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Instability of Standing Waves for Hamiltonian Wave Equations****

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Abstract This paper deals with the standing wave for a Hamiltonian nonlinear wave equation which can be viewed as a representative of the class of equations of interest. On the one hand, by proving a compactness lemma and solving a variational problem, the existence of the standing wave with ground state for the aforementioned equation is proved. On the other hand, the authors derive the instability of the standing wave by applying the potential well argument, the concavity method and an invariant region under the solution flow of the Cauchy problem for the equation under study, and the invariance of the region aforementioned can be shown by introducing an auxiliary functional and a supplementary constrained variational problem.

Keywords Hamiltonian wave equation, Ground state, Standing wave, Instability
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1 Introduction

Consider the Hamilton nonlinear wave equation

$$\phi_{tt} - \Delta\phi + a|x|^2\phi + \phi = |\phi|^{p-1}\phi, \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

where $\phi = \phi(t,x)$ is an unknown complex-valued function of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$, Δ is the Laplace operator on $\mathbb{R}^N$, $a \geq 0$, $N \geq 2$ and $1 < p < \frac{N+2}{N-2}$ (when $N = 2$, $1 < p < \infty$). Equation (1.1) can be viewed as an interaction between one or more discrete oscillators and a field or continuous medium and a representative of the class of equations of interest (see [1]).

When $a = 0$, (1.1) becomes a classical nonlinear model in field theory, namely

$$\phi_{tt} - \Delta\phi + \phi = |\phi|^{p-1}\phi, \quad t \geq 0,\ x \in \mathbb{R}^N. \tag{1.2}$$

Many works have been done on the study of (1.2) (see [2–9]). Pecher [2] proved the local well-posedness of the Cauchy problem for (1.2). Levine [3], Pagne and Sattinger [4] as well as Ball [5] obtained some blowup results for (1.2). Strauss [6] discussed the global existence of solutions

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to the Cauchy problem for (1.2) with the initial data sufficiently small. Moreover, Shatah [7, 8], Berestycki and Lions [9] studied the existence, stability and instability of the standing waves for (1.2).

When $a > 0$, for the general case of (1.1):

$$\phi_{tt} - \Delta\phi + V(x)\phi + \phi = \lambda f(\phi), \tag{1.3}$$

Soffer and Weinstein [1] showed that all small amplitude solutions of (1.3) decay to zero as time tends to infinity at an anomalously slow rate for generic nonlinear Hamiltonian perturbations, and obtained the local well-posedness of the Cauchy problem for (1.3) under some assumptions on $V(x)$ and the corresponding initial data. When $f(\phi) = |\phi|^{p-1}\phi$, (1.3) can be represented by

$$\phi_{tt} - \Delta\phi + V(x)\phi + \phi = \lambda|\phi|^{p-1}\phi, \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{1.4}$$

where $1 < p < \infty$ for $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$. By using global variational method in Strauss [6] and Lions [10] as well as local bifurcations method in [11], Rose and Weinstein [12] showed that (1.4) has time periodic and spatially localized solutions of the form $\mathrm{e}^{\mathrm{i}\omega t}u(x, \omega)$ with $u \in H^1(\mathbb{R}^N)$, which bifurcate from the zero solution at the point eigenvalue of $-\triangle + V(x) - \omega^2$. The stability of small amplitude bifurcating states of (1.4) can be proved by using the methods of Grillakis, Shatah and Strauss [13] as well as Weinstein [14].

Motivated by the study of the standing waves for the nonlinear Schrödinger equations with a harmonic potential (see [15–17]), we consider in this paper the standing wave of (1.1) with $a \geq 0$. Firstly, we prove the existence of standing wave with ground state for (1.1) by proving a compactness lemma and solving a variational problem. Next, we show an invariant region under the solution flow of the Cauchy problem for (1.1) by introducing an auxiliary functional and a supplementary constrained variational problem. Finally, gathering the former results, we show the instability of the standing wave by using the potential well argument in [4] as well as the concavity method in [3]. To our knowledge, this kind of results in the present paper are new for (1.1) and can be applied to the nonlinear Klein-Gordon equations without any potential.

At the end of this section, we recall the definition of the standing wave for equation (1.1). If a nontrivial real-valued function $u(x)$ satisfies the elliptic equation

$$-\triangle u + u + a|x|^2 u - |u|^{p-1}u = 0, \quad a \geq 0, \tag{1.5}$$

then $\phi(t, x) = u(x)$ satisfies (1.1) and we call it the standing wave solution of (1.1) and call the positive solution with minimal action of (1.5) the ground state of (1.5). From the physical point of view, the ground state of equation (1.5) plays an important role in the study of standing waves for (1.1).

For simplicity, throughout this paper we denote the strong convergence by $\rightarrow$, weak convergence by $\rightharpoonup$, $\|\cdot\|_{L^p(\mathbb{R}^N)}$ by $\|\cdot\|_p$, $\|\cdot\|_{H^1(\mathbb{R}^N)}$ by $\|\cdot\|_{H^1}$, $\int_{\mathbb{R}^N}\cdot\,\mathrm{d}x$ by $\int\cdot\,\mathrm{d}x$ and arbitrary positive constant by C.

