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The Tracial Rokhlin Property for Automorphisms on Non-simple C^*-Algebras**

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Abstract Let A be a unital AF-algebra (simple or non-simple) and let α be an automorphism of A. Suppose that α has certain Rokhlin property and A is α-simple. Suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^{J} = \mathrm{id}_{K_0(A)}$. The author proves that $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

Keywords Rokhlin property, Tracial rank zero, AF-algebra
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1 Introduction

The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by Connes [1]. It was adopted by Herman and Oeneanu [2] for UHF-algebras. Rørdam [13] and Kishimoto [6] introduced the Rokhlin property to a much more general context of C^*-algebras, then Osaka and Phillips studied integer group actions which satisfy certain type of Rokhlin property on some simple C^*-algebras (see [12]). More recently, Lin studied the Rokhlin property for automorphisms on simple C^*-algebras (see [10]).

Phillips proposed how to introduce appropriate Rokhlin property for automorphisms on non-simple C^*-algebras. In this paper, we attempt to introduce certain Rokhlin property for automorphisms on non-simple C^*-algebras; when C^*-algebra is simple, this Rokhlin property is weaker than the Rokhlin property in [10, 12]. If an integer group action of a C^*-algebra has this Rokhlin property, we can conclude that its crossed product is in the C^*-algebra class of tracial rank zero. In particular, these algebras all belong to the class known currently to be classifiable by K-theoretic invariants in the sense of the Elliott classification program. We hope that this case will lead us to more interesting in the Rokhlin property for automorphisms on non-simple C^*-algebras.

The organization of this paper is as follows. In Section 1, we briefly recall the notion of C^*-algebras, then we introduce certain Rokhlin property and discuss some property of crossed product $A \rtimes_\alpha \mathbb{Z}$ when an automorphism α of a C^*-algebra A has the Rokhlin property. In Section 2, we show that if A is a unital AF-algebra, suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property and A is α-simple, suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^{J} = \mathrm{id}_{K_0(A)}$. Then $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

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2 The Tracial Rokhlin Property

We will use the following convention:

(1) Let A be a C^*-algebra, $a \in A$ be a positive element and $p \in A$ be a projection. We write $[p] \leq [a]$ if there is a projection $q \in \overline{aAa}$ and a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(2) Let A be a C^*-algebra. We denote by $\mathrm{Aut}(A)$ the automorphism group of A. If A is unital and $u \in A$ is a unitary, we denote by $\mathrm{ad}u$ the inner automorphism defined by $\mathrm{ad}u(a) = u^*au$ for all $a \in A$.

(3) Let $x \in A$, $\varepsilon > 0$ and $\mathcal{F} \subset A$. We write $x \in_\varepsilon \mathcal{F}$, if $\mathrm{dist}(x, \mathcal{F}) < \varepsilon$, or there is a $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon$.

(4) Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. We say that A is α-simple if A does not have any non-trivial α-invariant closed two-sided ideals.

(5) A unital C^*-algebra is said to have real rank zero, written $\mathrm{RR}(A) = 0$, if the set of invertible self-adjoint elements is dense in self-adjoint elements of A. Note that every unital AF-algebra has real rank zero.

(6) A unital C^*-algebra A has the (SP)-property if every non-zero hereditary C^*-subalgebra of A has a non-zero projection. Note that every C^*-algebra A with real rank zero has the (SP)-property.

(7) Let $T(A)$ be the tracial state space of a unital C^*-algebra A. It is a compact convex set.

(8) We say that the order on projection over a unital C^*-algebra A is determined by traces, if for any two projections $p, q \in A, \tau(p) < \tau(q)$ for all $\tau \in T(A)$ implies that p is equivalent to a projection $p' \leq q$.

Definition 2.1 *We denote by $\mathcal{I}^{(0)}$ the class of all finite dimensional C^*-algebras, and denote by $\mathcal{I}^{(k)}$ the class of all unital C^*-algebras which are unital hereditary C^*-subalgebras of C^*-algebras of the form $C(X) \otimes F$, where X is a k-dimensional finite CW complex and $F \in \mathcal{I}^{(0)}$.*

We recall the definition of tracial topological rank of C^*-algebras.

Definition 2.2 (cf. [8]) *Let A be a unital simple C^*-algebra. Then A is said to have tracial (topological) rank no more than k if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any non-zero positive element $a \in A$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra $B \in \mathcal{I}^{(k)}$ with $1_B = p$ such that*

(1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$,

(3) $[1 - p] \leq [a]$.

If A has tracial rank no more than k, we will write $\mathrm{TR}(A) \leq k$. If furthermore, $\mathrm{TR}(A) \nleq k-1$, then we say $\mathrm{TR}(A) = k$.

Definition 2.3 *Let A be a unital C^*-algebra, $\alpha \in \mathrm{Aut}(A)$, $a \in A$ be a positive element, and $p \in A$ be a projection. We say $[p] \leq_\alpha [a]$ if there exist the mutually orthogonal projections p_i, the mutually orthogonal positive elements a_i and $s_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, n$ such that $p = \sum_{i=1}^{n} p_i$, $\{a_i\}_{i=1}^{n}$ belong to the hereditary C^*-subalgebra generated by a, and $[\alpha^{s_i}(p_i)] \leq [a_i]$, $i =$*

$1, \cdots, n$.

