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Gap Property of Bi-Lipschitz Constants of Bi-Lipschitz Automorphisms on Self-similar Sets***

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Abstract For a given self-similar set $E \subset \mathbb{R}^d$ satisfying the strong separation condition, let $\mathrm{Aut}(E)$ be the set of all bi-Lipschitz automorphisms on E. The authors prove that $\{f \in \mathrm{Aut}(E) : \mathrm{blip}(f) = 1\}$ is a finite group, and the gap property of bi-Lipschitz constants holds, i.e., $\inf\{\mathrm{blip}(f) \neq 1 : f \in \mathrm{Aut}(E)\} > 1$, where $\mathrm{lip}(g) = \sup\limits_{\substack{x,y\in E\\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|}$ and $\mathrm{blip}(g) = \max(\mathrm{lip}(g), \mathrm{lip}(g^{-1}))$.

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1 Introduction

Lipschitz equivalence of fractal is a very interesting topic, for example, Cooper and Pignatro [1], Falconer and Marsh [3–5], David and Semmes [2], Xi [13, 14] studied the shape of Cantor set, nearly Lipschitz equivalence, BPI equivalence and quasi-Lipschitz equivalence.

For self-similar sets, we suppose that $\{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^N$ are similitudes with contractive ratios $\{r_i\}_{i=1}^N \subset (0, 1)$, and let $K = \bigcup\limits_{i=1}^N S_i(K)$ be the corresponding self-similar set. If $\bigcup\limits_{i=1}^N S_i(K)$ is a pairwise disjoint union, we say that K satisfies the strong separation condition. Let $\mathrm{Aut}(K)$ denote the set of all bi-Lipschitz automorphisms on K. For $g \in \mathrm{Aut}(K)$, let $\mathrm{blip}(g)$ be the bi-Lipschitz constant of g defined by

$$\mathrm{blip}(g) = \max(\mathrm{lip}(g), \mathrm{lip}(g^{-1})),$$

where $\mathrm{lip}(f) = \sup\limits_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}$. Notice $\mathrm{blip}(g) \geq 1$ for any $g \in \mathrm{Aut}(K)$.

In [9], Lyapina discussed bi-Lipschitz automorphisms on Cantor set $C_r = (rC_r) \cup (rC_r + 1 - r)$, which is a self-similar set with ratio $r \in (0, \frac{1}{2})$, Then, in [9, 15], it was proved that $\{f \in \mathrm{Aut}(C_r) : \mathrm{blip}(f) = 1\} = \{f_1(x) \equiv x,\ f_2(x) \equiv 1 - x\}$, and

$$\inf\{\mathrm{blip}(f) > 1 : f \in \mathrm{Aut}(C_r)\} = \min\Big[\frac{1}{r}, \frac{1 - 2r^3 - r^4}{(1 - 2r)(1 + r + r^2)}\Big] > 1,$$

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which means that there is a gap in $\{\mathrm{blip}(f) : f \in \mathrm{Aut}(C_r)\}$ near 1.

Furthermore, in [7], the complete set $\mathcal{M}$ with suitable Moran-like structure in $\mathbb{R}^1$ was studied to get the gap property of bi-Lipschitz constants: there is a constant $c_0 > 1$ such that for any $f \in \mathrm{Aut}(\mathcal{M})$, $\mathrm{blip}(f) = 1$ or $\mathrm{blip}(f) \geq c_0$.

In this paper, we obtain the following results.

