H(\mathrm{d}x,\mathrm{d}t)$ for $h\in\mathcal{H}_1$, and let Λ be the class of smooth and cylindrical random variables of the form

$$F=f(B^H(h_1),\cdots,B^H(h_n)), \tag{2.9}$$

where $f\in C_b^\infty(\mathbb{R}^n)$ (the set of all bounded C^∞ functions with bounded derivatives of all orders) and $h_i\in\mathcal{H}_1$ ($i=1,\cdots,n$ and $n\in\mathbb{N}$). For each $F\in\Lambda$, define the derivative $\mathrm{D}_{t,x}F$ by

$$\mathrm{D}_{t,x}F:=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(B^H(h_1),\cdots,B^H(h_n))h_i(t,x).$$

Let $\mathcal{D}^{1,2}$ be the completion of Λ under the norm

$$\|F\|_{1,2}^2=\mathbf{E}[F^2+\|\mathrm{D}F\|_{\mathcal{H}_1}^2].$$

Then $\mathcal{D}^{1,2}$ is the domain of the closed operator D. For each $h\in\mathcal{H}_1$ and $F\in\Lambda$, define

$$\mathrm{D}_hF:=\lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}f(B^H(h_1)+\varepsilon\langle h_1,h\rangle_{\mathcal{H}_1},\cdots,B^H(h_n)+\varepsilon\langle h_n,h\rangle_{\mathcal{H}_1}),$$

which may be extended as a closed operator on $L^2(\Omega)$ (with the domain $\mathcal{D}_h$ being the closure of Λ) under the norm

$$\|F\|_h^2=\mathbf{E}[F^2+|\mathrm{D}_hF|^2].$$

Let $\{h_n : n \geq 1\}$ be an orthonormal basis of $\mathcal{H}_1$. Then $F \in \mathcal{D}^{1,2}$ if and only if $F \in \mathcal{D}_{h_n}$ for each $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \mathbf{E}|\mathrm{D}_{h_n} F|^2 < \infty.$$

In this case,

$$\mathrm{D}_h F = \langle \mathrm{D}F, h \rangle_{\mathcal{H}_1}.$$

On the other hand, the divergence operator δ is the adjoint of the derivative operator D characterized by

$$\mathbf{E}[F\delta(u)] = \mathbf{E}\langle \mathrm{D}F, u\rangle_{\mathcal{H}_1} \quad \text{for any } F \in \Lambda,$$

where $u \in L^2(\Omega, \mathcal{H}_1)$. Then, we denote the domain of δ by $\mathrm{Dom}\,\delta$, which is the set of all functions $u \in L^2(\Omega, \mathcal{H}_1)$ such that

$$\mathbf{E}|\langle \mathrm{D}F, u\rangle_{\mathcal{H}_1}| \leq C\|F\|_{L^2(\Omega)},$$

where C is some positive constant.

Proposition 2.2 *Let $A \in \mathcal{F}$, and let F be a square integrable random variable that is measurable with respect to the σ-field $\mathcal{F}_{A^c}$. Then*

$$\mathrm{D}F\mathbf{I}_A = 0, \quad a.s.$$

Remark 2.2 Let $\{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}\}$ be an $\{\mathcal{F}_t : t \in [0,T]\}$-adapted random field. According to Proposition 2.2, $\mathrm{D}_{s,y}u(t,x) = 0$, a.s. for any $0 \leq t < s \leq T$ and $x, y \in \mathbb{R}$.

Proposition 2.3 *Let $F \in \mathcal{D}^{1,2}$. If $\|\mathrm{D}F\|^2_{\mathcal{H}_1} > 0$, a.s., then the law of the random variable F is absolutely continuous with respect to the Lebesgue measure.*

Proof The proof is a standard argument like that in [23, Theorem 2.1.3].

3 Lyapunov Exponent Estimate of the Solution

In this section, we will establish the existence and uniqueness of the solution for (1.1) and then give a Lyapunov exponent estimate on the solution. We state the main theorem as follows.

Theorem 3.1 (Existence, Uniqueness and Lyapunov Exponent Estimate) *Let $h_1, h_2 \in (\frac{1}{2}, 1)$, $\lambda(2h_1 - 1) + h_2 > 1$ and $p \in (\frac{1}{2h_1-1}, \frac{\lambda}{1-h_2})$. If $u_0 \in L^\infty(\mathbb{R})$, then (1.1) has a unique solution $\{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}\}$ on S_ρ (for each $\rho < \frac{(2-p)\lambda - 2(1-h_2)p}{p\lambda}$). Moreover,*

$$\limsup_{t\to\infty} \log\Big(\sup_{x\in\mathbb{R}} \|u(t,x)\|^2_\rho\Big) t^{-\kappa} \leq \widetilde{C}(h_1, h_2)$$

for $\kappa = \frac{(4h_1+h_2-1)}{2}(\frac{4-(3-h_2)p}{2p} - \rho)^{-1}$, and $\widetilde{C}(h_1,h_2)$ is some positive constant.

In order to prove the theorem, we need the following technical lemmas.

Lemma 3.1 *For each $k \in \mathbb{N}$, define*

$$\begin{aligned}\Phi_k^{(\lambda)}(t,x,s,r) := \int_{\mathbb{R}^{2k}} \prod_{i=1}^{k} \varphi_{h_2}(u_i - v_i)|G_\lambda(t - s_k, x - u_k)| \cdots |G_\lambda(s_2 - s_1, u_2 - u_1)| \\ \times |G_\lambda(t - r_k, x - v_k)| \cdots |G_\lambda(r_2 - r_1, v_2 - v_1)| \mathrm{d}u_1 \cdots \mathrm{d}u_k \mathrm{d}v_1 \cdots \mathrm{d}v_k,\end{aligned}$$

where $(t,x)\in[0,T]\times\mathbb{R}$, $s=(s_1,\cdots,s_k)$ and $r=(r_1,\cdots,r_k)$. Then there exists a positive constant $C(\lambda,h_2)$ depending on λ and h_2 such that

$$\Phi_k^{(\lambda)}(t,x,s,r)\le[C(\lambda,h_2)]^k\prod_{i=1}^k(s_{i+1}-s_i)^{\frac{h_2-1}{\lambda}}(r_{i+1}-r_i)^{\frac{h_2-1}{\lambda}} \tag{3.1}$$

with $s_{k+1}=r_{k+1}=t$.

