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Koszul Differential Graded Algebras and BGG Correspondence II***

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Abstract The concept of Koszul differential graded (DG for short) algebra is introduced in [8]. Let A be a Koszul DG algebra. If the Ext-algebra of A is finite-dimensional, i.e., the trivial module ${}_Ak$ is a compact object in the derived category of DG A-modules, then it is shown in [8] that A has many nice properties. However, if the Ext-algebra is infinite-dimensional, little is known about A. As shown in [15] (see also Proposition 2.2), ${}_Ak$ is not compact if $H(A)$ is finite-dimensional. In this paper, it is proved that the Koszul duality theorem also holds when $H(A)$ is finite-dimensional by using Foxby duality. A DG version of the BGG correspondence is deduced from the Koszul duality theorem.

Keywords Koszul differential graded algebra, Koszul duality, BGG correspondence
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1 Introduction

In [8], we introduced the concept of Koszul DG algebras. Let k be a field. By a connected DG algebra we mean a cochain k-algebra A such that $A = \bigoplus\limits_{n\geq 0} A^n$ and $A^0 = k$. A connected DG algebra A is said to be Koszul if the trivial DG module ${}_Ak$ has a semifree resolution with a semifree basis concentrated in degree 0. Let A be a connected DG algebra. We write $D(A)$ to be the derived category of A, and $D^c(A)$ to be the full triangulated subcategory of $D(A)$ generated by ${}_AA$, that is, the smallest triangulated subcategory of $D(A)$ containing ${}_AA$ as an object and closed under isomorphisms. We say that a DG module ${}_AM$ is compact if ${}_AM \in D^c(A)$. Let A be a Koszul DG algebra, and let $E = \mathrm{Ext}_A^*({}_Ak, {}_Ak)$ be its Ext-algebra. If the trivial DG module ${}_Ak$ is compact, then the Koszul DG algebra A has nice properties such as (i) the Yoneda algebra of E is isomorphic to the cohomology algebra $H(A)$; (ii) there is a duality of triangulated categories between the bounded derived category of finite modules over E and $D^c(A)$. However, if ${}_Ak$ is not compact, little is known about E and A.

Examples of DG algebra from differential geometry and algebraic topology usually have the property that the cohomology algebra is finite-dimensional. A DG algebra with this property was said to be compact by Kontsevich and Soibelman [10]. In this case, the trivial module ${}_Ak$ must not be compact (see Proposition 2.2). Hence the results obtained in [8] can not be applied

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to these DG algebras. In this paper, by using Foxby duality, we show that a Koszul DG algebra A such that $H(A)$ is finite dimensional still has the same properties (see Theorems 3.1, 3.2 and 4.1) as in the case that ${}_A k$ is compact. The method of this paper is different from that of [8].

Throughout, k is a fixed field, unadorned $\otimes$ means $\otimes_k$. Let B be a graded algebra, M and N be graded B-modules. We write $\underline{\mathrm{Hom}}_B(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_B(M, N(n))$ to be the set of all graded B-module homomorphisms and $\underline{\mathrm{Ext}}_B(-,-)$ to be the derived functor of $\underline{\mathrm{Hom}}_B(-,-)$. For the notations and properties of DG algebras we refer to the references [1, 4, 8, 11].

2 Some Basic Properties of Koszul DG Algebras

Let A be a connected DG algebra. For convenience, we use $A^\natural$ to denote the underlying connected graded algebra of A. Similarly, if M is a DG A-module, we use $M^\natural$ to denote the underlying graded $A^\natural$-module. We use $M^\sharp$ to denote the graded vector space dual.

As in [4], let $R = B(A)$ be its bar construction for the augmented DG algebra A.

Lemma 2.1 (see [4]) (i) *The augmentation map $B(A;A) = A \otimes R \xrightarrow{\epsilon\otimes\epsilon} {}_A k$ is a quasi-isomorphism, and hence $A\otimes R$ is a semifree resolution of ${}_A k$.*

(ii) *The map $\varphi : R^\sharp \longrightarrow \mathrm{End}_A(A\otimes R)$ defined by*

$$\varphi(f)(1[a_1|\cdots|a_n]) = \sum_{i=0}^{n} (-1)^{|f|\omega_i} 1[a_1|\cdots|a_i] f([a_{i+1}|\cdots|a_n])$$

is a quasi-isomorphism of DG algebras, where $R^\sharp = \underline{\mathrm{Hom}}_k(R,k) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_k(R^i, k)$ and $\omega_i = |a_1| + \cdots + |a_i| + i$.