By this definition, we can compare nonzero positive elements with full positive elements by the action of α.

Example 2.1 Let $A = A_0 \oplus A_0$, where A_0 is an infinite dimensional unital simple C^*-algebra with real rank zero, and let $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(a_0, b_0) = (b_0, a_0)$, where $a_0, b_0 \in A_0$. Then for any non-zero projection $q \in A$, there exists a projection $p = (p_1, p_2) \in A$, $p_1 \neq 0$, $p_2 \neq 0$ such that $[p] \leq_\alpha [q]$.

Definition 2.4 *Let A be a unital C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. We say that α has the tracial Rokhlin property if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, every nonzero positive element $a \in A$, every finite set $\mathcal{F} \subset A$, $\mathcal{F} = \{p_1, \cdots, p_m, a_1, \cdots, a_s\}$, where $\{p_i\}, i = 1, \cdots, m$ are the mutually orthogonal projections, there are the mutually orthogonal projections $e_1, e_2, \cdots, e_n \in A$ such that*

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ *for* $1 \leq j \leq n-1$,

(2) $\|e_j b - b e_j\| < \varepsilon$ *for* $1 \leq j \leq n$ *and all* $b \in \mathcal{F}$,

(3) $\|e_1 p_j e_1\| \geq 1 - \varepsilon$ *for* $1 \leq j \leq m$,

(4) *with* $e = \sum\limits_{j=1}^{n} e_j$, $[1 - e] \leq_\alpha [a]$.

When A is a unital simple C^*-algebra, the tracial Rokhlin property of the above definition is weaker than the Rokhlin property as in [10, 12], we weak the condition (4) to only require that the positive element $1 - e$ can be compared with the given positive element a by the action of α.

We define a slightly stronger version of the tracial Rokhlin property.

Definition 2.5 *Let A be a unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the tracial cyclic Rokhlin property if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, every nonzero positive element $a \in A$, every finite set $\mathcal{F} \subset A$, $\mathcal{F} = \{p_1, \cdots, p_m, a_1, \cdots, a_s\}$, where $\{p_i\}$, $i = 1, \cdots, m$ are the mutually orthogonal projections, there are the mutually orthogonal projections $e_1, e_2, \cdots, e_n \in A$ such that*

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ *for* $1 \leq j \leq n$, *where* $e_{n+1} = e_1$,

(2) $\|e_j b - b e_j\| < \varepsilon$ *for* $1 \leq j \leq n$ *and all* $b \in \mathcal{F}$,

(3) $\|e_1 p_j e_1\| \geq 1 - \varepsilon$ *for* $1 \leq j \leq m$,

(4) *with* $e = \sum\limits_{j=1}^{n} e_j$, $[1 - e] \leq_\alpha [a]$.

The only difference between the tracial Rokhlin property and the tracial cyclic Rokhlin property is that in condition (1), we require $\|\alpha(e_n) - e_1\| < \varepsilon$.

Theorem 2.1 *Let A be a unital C^*-algebra with real rank zero, and let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then A is α-simple if and only if the crossed product $A \rtimes_\alpha \mathbb{Z}$ is simple.*

Proof Let I be an α-invariant norm closed two-sided ideal of A. Then, by [3, Lemma 1], $I \rtimes_\alpha \mathbb{Z}$ is a norm closed two-sided ideal of $A \rtimes_\alpha \mathbb{Z}$.

Conversely, let a be a positive element of the C^*-algebra A, $\mathcal{F} = \{a_i; i = 1, 2, \cdots, n\}$ elements of A, $s_i \in \mathbb{N}$, $i = 1, 2, \cdots, n$ and $\varepsilon > 0$. We prove that there exists a positive element

$x \in A$ with $\|x\| = 1$ such that

$$\|xax\| \geq \|a\| - \varepsilon, \quad \|xa_i\alpha^{s_i}(x)\| \leq \varepsilon, \quad \|xa_i - a_ix\| < \varepsilon, \quad i = 1, 2, \cdots, n. \qquad (*)$$

Because A has real rank zero, let $\varepsilon > 0$, by [9, Theorem 3.2.5], there are mutually orthogonal projections $p_1, p_2, \cdots, p_m$ and positive real numbers $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that $\left\| a - \sum_{i=1}^{m} \lambda_i p_i \right\| < \frac{\varepsilon}{3}$. Let $a_0 = \sum_{i=1}^{m} \lambda_i p_i$, $C = \max\{\|a_1\|, \|a_2\|, \cdots, \|a_n\|\}$, $N = \max\{s_1, s_2, \cdots, s_n\}$ and $\varepsilon_0 = \min\{\frac{\varepsilon}{3\|a_0\|}, \frac{\varepsilon}{(N+2)C}\}$.