Proof We first consider $\widetilde{\Phi}_1^{(\lambda)}(s_1,s_2,r_1,r_2,u_2,v_2)$ by

$$\widetilde{\Phi}_1^{(\lambda)}(s_1,s_2,r_1,r_2,u_2,v_2):=\int_{\mathbb{R}^2}\varphi_{h_2}(u_1-v_1)|G_\lambda(s_2-s_1,u_2-u_1)||G_\lambda(r_2-r_1,v_2-v_1)|\mathrm{d}u_1\mathrm{d}v_1.$$

By Lemma 2.2, we get

$$\widetilde{\Phi}_1^{(\lambda)}(s_1,s_2,r_1,r_2,u_2,v_2)\le\text{const.}\|G_\lambda(s_2-s_1,u_2-\,\cdot\,)\|_{L^{\frac{1}{h_2}}(\mathbb{R})}\|G_\lambda(r_2-r_1,v_2-\,\cdot\,)\|_{L^{\frac{1}{h_2}}(\mathbb{R})},$$

where

$$\begin{aligned}\|G_\lambda(s_2-s_1,u_2-\,\cdot\,)\|_{L^{\frac{1}{h}}(\mathbb{R})}&=\Big(\int_{\mathbb{R}}|G_\lambda(s_2-s_1,u_2-u_1)|^{\frac{1}{h_2}}\mathrm{d}u_1\Big)^{h_2}\\&=\Big(\int_{\mathbb{R}}(s_2-s_1)^{-\frac{1}{\lambda h_2}}|G_\lambda(1,(s_2-s_1)^{-\frac{1}{\lambda}}(u_2-u_1))|^{\frac{1}{h_2}}\mathrm{d}u_1\Big)^{h_2}\\&=\Big(\int_{\mathbb{R}}(s_2-s_1)^{\frac{1}{\lambda}-\frac{1}{\lambda h_2}}|G_\lambda(1,u_1)|^{\frac{1}{h_2}}\mathrm{d}u_1\Big)^{h_2}\\&\le(s_2-s_1)^{h_2(\frac{1}{\lambda}-\frac{1}{\lambda h_2})}\int_{\mathbb{R}}\Big(\frac{C}{1+|u_1|^2}\Big)^{\frac{1}{h_2}}\mathrm{d}u_1\\&\le C(\lambda,h_2)(s_2-s_1)^{\frac{1}{\lambda}(h_2-1)}.\end{aligned}$$

So

$$\widetilde{\Phi}_1^{(\lambda)}(s_1,s_2,r_1,r_2,u_2,v_2)\le C(\lambda,h_2)((s_2-s_1)(r_1-r_2))^{\frac{1}{\lambda}(h_2-1)}.$$

Since the estimate of $\widetilde{\Phi}_1(s_1,s_2,r_1,r_2,u_2,v_2)$ is independent of time parameter (u_2,v_2), one can prove (3.1) similarly for the cases $k\ge2$. Thus the proof of the lemma is completed.

Let us recall the definition of the symmetrical function. Note

$$\widetilde{f}_k^{(\lambda)}(t,x,(s_1,u_1),\cdots,(s_n,u_n))=\frac{1}{n!}\sum_\sigma f_k^{(\lambda)}(t,x,(s_{\sigma(1)},u_{\sigma(1)}),\cdots,(s_{\sigma(n)},u_{\sigma(n)})), \tag{3.2}$$

where the sum is taken over all permutations $\sigma=\{\sigma(1),\cdots,\sigma(n)\}$ of $\{1,2,\cdots,n\}$. For each $k\in\mathbb{N}\cup\{0\}$ and $(t,x)\in[0,T]\times\mathbb{R}$, let

$$\Psi_k^{(\lambda)}(t,x)=\mathbf{E}|I_k(\widetilde{f}_k^{(\lambda)}(t,x))|^2. \tag{3.3}$$

Recall Definition 2.2. By (2.7) and (3.3), we have

$$\Psi_k^{(\lambda)}(t,x)=k!\|\widetilde{f}_k^{(\lambda)}(t,x)\|_{\mathcal{H}_n}^2\le k!\|f_k^{(\lambda)}(t,x)\|_{\mathcal{H}_n}^2. \tag{3.4}$$

Lemma 3.2 *If $h_1, h_2 \in (\frac{1}{2}, 1)$, $\lambda(2h_1 - 1) + h_2 > 1$ and $p \in (\frac{1}{2h_1-1}, \frac{\lambda}{1-h_2})$, then there exists a positive constant $C(\lambda, h_1, h_2)$ depending on λ, h_1 and h_2, such that, for all $k \in \mathbb{N}$,*

$$\Psi_k^{(\lambda)}(t,x) \le [C(\lambda, h_1, h_2)]^k k! \frac{t^{\frac{2(\lambda h_1 + h_2 - 1)k}{\lambda}}}{[\Gamma(k(1 - \frac{p(1-h_2)}{\lambda}) + 1)]^{\frac{2}{p}}}, \quad (t,x) \in [0,T] \times \mathbb{R},$$

where $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} \mathrm{d}t$ is the Gamma function.

Proof This proof is similar to that of [2, Lemma 3.2], in which we only use $\frac{1}{\lambda}(h_2 - 1)$ instead of $\frac{h_2 - 1}{4}$. So the details are omitted.

Lemma 3.3 *Let $h_1, h_2 \in (\frac{1}{2}, 1)$, $\lambda(2h_1 - 1) + h_2 > 1$ and $p \in (\frac{1}{2h_1-1}, \frac{\lambda}{1-h_2})$. Then for each $n \in \mathbb{N}$, $u_n(t,x)$ in Definition 2.2 is an element on S_ρ with $\rho < \frac{(2-p)\lambda - 2(1-h_2)p}{p\lambda}$.*

Proof We will briefly prove this lemma. For more details, please refer to [2].