Proposition 2.1 *Let A be a Koszul DG algebra. If* $\mathrm{gl.dim}\, A^\natural < \infty$ *and* $\dim A^1 < \infty$, *then ${}_A k$ is compact.*

Proof By the previous lemma, $\mathrm{Ext}^n_A(k,k) \cong H^n(R^\sharp)$. Let us inspect the cohomology of R. Consider the following second quadrant double complex $\mathbf{P}$:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d^v"] & & & & \\
\cdots & \sum\limits_{i+j+k=4} A^i\otimes A^j\otimes A^k \arrow[l, "d^h"'] \arrow[d, "d^v"] & A^1\otimes A^1\otimes A^1 \arrow[l, "d^h"'] \arrow[d, "d^v"] & & & \\
\cdots & \sum\limits_{i+j=4} A^i\otimes A^j \arrow[l, "d^h"'] \arrow[d, "d^v"] & \sum\limits_{i+j=3} A^i\otimes A^j \arrow[l, "d^h"'] \arrow[d, "d^v"] & A^1\otimes A^1 \arrow[l, "d^h"'] \arrow[d, "d^v"] & & \\
\cdots & A^4 \arrow[l, "d^h"'] \arrow[d] & A^3 \arrow[l, "d^h"'] \arrow[d] & A^2 \arrow[l, "d^h"'] \arrow[d] & A^1 \arrow[l, "d^h"'] \arrow[d] & \\
\cdots & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & k \arrow[l]
\end{tikzcd}$$

where d^v and d^h are defined as follows:

$$d^h(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} (-1)^{\omega_i+1} a_1 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n,$$
$$d^v(a_1 \otimes \cdots \otimes a_n) = \sum_{i=2}^{n} (-1)^{\omega_i} a_1 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n,$$

in which $\omega_i = \sum\limits_{j<i}(|a_j| - 1)$.

It is clear that the bar complex $R = \mathrm{Tot}^{\oplus}\mathbf{P}$. Let $\widehat{R} = \mathrm{Tot}^{\Pi}\mathbf{P}$. Then R is a subcomplex of $\widehat{R}$. Choose a filtration on $\widehat{R}$ as follows:

$$F_0\widehat{R} = \widehat{R}, \quad F_n\widehat{R} = \mathrm{Tot}^{\Pi}\mathbf{P}(n) \ \text{ for } n < 0,$$

where $\mathbf{P}(n)$ is the double subcomplex of $\mathbf{P}$ by deleting the right n columns. Then this filtration is exhaustive and complete. The E_0-level of the spectral sequence induced by the filtration $\{F_n\widehat{R}\}$ is the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d^v"] & & & & \\
\cdots & \sum\limits_{i+j+k=4} A^i \otimes A^j \otimes A^k \arrow[d, "d^v"] & A^1 \otimes A^1 \otimes A^1 \arrow[d, "d^v"] & & & \\
\cdots & \sum\limits_{i+j=4} A^i \otimes A^j \arrow[d, "d^v"] & \sum\limits_{i+j=3} A^i \otimes A^j \arrow[d, "d^v"] & A^1 \otimes A^1 \arrow[d, "d^v"] & & \\
\cdots & A^4 \arrow[d] & A^3 \arrow[d] & A^2 \arrow[d] & A^1 \arrow[d] & \\
\cdots & 0 & 0 & 0 & 0 & k
\end{tikzcd}$$

It follows from $\mathrm{gl.dim}\, A^{\natural} < \infty$ that the spectral sequence is bounded. By the complete convergence theorem (see [20]), the spectral sequence converges to $H(\widehat{R})$. Since A^1 is finite-dimensional and $\mathrm{gl.dim}\, A^{\natural} < \infty$, it follows that $\dim H^0(\widehat{R}) < \infty$. Since R is a subcomplex of $\widehat{R}$, $H^0(R) = Z^0(R) \subseteq Z^0(\widehat{R}) = H^0(\widehat{R})$. Hence $\dim \mathrm{Ext}^0_A(k,k) = \dim(H^0(R^{\sharp})) = \dim H^0(R) \leq \dim H^0(\widehat{R}) < \infty$.

We do not know when ${}_Ak$ is compact in general. However, we have the following proposition which is proved in [15], as a corollary of some homological identities over DG algebras. For completeness we give a direct proof here.

Proposition 2.2 *Let A be a nontrivial connected DG algebra (that is, $H(A) \neq k$). If $\dim H(A) < \infty$, then the trivial module ${}_Ak$ is not compact.*