Theorem 1.1 *Suppose that the self-similar set K satisfies the strong separation condition. Then $\{f \in \mathrm{Aut}(K) : f$ is isometric$\}$ is a finite group.*

Example 1.1 Theorem 1.1 may be not valid for self-similar sets in other spaces. For example, when considering the symbolic system $\Sigma_2 = \{0,1\}^\infty$, which is a self-similar set satisfying the strong separation condition and equipped with a metric d satisfying $d(x_1x_2\cdots, y_1y_2\cdots) = 2^{-\min\{i:\ x_i \neq y_i\}}$ for $x_1x_2\cdots \neq y_1y_2\cdots$, we have

$$D^* \subset \{f \in \mathrm{Aut}(K) : f \text{ is isometric}\}, \tag{1.1}$$

where $D^* = \{$bijection $f : \Sigma_2 \to \Sigma_2 \mid$ for any $n \geq 0$ and any $u_1\cdots u_n \in \{0,1\}^n$, there exists $v_1\cdots v_n \in \{0,1\}^n$ such that $f([u_1\cdots u_n]) = [v_1\cdots v_n]\}$. Here $[w]$ is the cylinder with respect to the word w. It is easy to check $D^* \subset \{f \in \mathrm{Aut}(K) : f$ is isometric$\}$, however $\#D^* = \infty$ and thus $\{f \in \mathrm{Aut}(K) : f$ is isometric$\}$ is an infinite group.

Example 1.2 For any integer $n \geq 2$, there is a self-similar set $K_n \subset \mathbb{R}^2 = \mathbb{C}$ such that $\#\{f \in \mathrm{Aut}(K_n) : f$ is isometric$\} \geq n$. In fact, for a given n, let $z_j = \mathrm{e}^{\frac{2\pi j}{n}}$ for $j = 0, \cdots, n-1$. Then $G = \{z_0, \cdots, z_{n-1}\} = \{z \in \mathbb{C} : z^n = 1\}$ is a group. Suppose that $T_j : \mathbb{C} \to \mathbb{C}$ is defined by $T_j(z) = r(z - z_i)$, where $r \in (0,1)$ is small enough. Then we get a self-similar set $K_n = \bigcup\limits_{j=0}^{n-1} T_j K_n$. Let $f_i(z) = z_i z$ $(j = 0, \cdots, n-1)$. We conclude

$$\{f_i\}_{i=0}^{n-1} \subset \{f \in \mathrm{Aut}(K_n) : f \text{ is isometric}\}.$$

So we only need to verify $f_i K_n = K_n$, i.e., $f_i K_n = \bigcup\limits_{j=0}^{n-1} T_j(f_i K_n)$. In fact $z_i z_j = z_{i+j \pmod n}$, we have

$$\begin{aligned} f_i K_n &= f_i \bigcup_{j=0}^{n-1} T_j K_n = \bigcup_{j=0}^{n-1} z_i r(K_n - z_j) = \bigcup_{j=0}^{n-1} r(z_i K_n - z_{i+j \pmod n}) \\ &= \bigcup_{t=0}^{n-1} r(f_i K_n - z_t) = \bigcup_{t=0}^{n-1} T_j(f_i K_n). \end{aligned}$$

Theorem 1.2 *Suppose that the self-similar set K satisfies the strong separation condition. Then*

$$b_* := \inf\{\mathrm{blip}(f) \colon f \in \mathrm{Aut}(K) \text{ and } \mathrm{blip}(f) > 1\} > 1,$$

and there is an automorphism $f \in \mathrm{Aut}(K)$ such that $\mathrm{blip}(f) = b_*$.

In [7], there is a Moran set E constructed such that

$$b_* := \inf\{\mathrm{blip}(f) \colon f \in \mathrm{Aut}(E) \text{ and } \mathrm{blip}(f) > 1\} = 1. \tag{1.2}$$

Moreover, it is easy to see that equation (1.2) holds for any subset E of the Euclidean space which contains non-empty interior.

Let $O(K) = \{f \in \mathrm{Aut}(K) : f \text{ is isometric}\}$. The following property is useful:

$$\mathrm{blip}(hg) = \mathrm{blip}(gh) = \mathrm{blip}(g), \tag{1.3}$$

whenever $h \in O(K)$ and $g \in \mathrm{Aut}(K)$.