Using Stirling's formula

$$\Gamma(x+1) = K(x) x^x \mathrm{e}^{-x}, \quad x > -1,$$

where the function $K(x)$ satisfies $\theta^{-x} \le K(x) \le \theta^x$ for some constant $\theta > 0$, we can obtain the following estimates. Let $\beta_\lambda := 1 - \frac{p(1-h_2)}{\lambda} \in (0,1)$. Then we have

$$\frac{[k!]^{\rho+1}}{[\Gamma(k\beta_\lambda + 1)]^{\frac{2}{p}}} \le (\theta^{\frac{4\beta_\lambda}{p}})^k \Big[\frac{(\frac{2\beta_\lambda}{p} - (\rho+1))^{\frac{2\beta_\lambda}{p} - (\rho+1)}}{\beta_\lambda^{\frac{2\beta}{p}}} \Big]^k \frac{1}{\Gamma(k(\frac{2\beta_\lambda}{p} - (\rho+1)) + 1)}$$

and

$$\begin{aligned} \|u_n(t,x)\|_\rho^2 &= \sum_{k=0}^n [k!]^\rho \Psi_k^{(\lambda)}(t,x) \\ &\le \sum_{k=0}^\infty [C(\lambda, h_1, h_2)]^k (\theta^{\frac{4\beta_\lambda}{p}})^k \Big[\frac{(\frac{2\beta_\lambda}{p} - (\rho+1))^{\frac{2\beta_\lambda}{p} - (\rho+1)}}{\beta_\lambda^{\frac{2\beta_\lambda}{p}}} \Big]^k \frac{t^{\frac{2(\lambda h_1 + h_2 - 1)k}{\lambda}}}{\Gamma(k(\frac{2\beta_\lambda}{p} - (\rho+1)) + 1)} \\ &\le \mathrm{E}_{\frac{2\beta_\lambda}{p} - (\rho+1)}(\widetilde{C}(\lambda, h_1, h_2) t^{\frac{2(\lambda h_1 + h_2 - 1)}{\lambda}}), \end{aligned}$$

where

$$\widetilde{C}(\lambda, h_1, h_2) := C(\lambda, h_1, h_2) \theta^{\frac{4\beta_\lambda}{p}} \frac{(\frac{2\beta_\lambda}{p} - (\rho+1))^{\frac{2\beta_\lambda}{p} - (\rho+1)}}{\beta_\lambda^{\frac{2\beta_\lambda}{p}}} > 0$$

and $\mathrm{E}_r(z)$ is the Mittag-Leffer function with parameter $r > 0$. Note $\frac{2\beta_\lambda}{p} - \rho - 1 > 0$. Then by the asymptotic property of the Mittag-Leffer function (see, e.g., [24]), we obtain

$$\|u_n(t,x)\|_\rho^2 \le \frac{\text{const.}}{\frac{2\beta_\lambda}{p} - (\rho+1)} \exp(\widetilde{C}(\lambda, h_1, h_2) t^{\frac{2(\lambda h_1 + h_2 - 1)}{\lambda}(\frac{2\beta_\lambda}{p} - (\rho+1))^{-1}}) < +\infty, \tag{3.5}$$

if $\rho < \frac{(2-p)\lambda - 2(1-h_2)p}{p\lambda}$. Thus the proof of the lemma is completed.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1 It suffices to check that under the assumptions of Theorem 3.1, $\{u_n(t,x):(t,x)\in[0,T]\times D\}_{n\in\mathbb{N}}$ in Definition 2.2 is a Cauchy sequence on S_ρ for $\rho<\frac{(2-p)\lambda-2(1-h_2)p}{p\lambda}$. Through the lemmas above, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\|u_n(t,x)\|_\rho^2 &= \sum_{k=0}^{\infty}[k!]^\rho\Psi_k^{(\lambda)}(t,x)\\ &\le \frac{\text{const.}}{\frac{2\beta_\lambda}{p}-(\rho+1)}\exp(\widetilde{C}(\lambda,h_1,h_2)t^{\frac{2(\lambda h_1+h_2-1)}{\lambda}(\frac{2\beta_\lambda}{p}-(\rho+1))^{-1}})\\ &<\infty,\end{aligned}$$

where $\beta_\lambda=1-\frac{p(1-h_2)}{\lambda}$. Hence for $m,n\in\mathbb{N}$,

$$\|u_m(t,x)-u_n(t,x)\|_\rho^2=\sum_{k=m+1}^{n}[k!]^\rho\Psi_k^{(\lambda)}(t,x)\to 0,\quad \text{as } n,m\to\infty.$$

Since S_ρ is a Hilbert space under the norm $\|\cdot\|_\rho$, there exists a unique stochastic field $\{u(t,x):(t,x)\in[0,T]\times D\}$ on S_ρ such that

$$u(t,x)=\lim_{n\to\infty}u_n(t,x),\quad \text{in } S_\rho \text{ sense},$$

which is the unique solution of (1.1). Moreover, from (3.8) in Lemma 3.3, we immediately get the Lyapunov exponent estimate. This completes the proof of Theorem 3.1.

4 Hölder Continuity of the Solution

In this section, we will check the Hölder continuity of the solution $\{u(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ of (1.1) on both space and time variables.

Theorem 4.1 *Assume*

$$h_1,h_2\in(\frac{1}{2},1)\quad \textit{and}\quad (4h_1-3)\lambda+2h_2>2.$$

If $u_0\in L^\infty(\mathbb{R})$ *is an* α*-Hölder continuous function with* $\alpha\in(0,1)$*, then the solution* $\{u(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ *is* μ*-Hölder continuous in* t *and* ν*-Hölder continuous in* x*, where* $\mu\in(0,\min\{\frac{\alpha}{\lambda},\frac{\lambda h_1+h_2-1}{\lambda+1}\})$ *and* $\nu\in(0,\min\{\alpha,\lambda h_1+h_2-1\})$.