Proof Suppose that ${}_Ak$ is compact. Let P be a minimal semifree resolution of ${}_Ak$ with a semifree filtration $P(0) \subseteq P(1) \subseteq \cdots \subseteq P(i) \subseteq \cdots$ and a finite set of semifree basis. By adjusting the filtration, we may get a new semifree filtration $F(0) \subseteq F(1) \subseteq \cdots \subseteq F(i) \subseteq \cdots$ of P satisfying the following conditions: (i) there are graded vector spaces $0 \neq U(i)$ for $i = 0, \cdots, t$ such that $F(i)/F(i-1) = A \otimes U(i)$ for $i = 0, \cdots, t$ and $F(i)/F(i-1) = 0$ for $i > t$; (ii) the graded vector space $U(i)$ is concentrated in degree j_i for each i and $j_0 \leq j_1 \leq \cdots \leq j_t$. Since $H(A)$ is finite-dimensional, there is an integer n such that $H^n(A) \neq 0$ and $H^i(A) = 0$ for all $i > n$. By the truncation of A at the n-th position, we have a bounded DG module N such that $N^i = 0$ for $i > n$ or $i < 0$ and a quasi-isomorphism of DG modules $A \longrightarrow N$. Then we get a quasi-isomorphism of DG modules $P = A \otimes_A P \longrightarrow N \otimes_A P$. Write $M = N \otimes_A P$. There is a natural filtration $M(0) \subseteq M(1) \subseteq \cdots \subseteq M(i) \subseteq \cdots$ inheriting from P. Clearly, we have $M(i)/M(i-1) = N \otimes U(i)$ for $i = 0, \cdots, t$ and $M(i)/M(i-1) = 0$ for $i > t$. By inspecting the $(j_t + n)$-th cohomology of M, one can easily see that $H^{j_t+n}(M) \neq 0$, hence a contradiction. It follows that ${}_Ak$ is not compact.

Proposition 2.3 *Let A be a Koszul DG algebra and $E = \mathrm{Ext}^0_A(k,k)$ be its Ext-algebra. Then E is a local algebra with residue field k.*

Proof If ${}_Ak$ is compact the result was proved in [8]. We now assume that ${}_Ak$ is not compact, hence E is infinite dimensional. By [8, Theorem 3.1], $E = k \oplus \Big(\prod_{i\geq 1} E_i\Big)$ for some vector spaces E_i, such that each $F_n = \prod_{i \geq n} E_i$ $(n \geq 1)$ is an ideal of E and E is a filtered algebra with the filtration $E = F_0 \supset F_1 \supset \cdots$. The filtration defines a topology on E so that E is a complete topological algebra. Next, we show that $J = F_1$ is the Jacobson radical of E. To this end, it suffices to show that for any $0 \neq x \in J$, $1 + x$ has a left inverse. Since E is complete, we may write $x = x_1 + x_2 + \cdots$ with $x_i \in E_i$. Set $x_1^{(1)} = x_1$. We have $(1 - x_1^{(1)})(1 + x) = 1 + x_1 + \sum_{i\geq 2} x_i - x_1^{(1)} - x_1^{(1)}x = 1 + \sum_{i \geq 2} x_i - x_1^{(1)}x$. Since E is filtered, $x^{(2)} = \sum_{i\geq 2} x_i - x_1^{(1)}x \in F_2$. Hence, we may write $x^{(2)} = x_2^{(2)} + x_3^{(2)} + \cdots$ with $x_i^{(2)} \in E_i$ for $i \geq 2$. We have

$$\begin{aligned}(1 - x_2^{(2)})(1 - x_1^{(1)})(1 - x) &= (1 - x_2^{(2)})(1 + x^{(2)}) \\ &= 1 + x_2^{(2)} + \sum_{i\geq 3} x_i^{(2)} - x_2^{(2)} - x_2^{(2)}x^{(2)} \\ &= 1 + \Big(\sum_{i \geq 3} x_i^{(2)} - x_2^{(2)}x^{(2)}\Big).\end{aligned}$$

Now $x^{(3)} = \sum_{i\geq 3} x_i^{(2)} - x_2^{(2)}x^{(2)} \in F_3$. Similarly to the previous procedure, we may write $x^{(3)} = x_3^{(3)} + x_4^{(3)} + \cdots$ with $x_i^{(3)} \in E_i$ for $i \geq 3$.

Inductively, we have a sequence of elements $\{x_n^{(n)} \mid n \geq 1\}$ such that

$$(1 - x_n^{(n)})(1 - x_{n-1}^{(n-1)}) \cdots (1 - x_1^{(1)})(1 + x) = 1 + x_{n+1}^{(n+1)} + x_{n+2}^{(n+1)} + \cdots \tag{2.1}$$

with $x_i^{(n+1)} \in E_i$ for $i \geq n+1$. Set $y_n = \prod_{i=0}^{n-1}(1 - x_{n-i}^{(n-i)})$ and $z_n = 1 + \sum_{i \geq n+1} x_i^{(i)}$. We get two sequences of elements $\{y_n \mid n \geq 1\}$ and $\{z_n \mid n \geq 1\}$. Since $y_n - y_{n-1} = (1 - x_n^n)y_{n-1} - y_{n-1} = -x_n^n y_{n-1} \in F_n$, it follows that $\{y_n \mid n \geq 1\}$ is a Cauchy sequence. Hence $\{y_n \mid n \geq 1\}$ converges in E. Set $y = \lim\limits_{n\to\infty} y_n$. Taking limit on both sides of (2.1), we have

$$\lim_{n\to\infty}(y_n(1+x)) = \lim_{n\to\infty} z_n.$$

It follows

$$y(1+x) = \Big(\lim_{n\to\infty} y_n\Big)(1+x) = \lim_{n\to\infty} z_n = 1,$$

that is, $1+x$ has a left inverse. Hence E is local.