2 Preliminaries

We impose the initial data of (1.1) as follows:

$$\phi(0, x) = \phi_0(x) \in H^1(\mathbb{R}^N), \quad \phi_t(0, x) = \phi_1(x) \in L^2(\mathbb{R}^N). \tag{2.1}$$

For (1.1), we define the energy space Σ as

$$\Sigma := \{\varphi \in H^1(\mathbb{R}^N), \|x\varphi\|_2 < \infty\}, \tag{2.2}$$

and endow with the inner product in Σ as

$$(\varphi,\phi)_\Sigma := \int (\nabla\varphi\nabla\overline{\phi} + \varphi\overline{\phi} + a|x|^2\varphi\overline{\phi})\mathrm{d}x. \tag{2.3}$$

Then Σ becomes a Hilbert space, continuously embedded in $H^1(\mathbb{R}^N)$. By (2.2), for all $\phi \in \Sigma$, we define the norm $\|\phi\|_\Sigma$ as

$$\|\phi\|_\Sigma^2 := \|\phi\|_2^2 + \|\nabla\phi\|_2^2 + a\|x\phi\|_2^2. \tag{2.4}$$

In addition, the energy functional of equation (1.1) is defined by

$$\mathcal{E}(\phi,\phi_t) := \frac{1}{2}(\|\phi_t\|_2^2 + \|\phi\|_2^2 + \|\nabla\phi\|_2^2 + a\|x\phi\|_2^2) - \frac{1}{p+1}\|\phi\|_{p+1}^{p+1}. \tag{2.5}$$

Using the method of the papers [1, 18, 19], we can obtain the local well-posedness in energy space for the Cauchy problem (1.1)–(2.1), namely, for $(\phi_0,\phi_1) \in \Sigma \times L^2(\mathbb{R}^N)$, there is a $T > 0$ such that the Cauchy problem (1.1)–(2.1) has a unique solution ϕ with $(\phi,\phi_t) \in C([0,T);\Sigma \times L^2(\mathbb{R}^N))$, where either T is finite or T is infinite. If T is finite, then $\lim\limits_{t\to T}\|\phi(t)\|_\Sigma = \infty$. Moreover, for $t \in [0,T)$, $\phi(t,x)$ satisfies the conservation of energy:

$$\mathcal{E}(\phi,\phi_t) = \mathcal{E}(\phi_0,\phi_1) = \frac{1}{2}(\|\phi_1\|_2^2 + \|\phi_0\|_2^2 + \|\nabla\phi_0\|_2^2 + a\|x\phi_0\|_2^2) - \frac{1}{p+1}\|\phi_0\|_{p+1}^{p+1}. \tag{2.6}$$

Let $1 < p < \infty$ for $N = 1,2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$. For $u \in \Sigma$, define functionals $J(u)$ and $K(u)$ as

$$J(u) := \frac{1}{2}(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.7}$$

$$K(u) := \|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2 - \|u\|_{p+1}^{p+1}. \tag{2.8}$$

Furthermore, we define

$$\mathcal{M} := \{u \in \Sigma \setminus \{0\},\ K(u) = 0\}. \tag{2.9}$$

Remark 2.1 From Sobolev embedding theorem, we know that functionals $J(u)$ and $K(u)$ are well-defined. Moreover, by the definition of $\mathcal{M}$, we will prove that $\mathcal{M}$ is not empty at the end of this section.

Now we define a constrained variational problem

$$d_\mathcal{M} := \inf_{u\in\mathcal{M}} J(u). \tag{2.10}$$

Then the following three lemmas hold.

Lemma 2.1 *J is bounded below on $\mathcal{M}$ and $d_\mathcal{M} > 0$.*

Proof By (2.7)–(2.10), we obtain, on $\mathcal{M}$,

$$J(u) = \frac{p-1}{2(p+1)}(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2). \tag{2.11}$$

Using Sobolev embedding theorem and $K(u) = 0$, we can easily to check that

$$\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2 = \|u\|_{p+1}^{p+1} \leq C(\|u\|_2^2 + \|\nabla u\|_2^2)^{\frac{p+1}{2}}, \tag{2.12}$$

which implies that

$$\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2 \leq C(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2)^{\frac{p+1}{2}}, \tag{2.13}$$

that is,

$$(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2)^{\frac{p-1}{2}} \geq C > 0. \tag{2.14}$$

Thus (2.11), (2.14), $u \neq 0$ and $p > 1$ imply $J(u) \geq C > 0$ on $\mathcal{M}$. Therefore $d_{\mathcal{M}} \geq C > 0$ from (2.10).