Proof Recall Definition 2.2,

$$\begin{cases}u_0(t,x)=G_\lambda(t)*u_0(x),\\ u_n(t,x)=\sum_{k=0}^{n}I_k(\widetilde{f}_k^{(\lambda)}(t,x)),\quad n=1,2,\cdots.\end{cases}\tag{4.1}$$

It follows from the proof of Lemma 3.3 that if $p\in(\frac{1}{2h_1-1},\frac{2\lambda}{\lambda+2(1-h_2)})$, then $u_n\in S_\rho$ with $\rho<\frac{(2-p)\lambda-2(1-h_2)p}{p\lambda}$ for all $n=1,2,\cdots$. In particular, $u_n\in S_0=L^2(\Omega)$ since $\frac{(2-p)\lambda-2(1-h_2)p}{p\lambda}>0$. On the other hand, from (4.1), it follows that for each $n\in\mathbb{N}$,

$$\begin{aligned}u_n(t,x)&=I_0(\widetilde{f}_0^{(\lambda)}(t,x))+\sum_{k=1}^{n}I_k(\widetilde{f}_k^{(\lambda)}(t,x))\\ &=G_\lambda(t)*u_0(x)+\int_0^t\int_D G_\lambda(t-s,x-z)u_{n-1}(s,z)B^H(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

Then for $s, t \in [0,T]$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned} u_n(t,x) - u_n(s,y) =& G_\lambda(t) * u_0(x) - G_\lambda(s) * u_0(y) \\ &+ \int_s^t \int_{\mathbb{R}} G_\lambda(t-r, x-z) u_{n-1}(r,z) B^H(\mathrm{d}r, \mathrm{d}z) \\ &+ \int_0^s \int_{\mathbb{R}} (G_\lambda(t-r, x-z) - G_\lambda(s-r, y-z)) u_{n-1}(r,z) B^H(\mathrm{d}r, \mathrm{d}z) \\ :=& A_1 + A_2 + A_3. \end{aligned} \tag{4.2}$$

But by Azerad and Mellouk [1], we know

$$\mathbf{E}|A_1|^2 \le C(T)(|x-y|^{2\alpha} + |t-s|^{\frac{2\alpha}{\lambda}}). \tag{4.3}$$

Applying Lemma 2.2 and (3.5) with $\rho = 0$, we get

$$\begin{aligned} \mathbf{E}|A_2|^2 =& \int_{[s,t]^2} \int_{\mathbb{R}^2} \varphi_{h_1}(r-\overline{r}) \varphi_{h_2}(z-\overline{z}) |G_\lambda(t-r, x-z)| \\ & \times |G_\lambda(t-\overline{r}, x-\overline{z})| \mathbf{E}(|u_{n-1}(r,z) u_{n-1}(\overline{r}, \overline{z})|) \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}r \mathrm{d}\overline{r} \\ =& \int_{[s,t]^2} \int_{\mathbb{R}^2} \varphi_{h_1}(r-\overline{r}) \varphi_{h_2}(z-\overline{z}) |G_\lambda(t-r, x-z)| \\ & \times |G_\lambda(t-\overline{r}, x-\overline{z})| \|u_{n-1}(r,z)\|_0 \|u_{n-1}(\overline{r}, \overline{z})\|_0 \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}r \mathrm{d}\overline{r} \\ \le& C(T) \int_{[s,t]^2} \int_{\mathbb{R}^2} \varphi_{h_1}(r-\overline{r}) \varphi_{h_2}(z-\overline{z}) |G_\lambda(t-r, x-z)| |G_\lambda(t-\overline{r}, x-\overline{z})| \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}r \mathrm{d}\overline{r} \\ =& C(T) \int_{[s,t]^2} \varphi_{h_1}(r-\overline{r}) \mathrm{d}r \mathrm{d}\overline{r} \int_{\mathbb{R}^2} \varphi_{h_2}(z-\overline{z}) |G_\lambda(t-r, x-z)| |G_\lambda(t-\overline{r}, x-\overline{z})| \mathrm{d}z \mathrm{d}\overline{z} \\ \le& C(T, h_2) \int_{[s,t]^2} \varphi_{h_1}(r-\overline{r}) \|G_\lambda(t-r, x-\cdot)\|_{L^{\frac{1}{h_2}}(\mathbb{R})} \|G_\lambda(t-\overline{r}, x-\cdot)\|_{L^{\frac{1}{h_2}}(\mathbb{R})} \mathrm{d}r \mathrm{d}\overline{r} \\ \le& C(T, h_1, h_2) \Big[\int_s^t \Big(\int_{\mathbb{R}} |G_\lambda(t-r, x-z)|^{\frac{1}{h_2}} \mathrm{d}z \Big)^{\frac{h_2}{h_1}} \mathrm{d}r \Big]^{2h_1}. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} \int_{\mathbb{R}} |G_\lambda(t-r, x-z)|^{\frac{1}{h_2}} \mathrm{d}z &= (t-r)^{-\frac{1}{\lambda h_2}} \int_{\mathbb{R}} |G_\lambda(1, (t-r)^{-\frac{1}{\lambda}}(x-z))|^{\frac{1}{h_2}} \mathrm{d}z \\ &= (t-r)^{\frac{1}{\lambda}(1-\frac{1}{h_2})} \int_{\mathbb{R}} |G_\lambda(1,z)|^{\frac{1}{h_2}} \mathrm{d}z \\ &\le (t-r)^{\frac{1}{\lambda}(1-\frac{1}{h_2})} \int_{\mathbb{R}} \Big(\frac{C}{1+|z|^2} \Big)^{\frac{1}{h_2}} \mathrm{d}z \\ &= C(h_2)(t-r)^{\frac{1}{\lambda}(1-\frac{1}{h_2})}. \end{aligned}$$

Therefore, we obtain

$$\mathbf{E}|A_2|^2 \le C(T, h_1, h_2)|t-s|^{2h_1(\frac{1}{\lambda}(1-\frac{1}{h_2})\frac{h_2}{h_1}+1)} = C(T, h_1, h_2)|t-s|^{2(h_1 + \frac{1}{\lambda}(h_2-1))}. \tag{4.4}$$