3 Koszul Duality

Examples of DG algebra from differential geometry and algebraic topology usually have the property that the cohomology algebra is finite-dimensional. For example, let X be a finite CW complex, and let $B = C^*(X;k)$ be the singular cochain algebra on X. Then $H(B)$ is finite-dimensional. There is a connected DG algebra A which is weakly equivalent to B (see [4]), that is, there are finitely many DG algebras $D_1, \cdots, D_n$ which are connected by quasi-isomorphisms:

$$B \xrightarrow{\simeq} D_1 \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} D_n \xleftarrow{\simeq} A.$$

Of course, $H(A)$ is finite-dimensional. By Proposition 2.2, the trivial module ${}_Ak$ is not compact. Hence, the Koszul duality established in [8] is not applied to this class of DG algebras. In this section, we prove a version of Koszul duality theorem for Koszul DG algebras with finite-dimensional cohomology algebra by using Foxby duality (see [5]).

If A is a Koszul DG algebra, Proposition 2.3 says that $E = \mathrm{Ext}^0_A(k,k)$ is a local algebra with residue field k. Hence k is a left E-module by the trivial action. If we view E as a DG algebra concentrated in degree 0, then $A \otimes E$ is an augmented DG algebra.

For any DG A-modules M and N, let $\underline{\mathrm{Hom}}_A(M,N) = \underline{\mathrm{Hom}}_{A^\natural}(M^\natural, N^\natural)$. Let $\mathrm{R}\underline{\mathrm{Hom}}_A(-,N)$ be the right derived functor of $\underline{\mathrm{Hom}}_A(-,N)$.

Let $\mathcal{I}$ be a K-injective resolution of ${}_Ak$, and let $B = \underline{\mathrm{Hom}}_A(\mathcal{I},\mathcal{I})$. Then B is a DG algebra and $\mathcal{I}$ is a DG $A\otimes B$-module. Since A is Koszul, $H^i(B) = 0$ for $i \neq 0$. We have the following truncation:

$$B' := \cdots B^{-n} \longrightarrow B^{-n+1} \longrightarrow \cdots \longrightarrow B^{-1} \longrightarrow Z^0(B) \longrightarrow 0,$$

where $Z^0(B)$ is the 0-th cocycles of B. Then one can easily check that B' is a DG subalgebra of B, and the inclusion map $B' \hookrightarrow B$ is a quasi-isomorphism. Hence $\mathcal{I}$ is a DG $A\otimes B'$-module. Write

$$\begin{aligned} F &= \mathrm{R}\underline{\mathrm{Hom}}_A\,(-,\mathcal{I}) : \ D(A) \longrightarrow D(B'), \\ G &= \mathrm{R}\underline{\mathrm{Hom}}_{B'}(-,\mathcal{I}) : D(B') \longrightarrow D(A). \end{aligned}$$

Then F and G is a pair of adjoint contravariant functors.

Let

$$\mathcal{A}(A) = \{M \in D(A) \mid \text{the adjunction map } M \longrightarrow GFM \text{ is isomorphic}\}$$

be the Auslander class, and

$$\mathcal{B}(B') = \{N \in D(B') \mid \text{the adjunction map } N \longrightarrow FGN \text{ is isomorphic}\}$$

be the Bass class.

Lemma 3.1 (Foxby Duality) *If* (F, G) *is a pair of adjoint contravariant triangulated functors between triangulated categories* $\mathcal{C}$ *and* $\mathcal{D}$*, then*

(i) *the Auslander class and the Bass class are full triangulated subcategories,*

(ii) F *and* G *induce a pair of dualities between the Auslander class and the Bass class.*

By the Foxby duality, $F : \mathcal{A}(A) \longrightarrow \mathcal{B}(B')$ and $G : \mathcal{B}(B') \longrightarrow \mathcal{A}(A)$ is a pair of duality functors between triangulated categories.

Lemma 3.2 *The regular DG module* ${}_{B'}B'$ *is in the Bass class* $\mathcal{B}(B')$.

Proof In $D(B')$, we have

$$\begin{aligned} FG(B') &= \mathrm{R}\underline{\mathrm{Hom}}_A(\mathrm{R}\underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}), \mathcal{I}) \\ &= \underline{\mathrm{Hom}}_A(\underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}), \mathcal{I}) \\ &= \underline{\mathrm{Hom}}_A(\mathcal{I}, \mathcal{I}) \\ &\cong {}_{B'}B'. \end{aligned}$$

Hence ${}_{B'}B'$ is in the Bass class $\mathcal{B}(B')$.

Temporarily, we write $\langle {}_A k \rangle$ and $\langle {}_{B'}B' \rangle$ to be the full triangulated categories of $D(A)$ and $D(B')$ generated by ${}_A k$ and ${}_{B'}B'$ respectively.