Apply the tracial Rokhlin property with N in place of n, with ε_0 in place of ε. We can obtain $e_1, e_2, \cdots, e_N$, such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon_0$ for $1 \leq j \leq N - 1$,

(2) $\|e_ja_i - a_ie_j\| < \varepsilon_0$ for $1 \leq j \leq N$ and $1 \leq i \leq n$,

(3) $\|e_1p_je_1\| \geq 1 - \varepsilon_0$ for $1 \leq j \leq m$.

Then $\|e_1a_0e_1\| = \left\| \sum_{i=1}^{m} \lambda_i e_1 p_i e_1 \right\| \geq \|\lambda_i e_1 p_i e_1\| \geq \lambda_i(1 - \varepsilon_0),\ i = 1, 2, \cdots, m.$

We get

$$\|e_1a_0e_1\| \geq \|a_0\|(1 - \varepsilon_0) \geq \|a_0\| - \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned}
\|e_1ae_1\| &= \|e_1a_0e_1 + e_1ae_1 - e_1a_0e_1\| \geq \|e_1a_0e_1\| - \|e_1ae_1 - e_1a_0e_1\| \\
&\geq \|e_1a_0e_1\| - \frac{\varepsilon}{3} \geq \|a_0\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \geq \|a\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \|a\| - \varepsilon, \\
\|e_1a_i\alpha^{s_i}(e_1)\| &= \|e_1a_i\alpha^{s_i}(e_1) - e_1a_i\alpha^{s_i-1}(e_1) + e_1a_i\alpha^{s_i-1}(e_1) \\
&\quad - e_1a_i\alpha^{s_i-2}(e_1) + \cdots + e_1a_i\alpha^1(e_1)\| \\
&< \|e_1a_i\alpha^1(e_1)\| + (s_i - 1)\varepsilon_0\|a_i\| \\
&< \|a_ie_1\alpha^1(e_1)\| + s_i\varepsilon_0\|a_i\| < (s_i + 1)\varepsilon_0\|a_i\| < \varepsilon.
\end{aligned}$$

So we get $(*)$. Applying this condition and noticing that A is α-simple, we can complete the proof the same as [5, Theorem 3.1]. We omit the details.

Applying $(*)$ and the same proof of Theorem 4.2 in [4], we can get the following result.

Theorem 2.2 *Let A be a unital C^*-algebra with real rank zero and let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property and A is α-simple. Then any non-zero hereditary C^*-subalgebra of the crossed product $A \rtimes_\alpha \mathbb{Z}$ has a non-zero projection which is equivalent to a projection in A.*

Lemma 2.1 *Let $B = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(l)}$ be a finite dimensional C^*-subalgebra of a unital C^*-algebra A, and let $e_{i,j}^{(s)} \in B$ be a system of matrix units for $M_{r(s)},\ s = 1, 2, \cdots, l$. Then for any $\delta > 0$, there exists $\sigma > 0$ satisfying the following: If $\|pe_{i,i}^{(s)} - e_{i,i}^{(s)}p\| < \sigma$ and $\|pe_{i,i}^{(s)}p\| > \frac{1}{2}$ for $s = 1, 2, \cdots, l,\ i = 1, 2, \cdots, r(s)$, then there is a monomorphism $\varphi : B \to pAp$ such that $\|pbp - \varphi(b)\| < \delta\|b\|$ for all $b \in B$.*

Proof It follows from the arguments in [9, Section 2.5] and [11, Proposition 2.3].

Proposition 2.1 *Let A be a unital C^*-algebra. Suppose that $\alpha \in \mathrm{Aut}(A)$ is approximately inner and has the tracial Rokhlin property. If for any closed two-sided ideal I of C^*-algebra A,* *there is an $n \in \mathbb{N}$, here n only depends on I, such that $K_0(A/I)$ is not n-divisible, then A is α-simple.*

Proof Suppose that A is not α-simple, so there exists a closed two-sided ideal I of C^*-algebra A such that $\alpha(I) = I$. By the hypothesis, there is an $n \in \mathbb{N}$ such that $K_0(A/I)$ is not n-divisible.

Let $a \in I$ be a non-zero positive element, and $0 < \varepsilon < 1$. There are the mutually orthogonal projections $e_1, e_2, \cdots, e_n \in A$ such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $1 \le j \le n-1$,

(2) with $e = \sum\limits_{j=1}^{n} e_j$, $[1-e] \le_\alpha [a]$.

Because α is approximately inner and by (1), we have $[e_1] = [e_2] = \cdots = [e_n]$ in $K_0(A)$.

If $p \in A$ is a projection such that $[p] \le [b]$, where $b \in I$ is a positive element, then there is a $v \in A$ such that $v^*v = p$ and $vv^* \in \overline{bAb} \subset I$. If $\pi : A \to A/I$ denotes quotient map, $\pi(v)\pi(v^*) = 0$ in A/I, $\pi(v) = 0$ in A/I, then $p \in I$.

In (2), $[1-e] \le_\alpha [a]$. By the definition of $\le_\alpha$, $a \in I$ and the discussion above, we have $1 - e \in I$, so $\pi(1-e) = 0$, $[1-e] = 0$ in $K_0(A/I)$, then $n[e_1] = [1]$ in $K_0(A/I)$. This contradicts that $K_0(A/I)$ is not n-divisible.