Note

$$\begin{aligned}\mathbf{E}|A_3|^2 \le 2\Big[\mathbf{E}\Big|\int_0^s\int_D(G_\lambda(t-r,x-z)-G_\lambda(t-r,y-z))u_{n-1}(r,z)B^H(\mathrm{d}r,\mathrm{d}z)\Big|^2\\ +\mathbf{E}\Big|\int_0^s\int_D(G_\lambda(t-r,y-z)-G_\lambda(s-r,y-z))u_{n-1}(r,z)B^H(\mathrm{d}r,\mathrm{d}z)\Big|^2\Big]\\ := 2(\mathrm{I}+\mathrm{II}).\end{aligned} \tag{4.5}$$

Let $\gamma\in(0,\min\{\lambda h_1+h_2-1,1\})$. Then

$$\begin{aligned}\mathrm{I}\le&\int_{[0,s]^2}\int_{D^2}\varphi_{h_1}(r-\overline{r})\varphi_{h_2}(z-\overline{z})|G_\lambda(t-r,x-z)-G_\lambda(t-r,y-z)|\\ &\times|G_\lambda(t-\overline{r},x-\overline{z})-G_\lambda(t-\overline{r},y-\overline{z})|\mathbf{E}(|u_{n-1}(r,z)u_{n-1}(\overline{r},\overline{z})|)\mathrm{d}z\mathrm{d}\overline{z}\mathrm{d}r\mathrm{d}\overline{r}\\ \le& C(T)\int_{[0,s]^2}\int_{D^2}\varphi_{h_1}(r-\overline{r})\varphi_{h_2}(z-\overline{z})|G_\lambda(t-r,x-z)-G_\lambda(t-r,y-z)|\\ &\times|G_\lambda(t-\overline{r},x-\overline{z})-G_\lambda(t-\overline{r},y-\overline{z})|\mathrm{d}z\mathrm{d}\overline{z}\mathrm{d}r\mathrm{d}\overline{r}\\ \le& C(T)\|G_\lambda(t-\cdot,x-\cdot)-G_\lambda(t-\cdot,y-\cdot)\|^2_{\mathcal{H}_1}\\ =&C(T)\||G_\lambda(t-\cdot,x-\cdot)-G_\lambda(t-\cdot,y-\cdot)|^\gamma|G_\lambda(t-\cdot,x-\cdot)-G_\lambda(t-\cdot,y-\cdot)|^{1-\gamma}\|^2_{\mathcal{H}_1}\\ \le& C(T,\gamma)(\||G_\lambda(t-\cdot,x-\cdot)-G_\lambda(t-\cdot,y-\cdot)|^\gamma|G_\lambda(t-\cdot,x-\cdot)|^{1-\gamma}\|^2_{\mathcal{H}_1}\\ &+\||G_\lambda(t-\cdot,x-\cdot)-G_\lambda(t-\cdot,y-\cdot)|^\gamma|G_\lambda(t-\cdot,y-\cdot)|^{1-\gamma}\|^2_{\mathcal{H}_1})\\ :=& C(T,\gamma)(\mathrm{I}_1+\mathrm{I}_2).\end{aligned}$$

Using Lemma 2.1 and the mean-value theorem, we get

$$\begin{aligned}\mathrm{I}_1\le&\Big\|\Big|\frac{\partial G_\lambda(t-\cdot,\xi-\cdot)}{\partial x}\Big|^\gamma|x-y|^\gamma|G_\lambda(t-\cdot,x-\cdot)|^{1-\gamma}\Big\|^2_{\mathcal{H}_1}\\ =&|x-y|^{2\gamma}\int_{[0,T]^2}\int_{\mathbb{R}^2}\Big|\frac{\partial G_\lambda(t-r,\xi-z)}{\partial x}\Big|^\gamma|G_\lambda(t-r,x-z)|^{1-\gamma}\\ &\times\varphi_{h_1}(r-\overline{r})\varphi_{h_2}(z-\overline{z})\Big|\frac{\partial G_\lambda(t-\overline{r},\xi-\overline{z})}{\partial x}\Big|^\gamma|G_\lambda(t-\overline{r},x-\overline{z})|^{1-\gamma}\mathrm{d}z\mathrm{d}\overline{z}\mathrm{d}r\mathrm{d}\overline{r}\\ \le& C(h_1,h_2)|x-y|^{2\gamma}\\ &\times\Big(\int_0^T\Big(\int_{\mathbb{R}}\Big(\Big|\frac{\partial G_\lambda(t-r,\xi-z)}{\partial x}\Big|^\gamma|G_\lambda(t-r,x-z)|^{1-\gamma}\Big)^{\frac{1}{h_2}}\mathrm{d}z\Big)^{\frac{h_2}{h_1}}\mathrm{d}r\Big)^{2h_1}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\Big|\frac{\partial G_\lambda(t-r,\xi-z)}{\partial x}\Big|^\gamma|G_\lambda(t-r,x-z)|^{1-\gamma}\Big)^{\frac{1}{h_2}}\mathrm{d}z\\ =&\int_{\mathbb{R}}\Big((t-r)^{-\frac{2\gamma}{\lambda}}\Big|\frac{\partial G_\lambda(1,(t-r)^{-\frac{1}{\lambda}}(\xi-z))}{\partial x}\Big|^\gamma(t-r)^{-\frac{1-\gamma}{\lambda}}|G_\lambda(1,(t-r)^{-\frac{1}{\lambda}}(x-z))|^{1-\gamma}\Big)^{\frac{1}{h_2}}\mathrm{d}z\\ =&(t-r)^{\frac{1}{\lambda}-\frac{1+\gamma}{\lambda h_2}}\int_{\mathbb{R}}\Big(\Big|\frac{\partial G_\lambda(1,(\widetilde{\xi}-z))}{\partial x}\Big|^\gamma|G_\lambda(1,z)|^{1-\gamma}\Big)^{\frac{1}{h_2}}\mathrm{d}z\\ \le& C(h_2,\gamma)(t-r)^{\frac{1}{\lambda}-\frac{1+\gamma}{\lambda h_2}}.\end{aligned}$$