We give an equivalence relation $\sim$ on $\mathrm{Aut}(K)$ such that $f \sim g$ if and only if $f \circ g^{-1} \in O(K)$. Then we can induce a metric on $\mathrm{Aut}(K)/\sim$ such that

$$d([f_1],[f_2]) = \log(\mathrm{blip}(f_1 f_2^{-1})), \tag{1.4}$$

where $[f]$ is the equivalence relation with respect to f, and $d([f_1],[f_2])$ is independent of the choice of f_1 and f_2, since for any $f_i' = h_i \circ f_i \in [f_i]$ with $h_i \in O(K)$ for $i = 1, 2$, we have $\mathrm{blip}(f_1' f_2'^{-1})) = \mathrm{blip}(h_1 f_1 f_2^{-1} h_2^{-1})$ due to (1.3).

Then we get the following theorem.

Theorem 1.3 *Suppose that K is a self-similar set satisfying the strong separation condition. Then $d([f],[g])$ defined by (1.4) is a metric on $\mathrm{Aut}(K)/\sim$ and $(\mathrm{Aut}(K)/\sim, d)$ is discrete, and $\mathrm{Aut}(K)$ is also discrete under the following metric:*

$$D(f,g) = \log[\mathrm{blip}(fg^{-1})] + \sup_{x\in K} |f(x) - g(x)|.$$

The paper is organized as follows. Section 2 is a preliminary including Proposition 2.1 and Lemma 2.4, which are key points of this paper. Section 3 is devoted to proving Theorems 1.1–1.3. Finally, we give the proof of Proposition 2.1 in Appendix.

2 Preliminaries

Suppose that the similitudes $\{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^N$ with contractive ratios $\{r_i\}_{i=1}^N \subset (0,1)$ are given. Let $K = \bigcup_{i=1}^N S_i(K)$ be the corresponding self-similar set satisfying the strong separation condition

$$S_i(K) \cap S_j(K) = \emptyset, \quad \text{whenever } i \neq j.$$

Definition 2.1 *For a given subset X of the Euclidean space $\mathbb{R}^d$, let $\dim(X)$ be the dimension of the smallest hyperplane containing X. Here set $\dim(X) = d$ if X can not be contained in any hyperplane of $\mathbb{R}^d$.*

Lemma 2.1 *Suppose that H is the smallest hyperplane containing K, or $H = \mathbb{R}^d$ if K can not be contained in any hyperplane of $\mathbb{R}^d$. Then $S_iH = H$ for each i.*

Proof Suppose on the contrary that $S_iH \neq H$. We have $S_iK \subset S_iK \cap K \subset S_iH \cap H$, where $\dim(S_iH \cap H) < \dim(H) = \dim(K)$. Then $\dim(S_iK) < \dim(K)$, which contradicts the fact that $\dim(S_iK) = \dim(K)$ due to the similarity of S_i.

Without loss of generality, we assume $K \subset \mathbb{R}^d$ and

$$\dim(K) = d. \tag{2.1}$$

Otherwise, we get

$$T_i = S_i|_H \quad \text{with} \quad K = \bigcup_{i=1}^N T_iK,$$

where H is the smallest hyperplane containing K. Here by Lemma 2.1, we have $T_i : H \to H$ where H is isometric to the Euclidean space $\mathbb{R}^{\dim(H)}$.

The key points of the paper are the isometric extension theorem on Euclidean spaces (see Proposition 2.1) and Lemma 2.4 as its deduction. Proposition 2.1 may be known, but we will give its proof in Appendix only for the self-containedness of this paper.

Proposition 2.1 *Suppose that $A \subset \mathbb{R}^d$ with $\dim(A) = d$. Then each isometry on A exists a unique isometric extension on $\mathbb{R}^d$.*

Under assumption (2.1), we get the following lemma from Proposition 2.1.