3 Main Results

In the proof of Theorem 3.2, we first prove $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) \le 1$, then use the following Lemma 3.1 to prove $\mathrm{RR}(A \rtimes_\alpha \mathbb{Z}) = 0$. The following lemma is similar to [12, Lemma 2.5].

Lemma 3.1 *Let A be a unital C^*-algebra with real rank zero, and let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property. Suppose that A is α-simple and the order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces. Let $\iota : A \to A \rtimes_\alpha \mathbb{Z}$ be the inclusion map. Then for every finite set $F \subset A \rtimes_\alpha \mathbb{Z}$, every $\varepsilon > 0$, every nonzero positive element $z \in A \rtimes_\alpha \mathbb{Z}$, and every sufficiently large $n \in N$ (depending on F, ε and z), there exist a projection $e \in A \subset A \rtimes_\alpha \mathbb{Z}$, a unital subalgebra $D \subset e(A \rtimes_\alpha \mathbb{Z})e$, a projection $p \in D$, a projection $f \in A$, and an isomorphism $\varphi : M_n \otimes fAf \to D$, such that*

(1) with $(e_{j,k})$ being the standard system of matrix units for M_n, we have $\varphi(e_{1,1} \otimes a) = \iota(a)$ for all $a \in fAf$ and $\varphi(e_{k,k} \otimes 1) \in \iota(A)$ for $1 \le k \le n$,

(2) with $(e_{j,k})$ as in (1), we have $\|\varphi(e_{j,j} \otimes a) - \alpha^{j-1}(\iota(a))\| \le \varepsilon\|a\|$ for all $a \in fAf$,

(3) for every $a \in F$, there exist $b_1, b_2 \in D$ such that $\|pa - b_1\| < \varepsilon, \|ap - b_2\| < \varepsilon$, and $\|b_1\|, \|b_2\| \le \|a\|$,

(4) there is an $m \in \mathbb{N}$ such that $\frac{2m}{n} < \varepsilon$ and $p = \sum\limits_{j=m+1}^{n-m} \varphi(e_{j,j} \otimes 1)$,

(5) the projection $1 - p$ is Murray-von Neumann equivalent in $A \rtimes_\alpha \mathbb{Z}$ to a projection in the hereditary subalgebra of $A \rtimes_\alpha \mathbb{Z}$ generated by z and $\tau(1-p) < \varepsilon$ for all $\tau \in T(A \rtimes_\alpha \mathbb{Z})$.

Proof Let $\varepsilon > 0$, $F \subset A \rtimes_\alpha \mathbb{Z}$ be a finite set, and let $z \in A \rtimes_\alpha \mathbb{Z}$ be a nonzero positive element.

Let u be a standard unitary in the crossed product $A \rtimes_\alpha \mathbb{Z}$. We regard A as a subalgebra of $A \rtimes_\alpha \mathbb{Z}$ in the usual way. Choose $m \in \mathbb{N}$ such that for every $x \in F$ there are $a_l \in A$ for $-m \le l \le m$ such that $\left\| x - \sum\limits_{l=-m}^{m} a_l u^l \right\| < \frac{\varepsilon}{2}$. For each $x \in F$, choose one such expression, and let $S \subset A$ be a finite set which contains all the coefficients used for all elements of F. Let $M = 1 + \sup_{a \in S} \|a\|$.

Since $A \rtimes_\alpha \mathbb{Z}$ has (SP)-property and is simple, by Theorems 2.2 and 2.1, we can use [9, Lemma 3.5.7] to find nonzero orthogonal Murray-von Neumann equivalent projections $g_0, g_1, \cdots, g_{2m} \in \overline{z(A \rtimes_\alpha \mathbb{Z})z}$.

Since $A \rtimes_\alpha \mathbb{Z}$ is simple, g_0 is a nonzero projection, and the tracial state space $T(A \rtimes_\alpha \mathbb{Z})$ of $A \rtimes_\alpha \mathbb{Z}$ is weak-$*$ compact, we have $\delta = \inf_{\tau \in T(A \rtimes_\alpha \mathbb{Z})} \tau(g_0) > 0$. Now let $n \in \mathbb{N}$ be any integer such that $n > \max(\frac{1}{\delta}, (N+2)(2m+1), \frac{4m}{\varepsilon})$.

Set $\varepsilon_0 = \frac{\varepsilon}{10(2m+1)n^2 M}$.