Lemma 2.2 *Let $u \in \Sigma \setminus \{0\}$ and $u_\beta(x) = \beta u(x)$ with $\beta > 0$. Then there exists a $\lambda > 0$ uniquely depending on u such that $K(u_\lambda) = 0$. Moreover,*

$$\begin{aligned} &K(u_\beta) > 0 && \text{for } \beta \in (0, \lambda),\\ &K(u_\beta) < 0 && \text{for } \beta \in (\lambda, \infty),\\ &J(u_\lambda) \geq J(u_\beta) && \text{for all } \beta > 0. \end{aligned}$$

Proof From (2.7) and (2.8), we know

$$\begin{aligned} J(u_\beta) &= \frac{1}{2}\beta^2(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) - \frac{1}{p+1}\beta^{p+1}\|u\|_{p+1}^{p+1},\\ K(u_\beta) &= \beta^2(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) - \beta^{p+1}\|u\|_{p+1}^{p+1}. \end{aligned}$$

Then there exists a $\lambda > 0$ uniquely depending on u such that $K(u_\lambda) = 0$, and $K(u_\beta) > 0$ for $\beta \in (0, \lambda)$, $K(u_\beta) < 0$ for $\beta \in (\lambda, \infty)$.

Since

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\beta}\Big) J(u_\beta) = \beta^{-1} K(u_\beta),$$

from $K(u_\lambda) = 0$ it follows that for any $\beta > 0$, $J(u_\lambda) \geq J(u_\beta)$.

Lemma 2.3 *Let $1 \leq p < \infty$ for $N = 1, 2$ and $1 \leq p < \frac{N+2}{N-2}$ for $N \geq 3$. Then the embedding $\Sigma \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is compact.*

Proof We first prove the case $p = 1$.

From (2.2) and (2.3), it follows that Σ is continuously embedded in $H^1(\mathbb{R}^N)$. Using Sobolev embedding theorem, we know that Σ is continuously embedded in $L^{p+1}(\mathbb{R}^N)$. Now let (u_n) be a sequence in Σ, such that

$$u_n \rightharpoonup 0, \quad \text{in } \Sigma, \quad \text{as } n \to \infty. \tag{2.15}$$

By (2.2) and (2.3), we get

$$u_n \rightharpoonup 0, \quad \text{in } H^1(\mathbb{R}^N), \quad \text{as } n \to \infty \tag{2.16}$$

and

$$B := \sup_n \|u_n\|_\Sigma < \infty. \tag{2.17}$$

Let $\varepsilon > 0$. Then there exists $A > 0$ such that

$$\frac{1}{|x|^2} \leq \varepsilon, \quad \text{when } |x| \geq A. \tag{2.18}$$

For A, by (2.16) one has

$$u_n \to 0, \quad \text{in } L^2(\{|x| \le A\}), \quad \text{as } n \to \infty. \tag{2.19}$$

Thus there exists an m and when $n \ge m$, one gets

$$\int_{|x|\le A} |u_n|^2 \mathrm{d}x \le \varepsilon. \tag{2.20}$$

Therefore when $n \ge m$, we obtain by (2.17)–(2.20)

$$\begin{aligned} \int |u_n|^2 \mathrm{d}x &= \int_{|x|\le A} |u_n|^2 \mathrm{d}x + \int_{|x|\ge A} |u_n|^2 \mathrm{d}x \\ &= \int_{|x|\le A} |u_n|^2 \mathrm{d}x + \int_{|x|\ge A} \frac{|x|^2}{|x|^2} |u_n|^2 \mathrm{d}x \\ &\le \varepsilon + \varepsilon \int_{|x|\ge A} |x|^2 |u_n|^2 \mathrm{d}x \\ &\le \varepsilon + \varepsilon C B^2. \end{aligned}$$

So $u_n \to 0$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. From (2.15) we know that the embedding $\Sigma \hookrightarrow L^2(\mathbb{R}^N)$ is compact.

When $p > 1$, by using the conclusion of $p = 1$ and Gagliardo-Nirenberg inequality

$$\|u\|_{p+1}^{p+1} \le C \|\nabla u\|_2^{\frac{N(p-1)}{2}} \|u\|_2^{p+1-\frac{N(p-1)}{2}},$$

from (2.15), it follows that

$$u_n \to 0, \quad \text{in } L^{p+1}(\mathbb{R}^N), \quad \text{as } n \to \infty,$$

which implies that the embedding $\Sigma \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is compact.

At the end of this section, we define an auxiliary functional $I(u)$ as

$$\begin{aligned} I(u) &= J(u) - \frac{1}{\theta+1} K(u) \\ &= \frac{\theta-1}{2(\theta+1)} (\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) + \frac{p-\theta}{(\theta+1)(p+1)} \|u\|_{p+1}^{p+1}, \end{aligned} \tag{2.21}$$

and define a supplementary set $\mathcal{M}^-$ as

$$\mathcal{M}^- := \{u \in \Sigma \setminus \{0\}, K(u) \le 0\}, \tag{2.22}$$

where constant θ satisfies $1 < \theta \le p$. Moreover, we define a constrained variational problem

$$d_{\mathcal{M}^-} := \inf_{u \in \mathcal{M}^-} I(u). \tag{2.23}$$

Then the following two propositions hold.