Lemma 2.2 *For any isometry f on $K \subset \mathbb{R}^d$ with $\dim(K) = d$, there is a linear and isometric extension $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $F|_K = f$ and $F(x) = Qx + b$ where $Q_{d\times d}$ is an orthogonal matrix and $b \in \mathbb{R}^d$.*

Here some compact property of the function space $\mathrm{Aut}(K)$ under the uniform topology is needed. For $c \geq 1$, let

$$\mathrm{Aut}_c(K) := \big\{f \in \mathrm{Aut}(K) \colon \mathrm{blip}(f) \leq c\big\}.$$

Lemma 2.3 *Suppose that $\{f_i\}_{i\geq 1} \subset \mathrm{Aut}_c(K)$ and $\{f_i\}$ converges uniformly to a limit function f. Then $f \in \mathrm{Aut}_c(K)$ and*

$$\mathrm{blip}(f) \leq \liminf_{i\to\infty} \mathrm{blip}(f_i). \tag{2.2}$$

As a result, the set $\mathrm{Aut}_c(K)$ is compact with respect to the uniform topology.

Proof It is easy to check that $f \in \mathrm{Aut}_c(K)$ and $\mathrm{blip}(f) \leq \liminf_{i\to\infty} \mathrm{blip}(f_i)$. Therefore the set $\mathrm{Aut}_c(K)$ is closed under the uniform topology. Since $\mathrm{Aut}_c(K)$ is also equicontinuous, Arzelà-Ascoli theorem implies that $\mathrm{Aut}_c(K)$ is compact.

For each finite word $w = u_1 \cdots u_n \in \{1, \cdots, N\}^n$ of length $|w| = n$, we write

$$S_w = S_{u_1} \circ \cdots \circ S_{u_n} \quad \text{and} \quad K_w = S_w(K).$$

Let $\mathrm{conv}(B)$ denote the convex hull of $B \subset \mathbb{R}^d$. Notice that for any $B \subset \mathbb{R}^d$ and any linear mapping $T : \mathbb{R}^d \to \mathbb{R}^d$,

$$T[\mathrm{conv}(B)] = \mathrm{conv}[TB]. \tag{2.3}$$

Lemma 2.4 *There exists an integer $\gamma > 0$ depending on K such that if $f \in \mathrm{Aut}(K)$ is an isometry of K onto K with*

$$f(K_w) = K_w \quad \text{for any } w \in \{1, \cdots, N\}^\gamma, \tag{2.4}$$

then $f = \mathrm{id}$, i.e., $f(x) \equiv x$ for any $x \in K$.

Proof By the above discussion, we may assume $\dim(K) = d$, and thus we can pick $\{x_i\}_{i=0}^d \subset K$ such that $\{x_i - x_0\}_{i=1}^d$ are linearly independent. Let $C = \mathrm{conv}(K)$. Since the diameter of $S_w(C)$ tends to zero uniformly when $|w| \to \infty$, we can find an integer $\gamma > 0$ and $d + 1$ finite words $w^0, \cdots, w^d$ of length γ such that

(i) $x_i \in K_{w^i}$ for $0 \leq i \leq d$;

(ii) If $y_i \in S_{w^i}(C)$ for $0 \leq i \leq d$, then $\{y_i - y_0\}_{i=1}^d$ are linearly independent.

We will show that this integer γ is required. For this, suppose that the isometry f satisfies (2.4). According to Lemma 2.2, there is a linear and isometric extension $F : \mathbb{R}^d \to \mathbb{R}^d$ of f. It follows from (2.3) and (2.4) that for $0 \le i \le d$,

$$\begin{aligned} F(S_{w^i}(C)) &= F(S_{w^i}(\mathrm{conv}(K))) \\ &= \mathrm{conv}[F(S_{w^i}(K))] = \mathrm{conv}[f(K_{w^i})] = \mathrm{conv}(K_{w^i}) \\ &= \mathrm{conv}[S_{w^i}(K)] = S_{w^i}[\mathrm{conv}(K)] \\ &= S_{w^i}[C]. \end{aligned} \tag{2.5}$$