Choose $\varepsilon_1 > 0$ so small that whenever $e_1, e_2, \cdots, e_n$ are mutually orthogonal projections in a unital C^*-algebra B and $u \in B$ is a unitary such that $\|ue_ju^* - e_{j+1}\| < \varepsilon_1$ for $1 \le j \le n$, then there is a unitary $v \in B$ such that $\|v - u\| < \varepsilon_0$ and $ve_jv^* = e_{j+1}$ for $1 \le j \le n$. We can use [9, Lemma 3.5.7] to find nonzero orthogonal Murray-von Neumann equivalent projections $h_1, h_2, \cdots, h_{n+2} \in \overline{g_0(A \rtimes_\alpha \mathbb{Z})g_0}$ which are Murray-von Neumann equivalent in $A \rtimes_\alpha \mathbb{Z}$. Further use Theorem 2.2 to find a nonzero projection $q \in A$ which is Murray-von Neumann equivalent in $A \rtimes_\alpha \mathbb{Z}$ to a projection in $\overline{h_1(A \rtimes_\alpha \mathbb{Z})h_1}$.

Apply the tracial Rokhlin property with $n-1$ in place of n, with $\min(1, \varepsilon_0, \varepsilon_1)$ in place of ε, with S in place of F, and with q in place of x. Recall the resulting projections $e_1, e_2, \cdots, e_n$, and let $e = \sum_{j=1}^{n} e_j$, $[1-e] \le_\alpha [q]$. Apply the choice of ε_1 to these projections and the standard unitary u, and obtain a unitary $v \in A \rtimes_\alpha \mathbb{Z}$ as in the previous paragraph.

We can get Conditions (1)–(4) by the same proof of Lemma 2.5 in [12]. We omit them.

It remains to verify Condition (5) of the conclusion. We have

$$1 - p = 1 - e + \sum_{j=1}^{m} e_j + \sum_{j=n-m+1}^{n} e_j.$$

By construction, we have $[1-e] \le_\alpha [h_1] \le [g_0]$. Now let τ be any tracial state on $A \rtimes_\alpha \mathbb{Z}$. Then $\tau(e_j) = \tau(e_1)$ for all j, whence $\tau(e_j) \le \frac{1}{n}$. The inequality $n > \frac{1}{\delta} \ge \frac{1}{\tau(g_0)}$ therefore implies $\tau(e_j) < \tau(g_0)$. Since all g_j are Murray-von Neumann equivalent, it follows that for any tracial state τ on $A \rtimes_\alpha \mathbb{Z}$, we have $\tau(e_j) < \tau(g_j)$ and $\tau(e_{n-j}) < \tau(g_{m+j})$ for $1 \le j \le m$. So the order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces implies that $e_j \le g_j$ and $e_{n-j} \le g_{m+j}$ in $A \rtimes_\alpha \mathbb{Z}$ for $1 \le j \le m$. Thus $[1-p] \le_\alpha \Big[\sum_{j=0}^{2m} g_j\Big]$ which is a projection in the hereditary subalgebra $\overline{z(A \rtimes_\alpha \mathbb{Z})z}$.

$$\tau(1-p) = \tau(1-e) + \tau\Big(\sum_{j=1}^{m} e_j + \sum_{j=n-m+1}^{n} e_j\Big) \le \frac{1}{2m(n+2)} + \frac{2m}{n} < \varepsilon.$$

This is Condition (5) of the conclusion.

Theorem 3.1 *Let A be a unital C^*-algebra with real rank zero, and let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property. Suppose that A is α-simple and the order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces. Then $A \rtimes_\alpha \mathbb{Z}$ has real rank zero.*

Proof By applying Lemma 3.1 and the same proof of Theorem 4.5 in [12], we get the theorem.

Theorem 3.2 *Let A be a unital* AF*-algebra. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property and A is α-simple. Suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^J = \mathrm{id}_{K_0(A)}$. Then* $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.

Proof By Theorem 2.1, $A \rtimes_\alpha \mathbb{Z}$ is a unital simple C^*-algebra.

Let $0 < \varepsilon < 1$ and $\mathcal{F} \subset A \rtimes_\alpha \mathbb{Z}$ be a finite set. To simplify notation, without loss of generality, we may assume $\mathcal{F} = \mathcal{F}_0 \cup \{u\}$, where $\mathcal{F}_0 \subset A$ is a finite subset of the unit ball which contains 1_A and u is a unitary which implements α, i.e., $\alpha(a) = u^* a u$ for all $a \in A$. Choose an integer k which is a multiple of J such that $\frac{2\pi}{k-2} < \frac{\varepsilon}{16}$. Put $\mathcal{F}_1 = \mathcal{F}_0 \cup \{u^i a (u^*)^i : a \in \mathcal{F}_0, -k \leq i \leq k\}$.

Fix $b_0 \in (A \rtimes \mathbb{Z})_+ \backslash \{0\}$. It follows from Theorem 2.2 that there is a nonzero projection $r_0 \in A$ which is equivalent to a nonzero projection in the hereditary C^*-subalgebra generated by b_0.