Proposition 2.1 *If $u \ne 0$, then $I(s^{\frac{1}{2}} u)$ is a strict increasing function with respect to $s \in (0, \infty)$.*

Proof From (2.21), it follows that

$$I(s^{\frac{1}{2}} u) = \frac{\theta-1}{2(\theta+1)} s (\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) + \frac{p-\theta}{(\theta+1)(p+1)} s^{\frac{p+1}{2}} \|u\|_{p+1}^{p+1}. \tag{2.24}$$

Thus (2.24), $u \neq 0$ and $1 < \theta \le p$ yield

$$\frac{\mathrm{d}I(s^{\frac{1}{2}}u)}{\mathrm{d}s} = \frac{\theta-1}{2(\theta+1)}(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) + \frac{p-\theta}{2(\theta+1)}s^{\frac{p-1}{2}}\|u\|_{p+1}^{p+1} > 0.$$

Proposition 2.2 *$\mathcal{M}$ and $\mathcal{M}^-$ are not empty and*

$$d_{\mathcal{M}} = \inf_{u\in\mathcal{M}} J(u) = d_{\mathcal{M}^-} = \inf_{u\in\mathcal{M}^-} I(u). \tag{2.25}$$

Moreover,

$$\text{if } K(u) < 0, \text{ then } I(u) > d_{\mathcal{M}}. \tag{2.26}$$

Proof First we show that $\mathcal{M}^-$ is not empty. Choose $\forall u \in \Sigma$, $u \neq 0$ and consider

$$P(s) = K(s^{\frac{1}{2}}u) = s(\|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2) - s^{\frac{p+1}{2}}\|u\|_{p+1}^{p+1}.$$

Since $p > 1$, when $s > 1$ is sufficiently large, we obtain $P(s) < 0$. So $s^{\frac{1}{2}}u \in \mathcal{M}^-$.

Next we prove that $\mathcal{M}$ is not empty. Choosing $\forall u \in \Sigma$ such that $K(u) < 0$ and considering $K(s^{\frac{1}{2}}u)$, we get $K(s^{\frac{1}{2}}u) = K(u) < 0$ when $s = 1$, $K(s^{\frac{1}{2}}u) > 0$ when $s > 0$ and tends to 0. By continuity, there exists $s_0 \in (0,1)$ such that $K({s_0}^{\frac{1}{2}}u) = 0$, namely ${s_0}^{\frac{1}{2}}u \in \mathcal{M}$.

Then we prove that (2.25) is true. From Proposition 2.1, we obtain that $I(s^{\frac{1}{2}}u)$ is a strictly increasing function with respect to $s \in (0,\infty)$ especially for $K(u) \le 0$ and $u \neq 0$. By $K({s_0}^{\frac{1}{2}}u) = 0$ and (2.10), one has

$$d_{\mathcal{M}} \le J(s_0^{\frac{1}{2}}u) = I(s_0^{\frac{1}{2}}u) + \frac{1}{\theta+1}K(s_0^{\frac{1}{2}}u) = I(s_0^{\frac{1}{2}}u). \tag{2.27}$$

On the one hand, from Proposition 2.1 and (2.27), for $s_0 \in (0,1)$ one obtains

$$d_{\mathcal{M}} \le \inf_{s_0^{\frac{1}{2}}u\in\mathcal{M}^-} J(s_0^{\frac{1}{2}}u) = \inf_{s_0^{\frac{1}{2}}u\in\mathcal{M}^-} I(s_0^{\frac{1}{2}}u) \le \inf_{u\in\mathcal{M}^-} I(u) = d_{\mathcal{M}^-}. \tag{2.28}$$

On the other hand, by the definitions of $\mathcal{M}$, $\mathcal{M}^-$, (2.9), (2.21) and (2.22), we have $\mathcal{M} \subset \mathcal{M}^-$ and

$$d_{\mathcal{M}} = \inf_{u\in\mathcal{M}} J(u) = \inf_{u\in\mathcal{M}} \Big(I(u) + \frac{1}{\theta+1}K(u)\Big) = \inf_{u\in\mathcal{M}} I(u) \ge \inf_{u\in\mathcal{M}^-} I(u) = d_{\mathcal{M}^-}. \tag{2.29}$$

Thus (2.28) and (2.29) imply (2.25).

Finally, we show that (2.26) is true. Since $K(u) < 0$, by the proof that $\mathcal{M}$ is not empty we know that there exists $s \in (0,1)$ such that $K(s^{\frac{1}{2}}u) = 0$ and $u \neq 0$. By (2.21), one has

$$J(s^{\frac{1}{2}}u) = I(s^{\frac{1}{2}}u) \ge d_{\mathcal{M}}. \tag{2.30}$$

Thus Proposition 2.1, $s \in (0,1)$ and (2.30) imply

$$d_{\mathcal{M}} \le I(s^{\frac{1}{2}}u) < I(u). \tag{2.31}$$

Therefore, we get $I(u) > d_{\mathcal{M}}$ if $K(u) < 0$.

So far, the proof of Proposition 2.2 is completed.

3 Existence of Standing Wave with Ground State

In this section, we prove the existence of standing wave with ground state of the Hamiltonian nonlinear wave equation (1.1) by using variational argument and Lemmas 2.1–2.3, namely Theorem 3.1.