Lemma 3.3 *The trivial DG module* ${}_A k$ *is in the Auslander class* $\mathcal{A}(A)$*, and hence* F *and* G *induce a pair of duality functors between* $\langle {}_A k \rangle$ *and* $\langle {}_{B'}B' \rangle$.

Proof In $D(A)$, since we have $G({}_{B'}B') = \mathrm{R}\underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}) = \underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}) = \mathcal{I} \cong {}_A k$, by Foxby duality, ${}_A k$ is in the Auslander class $\mathcal{A}(A)$. Hence we have a pair of duality functors

$$\begin{aligned} F :&\ \ \langle {}_A k \rangle \longrightarrow \langle {}_{B'}B' \rangle, \\ G :&\ \langle {}_{B'}B' \rangle \longrightarrow \langle {}_A k \rangle. \end{aligned}$$

Since A is Koszul, $E = \mathrm{Ext}_A^0(k, k) = H^0(B) = H^0(B')$. Since B' is concentrated in negative degrees, there is a quasi-isomorphism of DG algebras

$$\varphi : B' \longrightarrow E.$$

Lemma 3.4 (see [11]) *Let* D *and* D' *be DG algebras. If there is a quasi-isomorphism of DG algebras* $\psi : D \longrightarrow D'$*, then the restriction of* ψ *induces an equivalence of triangulated categories* $\psi^* : D(D') \longrightarrow D(D)$ *with the inverse functor* $D' \otimes_D^L -$.

By the lemma above, we have a pair of quasi-inverse equivalences of triangulated categories $\varphi^* : D(E) \longrightarrow D(B')$ and $E \otimes_{B'}^L - : D(B') \longrightarrow D(E)$. Clearly $\varphi^*({}_E E) = {}_{B'}B'$. By the restriction of φ^* and $E \otimes_{B'}^L -$, we get a pair of quasi-inverse equivalences

$$\begin{aligned} \zeta :&\ \ \langle {}_E E \rangle \longrightarrow \langle {}_{B'}B' \rangle, \\ \xi :&\ \langle {}_{B'}B' \rangle \longrightarrow \langle {}_E E \rangle, \end{aligned}$$

such that $\zeta({}_E E) = {}_{B'}B'$.

Proposition 3.1 *Let A be a Koszul DG algebra. There is a pair of duality functors between triangulated categories*

$$\theta : D_{fd}(A) \longrightarrow \langle {}_E E \rangle,$$
$$\phi : \quad \langle {}_E E \rangle \longrightarrow D_{fd}(A),$$

where $D_{fd}(A)$ is the full triangulated subcategory of $D(A)$ consisting of DG modules M such that $\dim H(M) < \infty$.

Proof Let $\theta = \xi \circ F$ and $\phi = G \circ \zeta$. Then we have a pair of duality functors

$$\theta : \langle {}_A k \rangle \longrightarrow \langle {}_E E \rangle,$$
$$\phi : \langle {}_E E \rangle \longrightarrow \langle {}_A k \rangle.$$

By [8, Lemma 5.5], we have $\langle {}_A k \rangle = D_{fd}(A)$. Hence the result follows.

Theorem 3.1 (Koszul Duality on Ext-Algebra) *Let A be a Koszul DG algebra, E be its* Ext-*algebra. If* $\dim H(A) < \infty$, *then* $\mathrm{Ext}_E^*(k,k) \cong H(A)$.

Proof Since $\dim H(A) < \infty$, ${}_A A \in D_{fd}(A)$. By Proposition 3.1, we have

$$\begin{aligned} \theta({}_A A) &= \xi \circ F({}_A A) \\ &= E \otimes^L_{B'} \mathrm{R}\underline{\mathrm{Hom}}_A({}_A A, \mathcal{I}) \\ &= E \otimes^L_{B'} \mathcal{I} \\ &\overset{(a)}{\cong} {}_E k, \end{aligned} \tag{3.1}$$

where the isomorphism (a) holds for the following reason. Let $X = E \otimes^L_{B'} \mathcal{I}$. Since E and B' are quasi-isomorphic, $B'_{B'}$ is a semifree resolution of $E_{B'}$. Hence $X \cong B' \otimes^L_{B'} \mathcal{I} = \mathcal{I}$ as the complex of vector spaces. Hence $H^0(X) = k$ and $H^i(X) = 0$ for $i \neq 0$. Since E is concentrated in degree zero, the DG E-module X is exactly a cochain complex of E-modules. Hence, by suitable truncations, X is isomorphic to a simple E-module in $D(E)$. While Proposition 2.3 says that E is a local algebra with residue field k, hence there is a unique simple module in the category of E-modules. Thus X is isomorphic to ${}_E k$ in $D(E)$.