Let $\delta = \frac{\varepsilon}{16k^2}$. Since A is a unital AF-algebra, denoted by $A = \overline{\bigcup_{m=1}^{\infty} A_m}$, where A_m is a finite-dimensional C^*-algebra for $m = 1, 2, \cdots$, there is a lager enough $m \in \mathbb{N}$ such that $b \in_\delta A_m$ for all $b \in \mathcal{F}_1$ and $1_A \in A_m$. Let $A_m = M_{r(1)} \oplus M_{r(2)} \oplus M_{r(l)}$. Note $[(u^k)^* e u^k] = [e]$ in $K_0(A)$ for all projection $e \in A_m$. By [9, Theorem 3.4.6], there exists a unitary $w \in U(A)$ such that $w^*(u^k)^* b u^k w = b$ for all $b \in A_m$. Because A is an AF-algebra, $w \in U_0(A)$. By [10, Lemma 2.6], we have the unitaries w_i, $i = 1, 2, \cdots, k-1$ associated with finite dimensional C^*-subalgebra A_m such that $w = w_1 w_2 \cdots w_{k-1}$, $\|w_i - 1\| \leq \frac{\pi}{k-2}$. Since $b \in_\delta A_m$ for all $b \in \mathcal{F}_1$, there is an $a(b) \in A_m$ such that $\|a(b) - a\| < \delta$. Let $e_{ij}^{(s)}$ be a system of matrix units for $M_{r(s)}$ $(s = 1, 2, \cdots, l,\ i, j = 1, 2 \cdots r(s))$, and let $\mathcal{G}_0 = \{a(b) \,|\, b \in \mathcal{F}_1\} \cup \{e_{ij}^{(s)} \,|\, s = 1, 2, \cdots, l,\ i, j = 1, 2 \cdots r(s)\}$.

Define $\mathcal{F}_2 = \{u^i b u^{-i} : b \in \mathcal{G}_0, -k \leq i \leq k\}$ and let $w_k = 1$.

$$\mathcal{F}_3 = \{(w_{i_1} w_{i_1+1} \cdots w_i) a (w_{i_2} w_{i_2+1} \cdots w_i)^* : a \in \mathcal{F}_1 \cup \mathcal{F}_2,\ 1 \leq i, i_1, i_2 \leq k,\ i_1 \leq i,\ i_2 \leq i\}.$$

Note that $w, w_i \in \mathcal{F}_3$, $i = 1, 2, \cdots, k-1$.

Since α has the tracial cyclic Rokhlin property, $e_{i,j}^{(s)} \in A_m$ is a system of matrix units for $M_{r(s)}$, $s = 1, 2, \cdots, l$, let $\sigma > 0$ be associated with A_m and δ in Lemma 2.1, and let $\eta = \min\{\delta, \sigma\}$. Then there exist projections $e_1, e_2, \cdots, e_k \in A$ such that

(1) $\|\alpha(e_i) - e_{i+1}\| < \frac{\eta}{k}$ for $1 \leq i \leq k$, $e_{k+1} = e_1$,

(2) $\|e_i a - a e_i\| < \frac{\eta}{k}$ for $a \in \mathcal{F}_3$,

(3) $\|e_1 e_{jj}^{(s)} e_1\| \geq 1 - \frac{\eta}{k}$ for $s = 1, 2, \cdots, l$, $j = 1, 2, \cdots, r(s)$,

(4) $\left[1 - \sum_{i=1}^{k} e_i\right] \leq_\alpha [r_0]$.

Set $p = \sum_{i=1}^{k} e_i$. From (1) above, one can estimate

$$\|up - pu\| = \Big\| \sum_{i=1}^{k} u e_{i+1} - \sum_{i=1}^{k} e_i u \Big\| \leq \sum_{i=1}^{k} \|u e_{i+1} - e_i u\| = \sum_{i=1}^{k} \|u e_{i+1} - u\alpha(e_i)\| < \eta.$$

By (1) above, one can see that there is a unitary $v \in A$ such that $\|v - 1\| < \frac{2\eta}{k}$ and $v^* u^* e_i u v = e_{i+1}$, $i = 1, 2, \cdots, k$. Set $u_1 = uv$. Then $u_1^* e_i u_1 = e_{i+1}$, $i = 1, 2, \cdots, k$ and $e_{k+1} = e_1$. In particular, $u_1^k e_1 = e_1 u_1^k$. For any $a \in \mathcal{F}_3 \cap A_m$, since $w \in \mathcal{F}_3$,

$$e_1 w^* e_1 (u_1^k)^* e_1 a e_1 u_1^k e_1 w e_1 \approx_{\frac{3\eta}{k}} e_1 a e_1.$$

By (2) and (3) above, it follows from Lemma 2.1 that there is a monomorphism $\varphi : A_m \to e_1Ae_1$ such that $\|\varphi(a) - e_1ae_1\| < \delta\|a\|$ for all $a \in A_m$.

By applying [10, Lemma 2.9], we obtain unitaries $x, x_1, x_2, \cdots, x_{k-1} \in U_0(e_1Ae_1)$ such that $\|x - e_1we_1\| < \delta$, $\|x_i - e_1w_ie_1\| < \delta$, $x = x_1x_2\cdots x_{k-1}$ and $x^*(u_1^k)^*au_1^kx = a$ for all $a \in \varphi(A_m)$.