Therefore, if

$$\Big(\frac{1}{\lambda}-\frac{1+\gamma}{\lambda h_2}\Big)\frac{h_2}{h_1}+1>0$$

(that is, $\gamma < \lambda h_1 + h_2 - 1$), then

$$\mathrm{I}_1 \le C(T,\gamma,h_1,h_2)|x-y|^{2\gamma}\Big(\int_0^T (t-r)^{(\frac{1}{\lambda}-\frac{1+\gamma}{\lambda h_2})\frac{h_2}{h_1}}\mathrm{d}r\Big)^{2h_1} \le C(T,\gamma,\lambda,h_1,h_2)|x-y|^{2\gamma}.$$

Similarly,

$$\mathrm{I}_2 \le C(T,\gamma,\lambda,h_1,h_2)|x-y|^{2\gamma}. \tag{4.7}$$

From the estimates (4.6) and (4.7), it follows that

$$\mathrm{I} \le C(T,\gamma,\lambda,h_1,h_2)|x-y|^{2\gamma}. \tag{4.8}$$

On the other hand, we estimate the part "II". From Lemma 2.1, we know

$$G_\lambda(t,x) = t^{-\frac{1}{\lambda}}G_\lambda(1,t^{-\frac{1}{\lambda}}x).$$

So

$$\frac{\partial G_\lambda(t,x)}{\partial t} = -\frac{1}{\lambda}t^{-1-\frac{2}{\lambda}}\Big(t^{\frac{1}{\lambda}}G_\lambda(1,t^{-\frac{1}{\lambda}}x) + x\frac{\partial G_\lambda(1,t^{-\frac{1}{\lambda}}x)}{\partial x}\Big). \tag{4.9}$$

Recall (4.5). An analogue argument as that in (4.8), together with (4.9), shows that for $\eta \in (0, \frac{\lambda h_1+h_2-1}{\lambda+1})$,

$$\mathrm{II} \le C(T,\eta,\lambda,h_1,h_2)|t-s|^{2\eta}. \tag{4.10}$$

Apply Fatou's lemma to (4.2). (4.3), (4.4), (4.8) and (4.10) jointly imply the desired result. This proves Theorem 4.1.

5 Absolute Continuity of the Law of the Solution

In this section, we shall prove the absolute continuity of the law of the solution $\{u(t,x) : (t,x) \in [0,T]\times\mathbb{R}\}$ given in Section 3. We need the following proposition.

Proposition 5.1 *Assume*

$$h_1, h_2 \in \Big(\frac{1}{2}, 1\Big) \quad \text{and} \quad (4h_1-3)\lambda + 2h_2 > 2.$$

For $(t,x)\in[0,T]\times\mathbb{R}$*, the solution* $u(t,x)\in\mathcal{D}^{1,2}$*, and*

$$\mathrm{D}_{s,y}u(t,x) = \int_s^t\int_{\mathbb{R}} G_\lambda(t-r,x-z)\mathrm{D}_{s,y}u(r,z)B^H(\mathrm{d}r,\mathrm{d}z) + G_\lambda(t-s,x-y)u(s,y) \tag{5.1}$$

for all $s\le t$ *and* $y\in\mathbb{R}$.

Proof Let $\{u_n(t,x) : n\ge 1\}$ be defined as in (4.1). Then a similar argument as that in [29] shows that for each $n\in\mathbb{N}$ and $h\in\mathcal{H}_1$, $u_n(t,x)\in\mathcal{D}_h$ and satisfies

$$\mathrm{D}_h u_n(t,x) = \int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)\mathrm{D}_h u_{n-1}(s,y)B^H(\mathrm{d}s,\mathrm{d}y) + \langle G_\lambda(t-\cdot\,,x-\cdot\,)u_{n-1}(\cdot\,,\cdot\,),h\rangle_{\mathcal{H}_1}.$$

Since $u_n(t,x)\to u(t,x)$ as $n\to\infty$ in $L^2(\Omega)$ sense, there exists a random field $u_{(h)}(t,x)$ such that $\mathrm{D}_h u_n(t,x)\to u_{(h)}(t,x)$ as $n\to\infty$ uniformly on $(t,x)\in[0,T]\times\mathbb{R}$ (the proof is similar to that of Theorem 3.1) and the latter satisfies

$$u_{(h)}(t,x) = \int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)u_{(h)}(s,y)B^H(\mathrm{d}s,\mathrm{d}y) + \langle G_\lambda(t-\cdot\,,x-\cdot\,)u(\cdot\,,\cdot\,),h\rangle_{\mathcal{H}_1}.$$

Hence, from the closeness of the operator D_h, it follows that $u(t,x)\in\mathcal{D}_h,\ \mathrm{D}_h u(t,x)=u_{(h)}(t,x)$ and

$$\mathrm{D}_h u(t,x)=\int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)\mathrm{D}_h u(s,y)B^H(\mathrm{d}s,\mathrm{d}y)+\langle G_\lambda(t-\cdot,x-\cdot)u(\cdot,\cdot),h\rangle_{\mathcal{H}_1}. \tag{5.2}$$

Next, we proceed to prove $u(t,x)\in\mathcal{D}^{1,2}$. Recall $\{h_n:n\geq 1\}$ in Section 2. By (4.2),

$$\begin{aligned}\mathbf{E}|\mathrm{D}_{h_n}u(t,x)|^2 =&\,\mathbf{E}\Big|\int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)\mathrm{D}_{h_n}u(s,y)B^H(\mathrm{d}s,\mathrm{d}y)\\ &+\langle G_\lambda(t-\cdot,x-\cdot)u(\cdot,\cdot),h_n\rangle_{\mathcal{H}_1}\Big|^2\\ \leq&\, C\Big(\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)\mathrm{D}_{h_n}u(s,y)B^H(\mathrm{d}s,\mathrm{d}y)\Big]^2\\ &+\langle G_\lambda(t-\cdot,x-\cdot)u(\cdot,\cdot),h_n\rangle^2_{\mathcal{H}_1}\Big)\\ :=&\, C(\mathrm{I}_1+\mathrm{I}_2).\end{aligned} \tag{5.3}$$