The isomorphisms in (3.1) say that the trivial module ${}_E k$ is in $\langle {}_E E \rangle$. We have the following equalities:

$$\begin{aligned} \mathrm{Ext}_E^*(k,k) &= \bigoplus_{i \geq 0} \mathrm{Hom}_{D(E)}({}_E k, {}_E k[i]) \\ &\cong \Big(\bigoplus_{i \geq 0} \mathrm{Hom}_{D(A)}(\phi({}_E k), \phi({}_E k)[i]) \Big)^{\mathrm{op}} \\ &\cong \Big(\bigoplus_{i \geq 0} \mathrm{Hom}_{D(A)}({}_A A, {}_A A[i]) \Big)^{\mathrm{op}} \\ &\cong ((H(A))^{\mathrm{op}})^{\mathrm{op}} \\ &= H(A). \end{aligned}$$

Corollary 3.1 *Let A be a Koszul DG algebra, E be its* Ext-*algebra. If* $\dim H(A) < \infty$ *and E is Noetherian, then* $\mathrm{gl.dim}\, E < \infty$.

Proof Since E is local and Noetherian, $\mathrm{gl.dim}\, E = \mathrm{pd}\,_E k$. By Theorem 3.1, there is an integer n such that $\mathrm{Ext}_E^n(k,k) = 0$ since $\dim H(A) < \infty$.

Similarly to [8, Corollaries 3.8 and 3.9], we have the following results.

Corollary 3.2 *Let A be a Koszul DG algebra, E be its* Ext*-algebra. If* $\dim H(A) < \infty$*, then*

(i) *the local algebra E is quasi-Koszul* (*see* [6]) *if and only if $H(A)$ is generated by $H^1(A)$;*

(ii) *E is strongly quasi-Koszul* (*see* [6]) *if and only if $H(A)$ is a Koszul algebra.*

Theorem 3.2 (Koszul Duality) *Let A be a Koszul DG algebra, E be its Ext-algebra. If $\dim H(A) < \infty$ and E is Noetherian, then we have a pair of duality functors between triangulated categories*

$$\theta : D_{fd}(A) \longrightarrow D^b(\mathrm{mod}\, E), \quad \phi : D^b(\mathrm{mod}\, E) \longrightarrow D_{fd}(A),$$

where $\mathrm{mod}\, E$ *is the category of finitely generated left E-modules and* $D^b(\mathrm{mod}\, E)$ *is the bounded derived category of* $\mathrm{mod}\, E$.

Moreover, under these dualities $\theta(_A A) = {}_E k$ and $\theta(_A k) = {}_E E$.

Proof By Proposition 3.1, it suffices to show $\langle _E E\rangle = D^b(\mathrm{mod}\, E)$. This follows from Corollary 3.1 and the hypothesis that E is Noetherian.

For the rest of this section, assume that A is a compact Koszul DG algebra, E is its Ext-algebra and E is Noetherian. We next show that there is a natural t-structure on the triangulated category $D_{fd}(A)$ and the heart of the t-structure is a category of modules.

Define subclasses

$$D^{\geq n} = \{X \in D_{fd}(A) \mid \mathrm{Ext}_A^i(X,k) = 0 \text{ for all } i > -n\},$$
$$D^{\leq n} = \{X \in D_{fd}(A) \mid \mathrm{Ext}_A^i(X,k) = 0 \text{ for all } i < -n\}.$$

Proposition 3.2 $(D^{\geq 0}, D^{\leq 0})$ *is a t-structure on $D_{fd}(A)$.*

Proof Let

$$\mathcal{T}^{\geq n} = \{U \in D^b(\mathrm{mod}\, E) \mid H^i(U) = 0 \text{ for all } i < n\},$$
$$\mathcal{T}^{\leq n} = \{U \in D^b(\mathrm{mod}\, E) \mid H^i(U) = 0 \text{ for all } i > n\}.$$

Then $(\mathcal{T}^{\geq n}, \mathcal{T}^{\leq n})$ is a t-structure on $D^b(\mathrm{mod}\, E)$. By Theorem 3.2, there is an anti-equivalence

$$\phi : D^b(\mathrm{mod}\, E) \longrightarrow D_{fd}(A).$$

The anti-equivalence induces a t-structure $(\phi(\mathcal{T}^{\leq n}), \phi(\mathcal{T}^{\geq n}))$ on $D_{fd}(A)$. We next show $\phi(\mathcal{T}^{\geq n}) = D^{\leq -n}$ and $\phi(\mathcal{T}^{\leq n}) = D^{\geq -n}$. We check the following steps:

$$\begin{aligned} U \in \mathcal{T}^{\geq n} &\Longleftrightarrow H^i(U) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \mathrm{Hom}_{D^b(\mathrm{mod} E)}(E, U[i]) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \mathrm{Hom}_{D_{fd}(A)}(\phi(U), \phi(E)[i]) = 0 \quad \text{for all } i < n \\ &\overset{(a)}{\Longleftrightarrow} \mathrm{Hom}_{D_{fd}(A)}(\phi(U), k[i]) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \mathrm{Ext}_A^i(\phi(U), k_A) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \phi(U) \in D^{\leq -n}, \end{aligned}$$

where (a) holds by Theorem 3.2. Similarly, we have $\phi(\mathcal{T}^{\leq n}) = D^{\geq -n}$. Hence the proposition follows.