Theorem 3.1 *There exists* $Q \in M$ *such that*
(1) $J(Q) = \inf_{u \in \mathcal{M}} J(u) = d_{\mathcal{M}}$;
(2) Q *is a ground state solution of equation* (1.5).

Proof (1) Let $\{u_n, n \in \mathbf{N}\} \subset \mathcal{M}$ be a minimizing sequence of (2.10). Then

$$K(u_n) = 0, \quad J(u_n) \to d_{\mathcal{M}}, \quad \text{as } n \to \infty. \tag{3.1}$$

By (2.11) and (3.1), for any $n \in \mathbf{N}$, $\|u_n\|_\Sigma$ is bounded. So there exists a subsequence $\{u_{n_k}, k \in \mathbf{N}\} \subset \{u_n, n \in \mathbf{N}\}$ such that

$$u_{n_k} \rightharpoonup u^*, \quad \text{in } \Sigma, \quad \text{as } k \to \infty.$$

Here, for simplicity we still denote $\{u_{n_k}, k \in \mathbf{N}\}$ by $\{u_n, n \in \mathbf{N}\}$. Thus we get

$$u_n \rightharpoonup u^*, \quad \text{in } \Sigma, \quad \text{as } n \to \infty. \tag{3.2}$$

From Lemma 2.3, it follows that

$$\begin{cases} u_n \to u^*, & \text{in } L^2(\mathbb{R}^N), \quad \text{as } n \to \infty, \\ u_n \to u^*, & \text{in } L^{p+1}(\mathbb{R}^N), \quad \text{as } n \to \infty. \end{cases} \tag{3.3}$$

Therefore we assert $u^* \neq 0$ and we now prove this conclusion by contradiction.

If $u^* \equiv 0$, then (3.3) yields

$$\begin{cases} u_n \to 0, & \text{in } L^2(\mathbb{R}^N), \quad \text{as } n \to \infty, \\ u_n \to 0, & \text{in } L^{p+1}(\mathbb{R}^N), \quad \text{as } n \to \infty. \end{cases} \tag{3.4}$$

Since $u_n \in \mathcal{M}$, (3.4) and $K(u_n) = 0$ imply

$$\|u_n\|_2^2 + \|\nabla u_n\|_2^2 + a\|xu_n\|_2^2 \to 0, \quad \text{as } n \to \infty.$$

Using (3.4) again, we have

$$\|\nabla u_n\|_2^2 + a\|xu_n\|_2^2 \to 0, \quad \text{as } n \to \infty. \tag{3.5}$$

On the other hand, by (2.7) and (3.1) we get, as $n \to \infty$,

$$\|u_n\|_2^2 + \|\nabla u_n\|_2^2 + a\|xu_n\|_2^2 - \frac{1}{p+1}\|u_n\|_{p+1}^{p+1} \to d_{\mathcal{M}}.$$

Thanks to (3.4), one gets, as $n \to \infty$,

$$\|\nabla u_n\|_2^2 + a\|xu_n\|_2^2 \to d_{\mathcal{M}}. \tag{3.6}$$

Since $d_{\mathcal{M}} > 0$ by Lemma 2.1, one deduces that (3.5) and (3.6) are contradictory and thus $u^* \neq 0$.

Now according to Lemma 2.2, let $Q=(u^*)_\lambda$ and $\lambda>0$ be determined uniquely by $K(Q)=K[(u^*)_\lambda]=0$. Then (3.2) and (3.3) yield

$$\begin{cases}(u_n)_\lambda \to Q, & \text{in } L^2(R^N), \quad \text{as } n\to\infty\\ (u_n)_\lambda \to Q, & \text{in } L^{p+1}(R^N), \quad \text{as } n\to\infty,\\ (u_n)_\lambda \rightharpoonup Q, & \text{in } \Sigma, \quad \text{as } n\to\infty.\end{cases} \tag{3.7}$$

Applying Lemma 2.2 and $K(u_n)=0$, we obtain

$$J[(u_n)_\lambda]\le J(u_n), \tag{3.8}$$

which together with (3.7) implies

$$J(Q)\le \underline{\lim}_{n\to\infty} J[(u_n)_\lambda]\le \lim_{n\to\infty} J(u_n)=\inf_{\mathcal{M}} J. \tag{3.9}$$

Thus $Q\ne 0$ and $K(Q)=0$ imply $Q\in M$. Therefore, by (3.9) we know that Q satisfies

$$J(Q)=\min_{u\in\mathcal{M}} J(u). \tag{3.10}$$

Hence the proof of Theorem 3.1(1) is completed.

In the following, we prove Theorem 3.1(2).