Notice that C and $S_{w^i}(C)$ are compact and convex. Then it follows from (2.5) and Brouwer's Fixed Point Theorem that there exist $y_i \in S_{w^i}(C)$ such that

$$F(y_i) = y_i \quad \text{for } 0 \le i \le d.$$

By the property (ii) of γ, $\{y_i\}_{i=0}^d$ can not be contained in any hyperplane of $\mathbb{R}^d$, i.e., dim $(\{y_i\}_{i=0}^d)$ $= d$. Then by Proposition 2.1, since $F|_{\{y_i\}_{i=0}^d}(y_i) = y_i$, $F|_{\{y_i\}_{i=0}^d}$ has a unique isometric extension

$$F(x) \equiv x \quad \text{for } x \in \mathbb{R}^d,$$

that means $f = \mathrm{id}$.

3 Proofs of the Main Results

Proof of Theorem 1.1 It follows from Lemma 2.3 that $\mathrm{Aut}_1(K) = O(K)$ is compact with respect to the uniform topology. Suppose on the contrary that $\#O(K) = \infty$. Then we pick up a sequence $\{g_n\}_{n=1}^\infty \subset O(K)$ such that $g_i \ne g_j$ for any $i \ne j$ and $\{g_n\}_n$ converges to $g^* \in O(K)$ uniformly. Let $f_n = (g^*)^{-1} g_n \in O(K)$. Then

$$\{f_n\}_n \to \mathrm{id} \text{ uniformly and } f_i \ne f_j \text{ for any } i \ne j.$$

Suppose that γ is defined in Lemma 2.4. Notice that $\mathrm{id}(K_w) = K_w$ for any word w with $|w| = \gamma$ and the distance $d(K_w, K_{w'}) > 0$ for any different words w, w' of length γ. Since $\{f_n\}_n \to \mathrm{id}$ uniformly, there exists an integer β such that for any $n \ge \beta$,

$$f_n(K_w) = K_w \quad \text{for any word } w \text{ of length } \gamma.$$

Then it follows from Lemma 2.4 that

$$f_{n_1} = f_{n_2} = \mathrm{id}$$

for any $n_1, n_2 \ge \beta$, which is a contradiction.

Proof of Theorem 1.2 Let $\{f_i\}_{i\ge 1} \subset \mathrm{Aut}(K)$ such that $\mathrm{blip}(f_i) > 1$ for each i and

$$\lim_{i\to\infty} \mathrm{blip}(f_i) = b_*.$$

Without loss of generality, we suppose $\{f_i\}_{i\ge 1} \subset \mathrm{Aut}_c(K)$ with $c > 1$. According to Lemma 2.3, we may assume that $\{f_i\}_i$ converges uniformly to an automorphism $f_0 \in \mathrm{Aut}(K)$. We will distinguish two cases.

Case 1 $\mathrm{blip}(f_0) > 1$

In this case, it follows from inequality (2.2) that

$$\mathrm{blip}(f_0) \leq b_*,$$

and thus $\mathrm{blip}(f_0) = b_*$.

Case 2 $\mathrm{blip}(f_0) = 1$

For each $g \in \mathrm{Aut}(K)$ and each finite word w, we define an $N \times N$ matrix $M(g, w)$ to describe the sets $\{g(K_{wp})\}_{p=1}^N$, where $K_{wp} = S_w \circ S_p(K)$. For $1 \leq p, q \leq N$, let

$$M(g, w)_{p,q} = \begin{cases} 1, & \text{if } g(K_{wp}) \cap K_{wq} \neq \emptyset, \\ 0, & \text{if } g(K_{wp}) \cap K_{wq} = \emptyset. \end{cases} \tag{3.1}$$