Note that $(4h_1-3)\lambda+2h_2>2$ implies $\lambda(2h_1-1)+h_2>1$. Then

$$\begin{aligned}\mathrm{I}_1=&\int_{[0,t]^2}\int_{\mathbb{R}^2}\varphi_{h_1}(r-\overline{r})\varphi_{h_2}(z-\overline{z})|G_\lambda(t-r,x-z)|\\ &\times|G_\lambda(t-\overline{r},x-\overline{z})|\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)\mathrm{D}_{h_n}u(\overline{r},\overline{z})|)\mathrm{d}z\mathrm{d}\overline{z}\mathrm{d}r\mathrm{d}\overline{r}\\ \leq&\, C\int_{[0,t]^2}\int_{\mathbb{R}^2}\varphi_{h_1}(r-\overline{r})\varphi_{h_2}(z-\overline{z})|G_\lambda(t-r,x-z)|\\ &\times|G_\lambda(t-\overline{r},x-\overline{z})|\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)\mathrm{d}z\mathrm{d}\overline{z}\mathrm{d}r\mathrm{d}\overline{r}\\ \leq&\, C\int_{[0,t]^2}\sup_{z\in\mathbb{R}}\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)\varphi_{h_1}(r-\overline{r})\\ &\times\Big(\int_{\mathbb{R}^2}\varphi_{h_2}(z-\overline{z})|G_\lambda(t-r,x-z)||G_\lambda(t-\overline{r},x-\overline{z})|\mathrm{d}z\mathrm{d}\overline{z}\Big)\mathrm{d}r\mathrm{d}\overline{r}\\ \leq&\, C(h_2)\int_{[0,t]^2}\sup_{z\in\mathbb{R}}\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)\varphi_{h_1}(r-\overline{r})\\ &\times\|G_\lambda(t-r,x-\cdot)\|_{L^{\frac{1}{h_2}}(\mathbb{R})}\|G_\lambda(t-\overline{r},x-\cdot)\|_{L^{\frac{1}{h_2}}(\mathbb{R})}\mathrm{d}r\mathrm{d}\overline{r}\\ \leq&\, C(h_2)\int_{[0,t]^2}\sup_{z\in\mathbb{R}}\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)\varphi_{h_1}(r-\overline{r})(t-r)^{\frac{h_2-1}{\lambda}}(t-\overline{r})^{\frac{h_2-1}{\lambda}}\mathrm{d}r\mathrm{d}\overline{r}\\ \leq&\, C(h_2)\int_0^t\sup_{z\in\mathbb{R}}\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)(t-r)^{\frac{h_2-1}{\lambda}}\Big(\int_0^r(r-\overline{r})^{2h_1-2}(t-\overline{r})^{\frac{h_2-1}{\lambda}}\mathrm{d}\overline{r}\Big)\mathrm{d}r\\ \leq&\, C(T,\lambda,h_2)\int_0^t\sup_{z\in\mathbb{R}}\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)(t-r)^{\frac{h_2-1}{\lambda}}\Big(\int_0^r(r-\overline{r})^{2h_1-2+\frac{h_2-1}{\lambda}}\mathrm{d}\overline{r}\Big)\mathrm{d}r\\ \leq&\, C(T,\lambda,h_1,h_2)\int_0^t\sup_{z\in\mathbb{R}}\mathbf{E}(|\mathrm{D}_{h_n}u(r,z)|^2)(t-r)^{\frac{h_2-1}{\lambda}}\mathrm{d}r.\end{aligned} \tag{5.4}$$

On the other hand, similarly to (4.4), since $\lambda h_1+h_2>1$, we have

$$\|G_\lambda(t-\cdot,x-\cdot)u(\cdot,\cdot)\|^2_{\mathcal{H}_1}\leq C(T,\lambda,h_1,h_2). \tag{5.5}$$

Set

$$U_m(t)=\sup_{x\in\mathbb{R}}\mathbf{E}\sum_{n=1}^m|\mathrm{D}_{h_n}u(t,x)|^2.$$

By (5.4) and (5.5), we obtain

$$\begin{aligned}U_m(t) &\le C\int_0^t (t-s)^{\frac{h_2-1}{\lambda}}U_m(s)\mathrm{d}s + C\|G_\lambda(t-\cdot\,,x-\cdot\,)u(\,\cdot\,,\cdot\,)\|^2_{\mathcal{H}_1}\\ &\le C + C\int_0^t (t-s)^{\frac{h_2-1}{\lambda}}U_m(s)\mathrm{d}s. \end{aligned}\tag{5.6}$$

Then Gronwall's lemma yields

$$U_m(t) \le C\exp\Big(C\int_0^t (t-s)^{\frac{h_2-1}{\lambda}}\mathrm{d}s\Big),$$

where $C := C(T, \lambda, h_1, h_2)$ is independent of m. Let $m \to \infty$. Then we get

$$\sup_{x\in D}\mathbf{E}\sum_{n=1}^{\infty}|\mathrm{D}_{h_n}u(t,x)|^2 < \infty.$$

This means $u(t, x) \in \mathcal{D}^{1,2}$.

Since $u(t, x)$ is $\mathcal{F}_t$-adapted, there exists a measurable function $\mathrm{D}_{s,y}u(t, x) \in \mathcal{H}_1$ such that

$$\mathrm{D}_{s,y}u(t,x) = 0$$

for $s > t$, and for any $h \in \mathcal{H}_1$,

$$\mathrm{D}_h u(t,x) = \langle \mathrm{D}u(t,x), h\rangle_{\mathcal{H}_1}. \tag{5.7}$$