(2) Since Q is a solution of (3.10), there exists a Lagrange multiplier Λ such that

$$\delta_Q(J(Q)+\Lambda K(Q))=0, \tag{3.11}$$

where $\delta_u G$ denotes the variation of $G(u)$ with respect to u. By the formula

$$\delta_u G(u)=\frac{\partial}{\partial\eta}G(u+\eta\delta u)\Big|_{\eta=0},$$

we have

$$\delta_u(J(u)+\Lambda K(u))=\langle(1+2\Lambda)(-\Delta u+u+a|x|^2u)-[1+(p+1)\Lambda]|u|^{p-1}u,\delta u\rangle, \tag{3.12}$$

where δu denotes the variation of u and $\langle f,g\rangle$ denotes $\int fg\mathrm{d}x$. By (3.11) one has

$$(1+2\Lambda)(\|Q\|_2^2+\|\nabla Q\|_2^2+a\|xQ\|_2^2)=[1+(p+1)\Lambda]\|Q\|_{p+1}^{p+1}, \tag{3.13}$$

which together with $K(Q)=0$ yields $\Lambda=0$. Thus from (3.11) we conclude that

$$-\Delta Q+Q+a|x|^2Q-|Q|^{p-1}Q=0, \tag{3.14}$$

which implies that Q is a ground state solution of (1.5) by (3.10). So the proof of Theorem 3.1(2) is completed.

Gathering all the above discussions, the proof of Theorem 3.1 is completed.

4 Existence of Standing Wave with Ground State

In Section 3, we have obtained the existence of the standing wave for equation (1.1). In this section, we discuss the instability of the standing wave.

Theorem 4.1 *Let Q be a ground state of equation* (1.3). *Then* $\forall \epsilon > 0$*, there exists* $\phi_0(x) \in \Sigma$ *such that*

$$\|\phi_0 - Q\|_\Sigma < \varepsilon,$$

and the solution ϕ of the Cauchy problem (1.1) *with the initial data*

$$\phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = 0 \tag{4.1}$$

is defined on $T \in (0,\infty)$ such that $\phi \in C([0,T),\Sigma)$ and

$$\lim_{t\to T} \|\phi\|_\Sigma = \infty. \tag{4.2}$$

By (2.6) and (2.7), we rewrite the energy $\mathcal{E}$ defined by (2.5) as

$$\mathcal{E}(u,v) = \frac{1}{2}\|v\|_2^2 + J(u). \tag{4.3}$$

Moreover, we introduce a set $\mathcal{S}$ and a manifold $\mathcal{R}$ respectively as

$$\mathcal{S} := \{(u,v) : u \in \Sigma, v \in L^2(\mathbb{R}^N) \mid \mathcal{E}(u,v) < d_{\mathcal{M}}\}, \tag{4.4}$$

$$\mathcal{R} := \{(u,v) \in \mathcal{S} \mid K(u) < 0\}. \tag{4.5}$$

Now we give two lemmas which are key to the proof of Theorem 4.1.

Lemma 4.1 *$\mathcal{R}$ is invariant region under the solution flow of the Cauchy problem* (1.1) *and* (2.1).

Proof Let $(\phi_0, \phi_1) \in \mathcal{R}$ and $\phi(t)$ be the solution to (1.1) with the initial data. Assume that there exists a τ such that

$$\text{(A1)} \quad (\phi(\tau), \phi_t(\tau)) \notin \mathcal{R},$$

i.e., $K(\phi(\tau)) \geq 0$. Thus (2.6), (2.21) and (4.3) yield

$$I(\phi(\tau)) = J(\phi(\tau)) - \frac{1}{\theta+1}K(\phi(\tau)) \leq J(\phi(\tau)) \leq \mathcal{E}(\phi(\tau),\phi_t(\tau)) \leq d_{\mathcal{M}}. \tag{4.6}$$

Let

$$s = \inf\{0 \leq t \leq \tau \mid (\phi(t),\phi_t(t)) \notin \mathcal{R}\}. \tag{4.7}$$

Then Proposition 2.2 implies $I(\phi(s)) \leq d_{\mathcal{M}}$ and $I(\phi(t)) > d_{\mathcal{M}}$ for all $0 < t < s$.

On the other hand, from (2.6) and (2.21), we have

$$\begin{aligned} K(\phi(s)) &= \lim \inf_{t\to s^-} (\theta+1)(J(\phi(t)) - I(\phi(t))) \\ &\leq \lim \inf_{t\to s^-} (\theta+1)(\mathcal{E}(\phi(t),\phi_t(t)) - d_{\mathcal{M}}) \\ &\leq \lim \inf_{t\to s^-} (\theta+1)(\mathcal{E}(\phi_0,\phi_1) - d_{\mathcal{M}}) \\ &= (\theta+1)(\mathcal{E}(\phi_0,\phi_1) - d_{\mathcal{M}}) \\ &< 0, \end{aligned} \tag{4.8}$$

which together with the conclusion ($I(\phi(s)) > d_{\mathcal{M}}$ if $K(\phi(s)) < 0$ in Proposition 2.2) contradicts $I(\phi(s)) \leq d_{\mathcal{M}}$. Therefore, (A1) is not true. Thus $\mathcal{R}$ is an invariant region under the solution flow of the Cauchy problem (1.1) and (2.1).