Let $\mathcal{K}$ be the heart of the t-structure $(D^{\geq 0}, D^{\leq 0})$. If a DG module X is an object of $\mathcal{K}$, then $\mathrm{Ext}^i_A(X,k) = 0$ for all $i \neq 0$. We call such a DG module a Koszul DG module.

Theorem 3.3 $\mathrm{Ext}^0_A(-,k) : \mathcal{K} \longrightarrow \mathrm{mod}\, E$ *is an anti-equivalence of Abelian categories.*

Proof By Proposition 3.2, $\phi^{-1} : \mathcal{K} \longrightarrow \mathrm{mod}\, E$ is an anti-equivalence of Abelian categories. We have natural isomorphisms

$$\begin{aligned}\phi^{-1}(X) &= \mathrm{Hom}_E(E, \phi^{-1}(X)) \\ &= \mathrm{Hom}_{D^b(\mathrm{mod} E)}(E, \phi^{-1}(X)) \\ &\cong \mathrm{Hom}_{D_{fd}(A)}(X, \phi(E)) \\ &\cong \mathrm{Hom}_{D_{fd}(A)}(X, k) \\ &= \mathrm{Ext}^0_A(X,k).\end{aligned}$$

Hence $\phi^{-1} = \mathrm{Ext}^0_A(-,k)$.

4 BGG Correspondence

In this section, we form a correspondence of triangulated categories similar to the BGG correspondence (see [3, 8, 9, 16, 18]) for Koszul DG algebras with finite-dimensional cohomology algebra.

Throughout this section, A is a Koszul DG algebra with $H(A)$ finite-dimensional, and E is its Ext-algebra.

First of all, we recall some terminologies.

Definition 4.1 (see [7]) *Let R be a Noetherian local algebra with residue field k. R is said to be Gorenstein if there is an integer $d \geq 0$ such that*

$$\mathrm{Ext}^n_R(k,R) = \begin{cases} 0, & n \neq d, \\ k, & n = d. \end{cases}$$

Definition 4.2 (see [13]) *A connected DG algebra B is Frobenius if there is a quasi-isomorphism ${}_BB \longrightarrow {}_BB^\sharp[l]$ for some integer l.*

Proposition 4.1 *E is Gorenstein if and only if A is Frobenius.*

Proof By Proposition 3.1 and the isomorphisms in (3.1), we have the following equalities:

$$\begin{aligned}\mathrm{Ext}^n_E(k,E) &= \mathrm{Hom}_{D(E)}({}_Ek, {}_EE[n]) \\ &\cong \mathrm{Hom}_{D(A)}(\phi({}_EE), \phi({}_Ek)[n]) \\ &= \mathrm{Hom}_{D(A)}({}_Ak, {}_AA[n]).\end{aligned}$$

It follows that $\mathrm{Ext}^n_E(k,E) = 0$ for $n \neq d$ if and only if $\mathrm{Hom}_{D(A)}({}_Ak, {}_AA)[n] = 0$ for $n \neq d$, and $\mathrm{Ext}^d_E(k,E) = k$ if and only if $\mathrm{Hom}_{D(A)}({}_Ak, {}_AA)[d] = k$. We claim that

$$\mathrm{Hom}_{D(A)}({}_Ak, {}_AA[n]) = \begin{cases} 0, & n \neq d, \\ k, & n = d, \end{cases}$$

if and only if A is Frobenius. Since $H(A)$ is finite dimensional, the DG A-module $A^\sharp_A$ has minimal semifree resolution P with a set of semibasis $\{e_\alpha \mid \alpha \in \Lambda\}$. Then $P^\sharp$ is a K-injective resolution of ${}_AA$. If

$$\mathrm{Hom}_{D(A)}({}_Ak, {}_AA[n]) = \mathrm{Hom}_A({}_Ak, P^\sharp[n]) = \begin{cases} 0, & n \neq d, \\ k, & n = d, \end{cases}$$

then the index set Λ has only one element. Moreover $P^\sharp = A^\sharp[-d]$, that is, ${}_AA \cong {}_AA^\sharp[-d]$. Hence A is Frobenius. The other direction of the claim is clear.