Let $h_i = f_0^{-1} f_i \in \mathrm{Aut}_c(K)$ and thus $\{h_i\}_i \to \mathrm{id}$ uniformly in $\mathrm{Aut}_c(K)$. Here $f_0, f_0^{-1} \in O(K)$, it follows from (1.3) that $\mathrm{blip}(h_i) > 1$ for any i and

$$\lim_{i\to\infty} \mathrm{blip}(h_i) = \lim_{i\to\infty} \mathrm{blip}(f_i) = b_*. \tag{3.2}$$

As the discussion mentioned above, there is an integer α such that for any $i \geq \alpha$,

$$h_i(K_w) = K_w \quad \text{for any word } w \text{ of length } \gamma, \tag{3.3}$$

where γ is defined in Lemma 2.4. For each $i \geq \alpha$, since $h_i \neq \mathrm{id}$, we let

$$m(i) = \max\{m : h_i(K_{w'}) = K_{w'} \text{ for any word } w \text{ of length } m\}. \tag{3.4}$$

Then $m(i) \geq \gamma$; furthermore there is a word w^i with length $m(i)$ such that

$$M(h_i, w^i) \neq E, \tag{3.5}$$

where E is the identity matrix.

Suppose that u^i is the prefix of w^i such that $|u^i| = m(i) - \gamma \geq 0$. Write $w^i = u^i * v^i$ with $|v^i| = \gamma$, and let

$$H_i = S_{u^i}^{-1} \circ h_i \circ S_{u^i}, \tag{3.6}$$

where $S_{u^i} = \mathrm{id}$ if u^i is the empty word.

For each $i \geq \alpha$, it follows from (3.4), (3.5) and (3.6) that

(a1) $H_i(K_w) = K_w$ for any w with $|w| = \gamma$;

(a2) $\mathrm{blip}(H_i) \leq \mathrm{blip}(h_i)$;

(a3) $M(H_i, v^i) = M(h_i, w^i) \neq E$ with $|v^i| = \gamma$.

Here $\{H_i\}_{i\geq\alpha} \subset \mathrm{Aut}_c(K)$. By Lemma 2.3, we pick a subsequence $\{H_{n_i}\}_{i=1}^\infty$ such that for any i,

$$v^{n_i} \equiv v^{n_1}, \quad M(H_{n_i}, v^{n_i}) \equiv M(H_{n_1}, v^{n_1})$$

and

$$H_{n_i} \to H \in \mathrm{Aut}_c(K) \text{ uniformly as } i \to \infty.$$

Therefore, we get

(b1) $H(K_w) = K_w$ for any w with $|w| = \gamma$;

(b2) $\text{blip}(H) \le b_*$;

(b3) $M(H, v^{n_1}) \ne E$.

Now, by (b2) we only need to show $\text{blip}(H) > 1$. Otherwise, if $\text{blip}(H) = 1$, it follows from (b1) and Lemma 2.4 that $H = \text{id}$, which contradicts (b3).

Proof of Theorem 1.3 Here $d([f_1], [f_2]) = 0$ if and only if $f_1 f_2^{-1}$ is isometric, i.e., $[f_1] = [f_2]$. Since $\text{blip}(g_1) = \underline{\text{lip}}(g_1^{-1})$, we get $d([g], [f]) = d([f], [g])$. For f_1, f_2 and f_3, since $\text{blip}(g_1 \circ g_2) \le \underline{\text{lip}}(g_1)\text{blip}(g_2)$, we have

$$\begin{aligned} d([f_1], [f_3]) &= \log \text{blip}(f_1 f_3^{-1}) \\ &= \log \text{blip}[(f_1 f_2^{-1}) \circ (f_2 f_3^{-1})] \\ &\le \log \text{blip}(f_1 f_2^{-1}) + \log \text{blip}(f_2 f_3^{-1}) \\ &= d([f_1], [f_2]) + d([f_2], [f_3]). \end{aligned}$$

Therefore, $d([f], [g])$ is a metric. Notice

$$d([f], [g]) = 0 \quad \text{or} \quad d([f], [g]) \ge \log(b_*) (> 0).$$

Then $(\text{Aut}(K)/\sim, d)$ is discrete. Similarly, we have

$$D(f, g) = 0 \quad \text{or} \quad D(f, g) \ge \min \Big[\log(b_*), \min_{\substack{g \in O(K) \\ g \ne \text{id},}} \sup_{x \in K} |g(x) - x| \Big],$$

which implies that $(\text{Aut}(K), D)$ is discrete.