Let $Z = \sum_{i=1}^{k} e_iu_1^{k+1-i}x_i(u_1^{k-i})^* + (1-p)u_1$. Define $B = \varphi(A_m)$. Then

$$\|Z - u_1\| \le \max_i\{\|x_i - e_1\|\} \le \max_i\{\|x_i - e_1w_ie_1\| + \|e_1w_ie_1 - e_1\|\} < \delta + \frac{\eta}{k} + \frac{\pi}{k-2},$$

$(Z^k)^*bZ^k = b$ for all $b \in B$ and

$$(Z^i)^*e_1Z^i \le e_{i+1}, \quad Z^i = u_1^k(x_1x_2\cdots x_i)(u_1^{k-i})^*, \quad i = 1, 2, \cdots, k,\ e_{k+1} = e_1.$$

Write $B = C_1 \oplus C_2 \oplus \cdots C_N$, let $\{c_{is}^{(j)}\}$ be the matrix units for C_j, $j = 1, 2, \cdots, N$, where $C_j = M_{R(j)}$ and put $q = 1_B$. Define $D_0 = B \bigoplus \bigoplus_{i=1}^{k-1} Z^{i*}BZ^i$, and denote by D_1 the C^*-subalgebra generated by B and $c_{ss}^{(j)}Z^i$, $s = 1, 2, \cdots, R(j)$, $j = 1, 2, \cdots, N$ and $i = 0, 1, 2, \cdots, k-1$. Then $D_1 \cong B \otimes M_k$ and $D_1 \supset D_0$.

Define $q_{ss}^{(j)} = \sum_{i=0}^{k-1} Z^{i*}c_{ss}^{(j)}Z^i$, $q^{(j)} = \sum_{s=1}^{R(j)} q_{ss}^{(j)}$ and $Q = \sum_{j=1}^{N} q^{(j)} = 1_{D_1}$. Note $Q = \sum_{i=0}^{k-1}(Z^i)^*qZ^i$ and

$$q_{ss}^{(j)}Z = \Big(\sum_{i=0}^{k-1} Z^{i*}c_{ss}^{(j)}Z^i\Big)Z = Z\sum_{i=0}^{k-1}(Z^{i+1})^*c_{ss}^{(j)}Z^{i+1} = Z\Big(\sum_{i=1}^{k-1} Z^{i*}c_{ss}^{(j)}Z^i + c_{ss}^{(j)}\Big) = Zq_{ss}^{(j)}.$$

It follows from [10, Lemma 2.11] that $c_{11}^{(j)}, c_{11}^{(j)}Z^i$ and $c_{11}^{(j)}Z^kc_{11}^{(j)}$ generate a C^*-subalgebra which is isomorphic to $C(X_j) \otimes M_k$ for some compact subset $X_j \subset S^1$. Moreover, $q_{ss}^{(j)}Zq_{ss}^{(j)}$ is in the C^*-subalgebra. Let D be the C^*-subalgebra generated by D_1 and $c_{11}^{(j)}Z^kc_{11}^{(j)}$. Then $D \cong \bigoplus_{j=1}^{N} C(X_j) \otimes B \otimes M_k$. It follows that $q^{(j)}$ and Q commutes with Z. Therefore $QZQ \in D$. Thus,

$$\begin{aligned}\|Qu - uQ\| &\le \|Qu - Qu_1\| + \|Qu_1 - QZ\| + \|ZQ - u_1Q\| + \|u_1Q - uQ\| \\ &< \frac{4\eta}{k} + 2\delta + \frac{2\pi}{k-2} < \varepsilon.\end{aligned}$$

From $QZQ \in D$, we also have $QuQ \in_\varepsilon D$.

For $b \in \mathcal{F}_0$, we compute

$$\begin{aligned}(Z^i)^*q(Z^i)b &= (Z^i)^*qu_1^k(x_1x_2\cdots x_i)(u_1^{k-i})^*b \\ &\approx_{k\delta+2\eta} (Z^i)^*qu_1^k(w_1w_2\cdots w_i)(u^{k-i})^*b \\ &= (Z^i)^*qu_1^k(w_1w_2\cdots w_i)(u^{k-i})^*bu^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^* \\ &\quad\cdot [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^*.\end{aligned}$$

Let $c_i = (u^{k-i})^*bu^{k-i}$. Then $c_i \in \mathcal{F}_1$. There is an $a_i \in \mathcal{G}_0 \subset A_m$ such that $\|c_i - a_i\| < \delta$. Since $(w_1w_2\cdots w_i)\mathcal{F}_1(w_1w_2\cdots w_i)^* \subset \mathcal{F}_3$, we have

$$(Z^i)^*qu_1^k(w_1w_2\cdots w_i)(u^{k-i})^*bu^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*[u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^*$$