Lemma 4.2 Let $u_\lambda(x) = \lambda u(x)$, $u \in \Sigma \setminus \{0\}$ and $0 < \lambda < 1$ such that $K(u_\lambda) = 0$. Then

$$J(u) - J(u_\lambda) \geq \frac{1}{2}K(u). \tag{4.9}$$

Proof From Lemma 2.2, we have

$$K(u_\lambda) = A\lambda^2 - B\lambda^{p+1}, \quad J(u_\lambda) = \frac{1}{2}A\lambda^2 - B\frac{1}{p+1}\lambda^{p+1},$$

where

$$A = \|u\|_2^2 + \|\nabla u\|_2^2 + a\|xu\|_2^2, \quad B = \|u\|_{p+1}^{p+1}.$$

Thus $K(u_\lambda) = 0$ implies

$$A\lambda^2 = B\lambda^{p+1}. \tag{4.10}$$

Therefore

$$J(u_\lambda) = \frac{1}{2}B\lambda^{p+1} - B\frac{1}{p+1}\lambda^{p+1} = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)B\lambda^{p+1} = \frac{p-1}{2(p+1)}B\lambda^{p+1}. \tag{4.11}$$

Since $K(u) = A - B$, $J(u) = \frac{1}{2}A - \frac{1}{p+1}B$ and $0 < \lambda < 1$, we obtain

$$\begin{aligned} J(u) - J(u_\lambda) &= \frac{1}{2}A - \frac{1}{p+1}B - \frac{p-1}{2(p+1)}B\lambda^{p+1} \\ &> \frac{1}{2}A - \frac{1}{p+1}B - \frac{p-1}{2(p+1)}B \\ &= \frac{1}{2}A - \frac{1}{2}B = \frac{1}{2}K(u), \end{aligned}$$

which yields the desired result of Lemma 4.2.

Now we begin to prove Theorem 4.1.

Proof of Theorem 4.1 For initial data (4.1), from (2.5)–(2.7) it follows that

$$\mathcal{E}(\phi_0, \phi_1) = J(\phi_0). \tag{4.12}$$

Let

$$\phi_0(x) = \lambda Q(x), \quad \lambda > 1. \tag{4.13}$$

Then $\forall\, \varepsilon > 0$, we can always choose a $\lambda > 1$ such that

$$\|\phi_0 - Q\|_\Sigma = (\lambda - 1)\|Q\|_\Sigma < \varepsilon.$$

Since $\lambda > 1$, using (4.13) and Lemma 2.2, we obtain

$$K(\phi_0) < K(Q) = 0, \tag{4.14}$$

$$J(\phi_0) < d_{\mathcal{M}}, \tag{4.15}$$

which together with (4.12) yield

$$E(\phi_0, \phi_1) < d_{\mathcal{M}}. \tag{4.16}$$

Thus (4.14)–(4.16) and Lemma 4.1 imply

$$K(\phi(t)) < 0. \tag{4.17}$$

Let $\phi(t)$ be a solution of the Cauchy problem (1.1)–(4.1) in $C([0,T);\Sigma)$. Putting

$$F(t) := \|\phi\|_2^2, \tag{4.18}$$

we have

$$F'(t) = \int (\phi_t\overline{\phi} + \phi\overline{\phi}_t)\mathrm{d}x = 2\mathrm{Re}\int \phi_t\overline{\phi}\mathrm{d}x, \tag{4.19}$$

$$F''(t) = 2\|\phi_t\|_2^2 - 2\|\phi\|_2^2 - 2\|\nabla\phi\|_2^2 - 2\|x\phi\|_2^2 + 2\|\phi\|_{p+1}^{p+1} = 2\|\phi_t\|_2^2 - 2K(\phi). \tag{4.20}$$

On the other hand, (2.5) and (2.6) imply

$$F''(t) = (p+3)\|\phi_t\|_2^2 + (p-1)(\|\phi\|_2^2 + \|\nabla\phi\|_2^2 + a\|x\phi\|_2^2) - 2(p+1)\mathcal{E}(\phi_0,\phi_1). \tag{4.21}$$

From (4.17) and (4.20), it follows that $F(t)$ is a convex function about t. Thus if there exists a time t^* such that $F'(t)|_{t=t^*} > 0$, then $F(t)$ is increasing for all $t > t_1$ (within the interval of existence). In this case, $(p-1)\|\phi\|_2^2 - 2(p+1)\mathcal{E}(\phi_0,\phi_1)$ will eventually become positive, and will remain positive thereafter. Thus by (4.21), we would have for t large enough

$$F''(t) \geq (p+3)\|\phi_t\|_2^2. \tag{4.22}$$

From (4.18), (4.19) and (4.22), using Hölder inequality, we have

$$F(t)F''(t) \geq \frac{p+3}{4}[F'(t)]^2. \tag{4.23}$$

Since

$$[F^{-\frac{p-1}{4}}(t)]'' = -\frac{p-1}{4}F^{-\frac{p+1}{4}}(t)\Big\{F(t)F''(t) - \frac{p+1}{4}[F'(t)]^2\Big\},$$