Appendix Proof of Proposition 2.1

Let f be an isometry of A onto A, we first construct an isometric extension $f^* : \mathbb{R}^d \to \mathbb{R}^d$ of f. Here $\dim(A) = d$, i.e., A can not be contained in any hyperplane of $\mathbb{R}^d$. Then there are $\{x_i\}_{i=0}^d \subset A$ such that $\{x_i - x_0\}_{i=1}^d$ are linearly independent. For each $x \in \mathbb{R}^d$, suppose $x = x_0 + \sum_{i=1}^{d} a_i (x_i - x_0)$, where $a_i \in \mathbb{R}$ is uniquely determined by x for each $1 \le i \le d$. Then $f^* : \mathbb{R}^d \to \mathbb{R}^d$ can be defined as

$$f^* : x \mapsto f(x_0) + \sum_{i=1}^{d} a_i (f(x_i) - f(x_0)).$$

We will show that f^* is the unique isometric extension of f. For this, we need to verify the following three claims:

(1) f^* is an isometry of $\mathbb{R}^d$ onto $\mathbb{R}^d$;

(2) $f^*(x) = f(x)$ for any $x \in A$;

(3) If g is also an isometric extension of f, then $g = f^*$.

For claim (1), since $|f(x_p) - f(x_q)| = |x_p - x_q|$ for $0 \le p, q \le d$, we notice that the triangle $\Delta x_0 x_i x_j$ is isometric to triangle $\Delta f(x_0) f(x_i) f(x_j)$ with $1 \le i \ne j \le d$, which means for all $1 \le i, j \le d$,

$$(f(x_i) - f(x_0)) \cdot (f(x_j) - f(x_0)) = (x_i - x_0) \cdot (x_j - x_0). \tag{$*$}$$

Assume $x, x' \in \mathbb{R}^d$ with $x = x_0 + \sum_{i=1}^{d} a_i(x_i - x_0)$, $x' = x_0 + \sum_{i=1}^{d} a'_i(x_i - x_0)$. We have

$$\begin{aligned}
|f^*(x) - f^*(x')|^2 &= \Big|\sum_{i=1}^{d}(a_i - a'_i)(f(x_i) - f(x_0))\Big|^2 \\
&= \sum_{1\le i,j\le d}(a_i - a'_i)(a_j - a'_j)[(f(x_i) - f(x_0)) \cdot (f(x_j) - f(x_0))] \\
&= \sum_{1\le i,j\le d}(a_i - a'_i)(a_j - a'_j)[(x_i - x_0) \cdot (x_j - x_0)] \quad (\text{by } (*)) \\
&= |x - x'|^2.
\end{aligned}$$

Therefore f^* is an isometry.

To prove claims (2) and (3), we notice $f^*(x_i) = f(x_i)$ for $0 \le i \le d$, so we need only to show that for $y, y' \in \mathbb{R}^d$,

$$|y - f(x_i)| = |y' - f(x_i)|, \ \forall\, 0 \le i \le d \Rightarrow y = y'.$$

If this is false, we may assume $y \ne y'$. Then the set $\{f^*(x_i)\}_{i=0}^d = \{f(x_i)\}_{i=0}^d$ is contained in the hyperplane

$$\{x \in \mathbb{R}^d \colon |y - x| = |y' - x|\}.$$

And thus $\dim(\{f^*(x_i)\}_{i=0}^d) < d$ for linear isometry $f^* : \mathbb{R}^d \to \mathbb{R}^d$, which contradicts the fact $\dim(\{x_i\}_{i=0}^d) = d$.

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