Lemma 4.1 *A connected DG algebra B is Frobenius if and only if $H(B)$ is graded Frobenius.*

Proof If B is Frobenius, then there is a quasi-isomorphism of DG modules ${}_BB \longrightarrow {}_BB^\sharp[l]$, which implies a graded module isomorphism ${}_{H(B)}H(B) \longrightarrow {}_{H(B)}H(B^\sharp)[l]$. Hence $H(B)$ is graded Frobenius. Conversely, if $H(B)$ is graded Frobenius, then

$$_{H(B)}H(B) \longrightarrow {}_{H(B)}H(B^\sharp)[l] \tag{4.1}$$

is a free resolution of $H(B^\sharp)$. The Eilenberg-Moore resolution (see [4]) of ${}_BB^\sharp[l]$ defined by (4.1) is

$$_BB \longrightarrow {}_BB^\sharp[l].$$

Hence the DG algebra B is Frobenius.

Corollary 4.1 *E is Gorenstein if and only if $H(A)$ is a graded Frobenius algebra.*

Proof The proof is directly from Proposition 4.1 and Lemma 4.1.

Now suppose that E is Noetherian. Let J be its Jacobson radical. An E-module M is called a J-torsion module if for any element $m \in M$ there is an integer n such that $J^n m = 0$. Let $\mathrm{tor}\, E$ be the full subcategory of $\mathrm{mod}\, E$ consisting of all the J-torsion modules. Since E is Noetherian, $\mathrm{tor}\, E$ is a thick Abelian subcategory of $\mathrm{mod}\, E$. Write $\mathrm{qmod}\, E$ to be the quotient category $\mathrm{mod}\, E/\mathrm{tor}\, E$. Since E is Noetherian, $\mathrm{tor}\, E$ is exactly the category of all finite dimensional E-modules.

Theorem 4.1 (BGG Correspondence) *If E is Noetherian, then we have an anti-equivalence of triangulated categories*

$$D^b(\mathrm{qmod}\, E) \longrightarrow D_{fd}(A)/D^c(A).$$

Proof By Theorem 3.2, there is an anti-equivalence

$$D^b(\mathrm{mod}\, E) \longrightarrow D_{fd}(A).$$

Under this anti-equivalence, ${}_Ek$ is corresponding to ${}_AA$. Hence we have an anti-equivalence

$$D^b(\mathrm{mod}\, E)/\langle {}_Ek\rangle \longrightarrow D_{fd}(A)/D^c(A).$$

It is clear $\langle {}_Ek\rangle = D^b_{\mathrm{tor}\, E}(\mathrm{mod}\, E)$, the full triangulated subcategory of $D^b(\mathrm{mod}\, E)$ consisting of complexes M such that each cohomology $H^\bullet(M)$ is a J-torsion module. By [17],

$$D^b(\mathrm{mod}\, E)/D^b_{\mathrm{tor}\, E}(\mathrm{mod}\, E) \cong D^b(\mathrm{qmod}\, E).$$

Hence the result follows.

Remark 4.1 The BGG correspondence established in Theorem 4.1 also implies the classical one in [3].

In fact, as we see in [8], if A is a Koszul Adams connected DG algebra, then its Ext-algebra E is a connected graded algebra. Following the notations in [8], we write $AD_{dg}(A)$ to be the derived category of left DG A-modules, $AD^c(A)$ to be the full triangulated subcategory of $AD_{dg}(A)$ generated by ${}_AA$, and $AD_{fd}(A)$ to be the full triangulated subcategory of $AD_{dg}(A)$ consisting of objects with finite-dimensional cohomologies. Now the Koszul duality is of the following form:

$$D^b(\operatorname{gr} E) \xrightarrow{\cong} AD_{fd}(A), \tag{4.2}$$

and the BGG correspondence is of the following form:

$$D^b(\operatorname{qgr} E) \xrightarrow{\cong} AD_{fd}(A)/AD^c(A). \tag{4.3}$$

Now let R be a Noetherian Koszul AS-regular algebra. Then its Yoneda algebra $S = R^!$ is a graded Frobenius algebra. Let A be the Adams connected DG algebra given as follows: $A^i_i = S_i$ for $i \geq 0$ and $A^i_j = 0$ for $i \neq j$. Then A is a Koszul Adams connected DG algebra. The Ext-algebra of A is R. Since $AD_{dg}(A) = D(S)$ and S is finite dimensional, it follows that $AD_{fd}(A) = D^b(\operatorname{gr} S)$ and $AD^c(A) = D^b(\operatorname{proj} S)$, where $\operatorname{proj} S$ is the category of finitely generated graded projective S-modules. Hence the Koszul duality (4.2) can be written as

$$D^b(\operatorname{gr} R) \xrightarrow{\cong} D^b(S),$$

and the BGG correspondence (4.3) as

$$D^b(\operatorname{qgr} R) \xrightarrow{\cong} D^b(S)/D^b(\operatorname{proj} S).$$

By [19], $D^b(S)/D^b(\operatorname{proj} S) \cong \overline{\operatorname{gr}}\, S$ since S is graded Frobenius. Hence the BGG correspondence is of the form

$$D^b(\operatorname{qgr} R) \xrightarrow{\cong} \overline{\operatorname{gr}}\, S,$$

which was proved in [3, 9, 18].

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