$$
\begin{aligned}
&= (Z^i)^* q u_1^k (w_1w_2\cdots w_i) c_i (w_1w_2\cdots w_i)^* (u_1^k)^* [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^* \\
&\approx_\delta (Z^i)^* q u_1^k (w_1w_2\cdots w_i) a_i (w_1w_2\cdots w_i)^* (u_1^k)^* [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^* \\
&\approx_\delta (Z^i)^* e_1 u_1^k (w_1w_2\cdots w_i) a_i (w_1w_2\cdots w_i)^* (u_1^k)^* [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^* \\
&\approx_{\frac{\eta}{k}} (Z^i)^* u_1^k (w_1w_2\cdots w_i) a_i (w_1w_2\cdots w_i)^* (u_1^k)^* e_1 [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^* \\
&\approx_\delta (Z^i)^* u_1^k (w_1w_2\cdots w_i) c_i (w_1w_2\cdots w_i)^* (u_1^k)^* q [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^* \\
&\approx_\delta (Z^i)^* u_1^k (w_1w_2\cdots w_i) (u^{k-i})^* b u^{k-i} (w_1w_2\cdots w_i)^* (u_1^k)^* q [u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^*]^* \\
&\approx_{k\delta+2\eta} b (Z^i)^* q Z^i.
\end{aligned}
$$

Hence

$$
\|(Z^i)^* q Z^i b - b (Z^i)^* q Z^i\| < 2(k\delta + 2\eta + \delta + \delta) + \frac{\eta}{k} < \frac{\varepsilon}{k}, \quad i = 0, 1, \cdots, k-1.
$$

Therefore, for $b \in \mathcal{F}_0$, $\|Qb - bQ\| < k \cdot (\frac{\varepsilon}{k}) = \varepsilon$. It follows that $\|Qa - aQ\| < \varepsilon$ for all $a \in \mathcal{F}$.

For any $b \in \mathcal{F}_0$, a same estimation shows

$$
\|qZ^i b (Z^i)^* q - q u_1^k (w_1w_2\cdots w_i)(u^{k-i})^* b u^{k-i} (w_1w_2\cdots w_i)^* (u_1^k)^* q\| < 2k\delta + 4\eta.
$$

However, $q u_1^k (w_1w_2\cdots w_i)(u^{k-i})^* b u^{k-i}(w_1w_2\cdots w_i)^*(u_1^k)^* q \in_{\delta+2\delta+\frac{4\eta}{k}} B$. It follows that, for $b \in \mathcal{F}_0$,

$$
(Z^i)^* q Z^i b (Z^i)^* q Z^i \in_{\frac{\varepsilon}{k}} (Z^i)^* B Z^i, \quad i = 0, 1, \cdots, k-1.
$$

We obtain $QbQ \in_\varepsilon D_1 \subset D$ and then $QaQ \in_\varepsilon D$ for all $a \in \mathcal{F}$.

Because $\left[1 - \sum_{i=1}^{k} e_i\right] = [1-p] \leq_\alpha [r_0]$ in A, there exist the mutually orthogonal projections p_i, the mutually orthogonal positive elements a_i and $s_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, n$ such that $p = \sum_{i=1}^{n} p_i$, $\{a_i\}_{i=1}^n$ belong to the hereditary C^*-subalgebra generated by r_0, and $[\alpha^{s_i}(p_i)] \leq [a_i]$, $i = 1, \cdots, n$. Because $[\alpha^{s_i}(p_i)] = [u^{s_i} p_i (u^{s_i})^*] = [p_i]$ in $A \rtimes_\alpha \mathbb{Z}$, we obtain $\left[1 - \sum_{i=1}^{k} e_i\right] \leq [r_0]$ in $A \rtimes_\alpha \mathbb{Z}$.

By computation we can get

$$
[1-Q] \leq \left[1 - \sum_{i=1}^{k} e_i\right] \leq [r_0] \leq [b_0].
$$

So $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) \leq 1$. The order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces by [9, Theorem 3.7.2].

By applying Theorem 3.1, we have $\mathrm{RR}(A \rtimes_\alpha \mathbb{Z}) = 0$. By [10, Lemma 3.2], we conclude $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.

Corollary 3.1 *Let A be a unital* AF-*algebra. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property and A is α-simple. Suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^J = \mathrm{id}_{K_0(A)}$. Then the restriction map is a bijection from the tracial states of $A \rtimes_\alpha \mathbb{Z}$ to the α-invariant tracial states of A.*

Proof Since A has real rank zero and $A \rtimes_\alpha \mathbb{Z}$ also has real rank zero by Theorem 3.2, the corollary follows from [7, Proposition 2.2].

Example 3.1 Let $A = A_0 \oplus A_0$, where A_0 is an infinite dimensional unital simple AF-algebra. Let $\beta \in \mathrm{Aut}(A_0)$ be an approximately inner automorphism of A_0 and have the traical cyclic Rokhlin property as in [10]. Define $\alpha \in \mathrm{Aut}A$ by $\alpha(a, b) = (\beta(b), \beta(a))$.

Obviously, A is α-simple. Because β is an approximately inner automorphism of A_0, therefore $\beta_{*0} = \mathrm{id}_{K_0(A_0)}$, then we have $\alpha^2_{*0} = \mathrm{id}_{K_0(A)}$.

Because β is an approximately inner automorphism of A_0 and has the traical cyclic Rokhlin property as in [10], furthermore by applying [10, Lemma 2.8], it is easy to verify that α has the traical cyclic Rokhlin property in this paper.

So (A, α) satisfies the conditions of Theorem 3.2, then we have $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.

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