we get by (4.23) that $[F^{-\frac{p-1}{4}}(t)]'' \leq 0$. Therefore $F^{-\frac{p-1}{4}}(t)$ is concave for t large sufficiently, and there is a finite time T^* such that $\lim\limits_{t\to T^*} F^{-\frac{p-1}{4}}(t) = 0$, which together with $p > 1$ implies that there exists $T < \infty$ such that

$$\lim_{t\to T}\|\phi\|_\Sigma = \infty.$$

The proof of Theorem 4.1 will be completed once we have shown that $F'(t)|_{t=t^*} > 0$ for some t^* and we prove it by contradiction. Suppose for all t,

$$F'(t) < 0. \tag{4.24}$$

Then since $F(t) > 0$ and $F(t)$ is a convex function with respect to t, $F(t)$ must tend to a finite, nonnegative limit l as $t \to \infty$ and we assert $l > 0$ by Lemma 4.1. Therefore, we get as $t \to \infty$, $F(t) \to l > 0$, $F'(t) \to 0$ and $F''(t) \to 0$, which together with (4.20) imply

$$\lim_{t\to\infty}\|\phi_t\|_2^2 = 0, \tag{4.25}$$

$$\lim_{t\to\infty} K(\phi(t)) = 0. \tag{4.26}$$

On the other hand, for fixed $t > 0$, $K(\phi(t)) < 0$ and Lemma 2.2 imply that there exists $0 < \lambda < 1$ such that $K(\lambda\phi(t)) = 0$. Moreover, we obtain by Lemma 4.2,

$$J(\phi) - J(\lambda\phi) \geq \frac{1}{2}K(\phi) \tag{4.27}$$

and $J(\lambda\phi) \geq d_{\mathcal{M}}$ by Theorem 3.1, which together with (4.26) implies that $J(\phi(t)) \geq d_{\mathcal{M}}$ as $t \to \infty$. This contradicts Lemma 4.1. Thus (4.24) is not true, namely there exists some $t^* > 0$ such that $F'(t) > 0$.

Gathering the above arguments, we complete the proof of Theorem 4.1.

References

[1] Soffer, A. and Weinstein, M. I., Resonance, radiation damping and instability in Hamiltonian nonlinear wave equations, *Invent. Mat.*, **136**, 1999, 9–74.

[2] Pecher, H., L^p-Abschätzungen und klassiche für nichtlineare Wellengeichungen. I, *Math. Z.*, **150**, 1976, 159–183.

[3] Levine, H. A., Instability and non-existence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$, *Trans. Amer. Math. Soc.*, **192**, 1974, 1–21.

[4] Payne, L. E. and Sattinger, D. H., Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22**(3–4), 1975, 273–303.

[5] Ball, J. M., Finite time blow-up in nonlinear problems, Nonlinear Evolution Equations, M. G. Grandall (ed.), Academic Press, New York, 1978, 189–205.

[6] Strauss, W. A., Nonlinear wave equations, CBMS Reg. Conf. Ser. Math., Vol. 73, A. M. S., Providence, RI, 1989.

[7] Shatah, J., Stable standing waves of nonlinear Klein-Gordon equations, *Comm. Math. Phys.*, **91**, 1983, 313–327.

[8] Shatah, J., Unstable ground state of nonlinear Klein-Gordon equations, *Trans. Amer. Math. Soc.*, **290**(2), 1985, 701–710.

[9] Berestycki, H. and Lions, P. L., Nonlinear scalar field equations I, II, *Arch. Rat. Mech. Anal.*, **82**, 1983, 313–375.

[10] Lions, P. L., The concentration compactness principle in the calculus of variations, I, The locally compact case, *Ann. Inst. Inst. H. Poincaré, Anal. Non Linéaire*, **1**, 1984, 109–145.

[11] Nirenberg, L., Topics in nonlinear functional analysis, Courant Institute Lecture Notes, New York University, New York, 1974.

[12] Rose, H. A. and Weinstein, M. I., On the bound states of the nonlinear Schrödinger equation with a linear potential, *Phys. D*, **30**, 1988, 207–218.

[13] Grillakis, M., Shatah, J. and Strauss, W., Stability theory of solitary waves in the presence of symmetry, I, *J. Funct. Anal.*, **74**, 1987, 160–197.

[14] Weinstein, M. I., Lyapunov stability of ground states of nonlinear dispersive ebolution equations, *Comm. Pure Appl. Math.*, **39**, 1986, 51–68.

[15] Zhang, J., Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potential, *Commun. Part. Diff. Eqs.*, **30**, 2005, 1429–1443.

[16] Zhang, J., Stability of attractive Bose-Einstein condensates, *J. Stat. Phys.*, **101**, 2000, 731–746.

[17] Fukuizumi, R., Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential, *Disc. Conti. Dyn. Sys.*, **7**(3), 2001, 525–544.

[18] Oh, Y. G., Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials, *J. Diff. Eqs.*, **81**, 1989, 255–274.

[19] Cazenave, T., An Introduction to Nonlinear Schrödinger Equations, Textos de Metodos Matematicos, Vol. **22**, Instituto de Mathemática-UFRJ, Rio de Janeiro, 1989.