From (4.2), (4.5) and Fubini's theorem, it follows that

$$\begin{aligned}&\langle \mathrm{D}u(t,x), h\rangle_{\mathcal{H}_1}\\ &= \int_0^t\int_{\mathbb{R}} G_\lambda(t-r,x-z)\mathrm{D}_h u(r,z)B^H(\mathrm{d}r,\mathrm{d}z) + \langle G_\lambda(t-\cdot\,,x-\cdot\,)u(\,\cdot\,,\cdot\,),h\rangle_{\mathcal{H}_1}\\ &= \int_0^t\int_{\mathbb{R}} G_\lambda(t-r,x-z)\langle \mathrm{D}u(r,z),h\rangle_{\mathcal{H}_1}B^H(\mathrm{d}r,\mathrm{d}z) + \langle G_\lambda(t-\cdot\,,x-\cdot\,)u(\,\cdot\,,\cdot\,),h\rangle_{\mathcal{H}_1}\\ &= \int_0^t\int_{\mathbb{R}} G_\lambda(t-r,x-z)B^H(\mathrm{d}r,\mathrm{d}z)\int_0^r\int_0^r\int_{\mathbb{R}}\int_{\mathbb{R}}\mathrm{D}_{s,y}u(r,z)h(\overline{s},\overline{y})\\ &\quad\times\varphi_{h_1}(s-\overline{s})\varphi_{h_2}(y-\overline{y})\mathrm{d}y\mathrm{d}\overline{y}\mathrm{d}s\mathrm{d}\overline{s} + \langle G_\lambda(t-\cdot\,,x-\cdot\,)u(\,\cdot\,,\cdot\,),h\rangle_{\mathcal{H}_1}\\ &= \int_0^t\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} h(\overline{s},\overline{y})\varphi_{h_1}(s-\overline{s})\varphi_{h_2}(y-\overline{y})\mathrm{d}y\mathrm{d}\overline{y}\mathrm{d}s\mathrm{d}\overline{s}\\ &\quad\times\int_s^t\int_{\mathbb{R}} G_\lambda(t-r,x-z)\mathrm{D}_{s,y}u(r,z)B^H(\mathrm{d}r,\mathrm{d}z) + \langle G_\lambda(t-\cdot\,,x-\cdot\,)u(\,\cdot\,,\cdot\,),h\rangle_{\mathcal{H}_1}.\end{aligned}$$

Therefore

$$\mathrm{D}_{s,y}u(t,x) = \int_s^t\int_{\mathbb{R}} G_\lambda(t-r,x-z)\mathrm{D}_{s,y}u(r,z)B^H(\mathrm{d}r,\mathrm{d}z) + G_\lambda(t-s,x-y)u(s,y).$$

Thus the proof of the proposition is completed.

Now we state the main theorem in this section.

Theorem 5.1 *Assume*

$$h_1, h_2 \in \Big(\frac{1}{2}, 1\Big) \quad \textit{and} \quad (4h_1 - 3)\lambda + 2h_2 > 2.$$

If

$$u_0(x) \geq \varepsilon_0 > 0 \quad \textit{for } x \in \mathbb{R},$$

then for $t \leq T$ and $x \in \mathbb{R}$, the law of the solution $u(t,x)$ of (1.1) *is absolutely continuous with respect to Lebesgue measure.*

Proof By Proposition 2.3, it suffices to check that

$$\|\mathrm{D}u(t,x)\|_{\mathcal{H}_1} > 0, \quad \text{a.s.}$$

Note

$$\|\mathrm{D}u(t,x)\|_{\mathcal{H}_1} > 0 \iff \|\mathrm{D}u(t,x)\|_{L^2([0,t]\times\mathbb{R})} > 0.$$

Hence we only need to prove

$$\|\mathrm{D}u(t,x)\|_{L^2([0,t]\times\mathbb{R})} > 0, \quad \text{a.s.}$$

Recalling the proof of Theorem 4.1, we can formally express $u(t,x)$ as follows:

$$u(t,x) = G_\lambda(t) * u_0(x) + \int_0^t \int_{\mathbb{R}} G_\lambda(t-r, x-z) u(r,z) B^H(\mathrm{d}r, \mathrm{d}z).$$

On the other hand, it is easy to check that

$$G_\lambda(t-r, x-z)u(r,z) \quad \text{and} \quad G_\lambda(t-r, x-z)\mathrm{D}_{s,y}u(r,z)$$

belong to $\mathcal{H}_1$. Then from Propositions 2.1 and 5.1, we have

$$\begin{aligned}
\mathbf{E}(\mathrm{D}_{s,y}u(t,x))^2 &= \mathbf{E}\Big[\int_s^t \int_{\mathbb{R}} G_\lambda(t-r, x-z)\mathrm{D}_{s,y}u(r,z)B^H(\mathrm{d}r, \mathrm{d}z) + G_\lambda(t-s, x-y)u(s,y)\Big]^2 \\
&= \mathbf{E}\Big[\int_s^t \int_{\mathbb{R}} G_\lambda(t-r, x-z)\mathrm{D}_{s,y}u(r,z)B^H(\mathrm{d}r, \mathrm{d}z)\Big]^2 \\
&\quad + G_\lambda^2(t-s, x-y)\mathbf{E}[u(s,y)]^2 \\
&\geq G_\lambda^2(t-s, x-y)\mathbf{E}[u(s,y)]^2 \\
&\geq G_\lambda^2(t-s, x-y)\mathbf{E}[G_\lambda(t) * u_0(x)]^2 \\
&= G_\lambda^2(t-s, x-y)\mathbf{E}\Big[\int_{\mathbb{R}} G_\lambda(t, x-y)u_0(y)\mathrm{d}y\Big]^2 \\
&\geq \varepsilon_0 G_\lambda^2(t-s, x-y).
\end{aligned}$$

Since $G_\lambda(t,x) > 0$, we have

$$\begin{aligned}
\mathbf{E}\Big[\int_0^t \int_{\mathbb{R}} (\mathrm{D}_{s,y}u(t,x))^2 \mathrm{d}y\mathrm{d}s\Big] &\geq \int_0^t \int_{\mathbb{R}} G_\lambda^2(t-s, x-y)\mathbf{E}[u(s,y)]^2 \mathrm{d}y\mathrm{d}s \\
&\geq \varepsilon_0 \int_0^t \int_{\mathbb{R}} G_\lambda^2(t-s, x-y)\mathrm{d}y\mathrm{d}s \\
&> 0, \qquad (5.8)
\end{aligned}$$

which implies

$$\int_0^t \int_{\mathbb{R}} (\mathrm{D}_{s,y} u(t,x))^2 \mathrm{d}y \mathrm{d}s > 0, \quad \text{a.s.} \tag{5.9}$$

Then by Proposition 2.3, the proof of the theorem is completed.

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