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High-Order Energy Decay for Structural Damped Systems in the Electromagnetical Field****

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Abstract This paper is concerned with the decay estimate of high-order energy for a class of special time-dependent structural damped systems represented by Fourier multipliers. This model is widely used in the fields of semiconductivity, superconductivity, electromagnetic waves, electrolyte and electrode materials, etc.

Keywords High-order energy, Viscoelasticity, Structural dissipation, Electromagnetical field, Superconductivity
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1 Introduction

Fractional order equations are useful models for the description of anomalous dynamic behaviors, such as charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative and porous media, transport on fractal geometries, diffusion of a scalar tracer in an array of convection rolls, dynamics of a bead in a polymeric network, transport in viscoelastic materials, etc. For the Cauchy problem of the fractional order operator $\mathscr{L} = \partial_t^2 + (-\Delta)^\sigma$ in the sense of Fourier multipliers, we define the σ-energy ($\sigma > 0$) of the solution as

$$E_\sigma(u)(t) := \frac{1}{2}\|u\|^2_{\dot{H}^\sigma(\mathbb{R}^N)} + \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{R}^N)}.$$

It is clear that the energy conservation law holds as in the case of classical wave equations. Generally speaking, a moderate damping term will engender energy decay; for instance, in the field of electromagnetic waves, a kind of polynomial decay has been observed for the damped wave equations which describe the voltage and the current on an electrical transmission line with distance and time:

$$\mathbf{E}_{tt} - \frac{1}{LC}\Delta\mathbf{E} + \frac{R}{L}\mathbf{E}_t = 0,$$

where $\mathbf{E} = (V, I)^{\mathrm{T}}$, V denotes the voltage, I the current, L the distributed inductance, C the capacitance, R the distributed resistance of the conductors. We know that, except for superconductors below critical temperature, in the physical world, metals, semiconductors, insulators, ionic liquids and electrolytes all generate distributed resistance and inductance arising from the movement of electrons. As times goes on, the electromagnetic field energy will be changed into

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thermal energy. At the same time, the electric resistance of a typical metal increases linearly with rising temperature, while the electrical resistance of a typical semiconductor decreases with rising temperature. Consequently, R is not a constant any more, but is a function of time. With [7, 8] one has considered the energy decay of the operator $\mathscr{L} = \partial_t^2 - \Delta + \theta(t)\partial_t$ which explains the above phenomenon. As a matter of fact, when we apply a certain viscoelastic effect to the damped term, the situation will change. This is demonstrated in [5]. For $E_1(u)(t)$ of the operator $\mathscr{L} = \partial_t^2 - \Delta - \Delta\partial_t$, there is no decay any more due to the viscoelastic influence. This shows an important fact of the alkaline cells. The more electrolyte and electrode material there is in the cell, the greater the capacity of the cell, and the longer the discharge time.

Denote $A = -\Delta$. In this paper, we mainly consider the time-dependent viscoelastic damped system of fractional orders:

$$\begin{cases} u_{tt} + \theta(t)A^\sigma u_t + A^{2\sigma}u = 0, & \text{in } (0,\infty)\times\mathbb{R}^N, \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)), & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

Next we introduce three types of $\theta(t)$ as a classification of damping effects.

$\Gamma^{(0)}$ (Constant Structural Dissipation): $\theta(t) \equiv \mu$, $\mu \in (0, +\infty)$;

$\Gamma^{(1)}$ (Strictly Decreasing Structural Dissipation): $\theta(0) \in (0,2)$, $\theta(t) \in C^1[0,\infty)$ strictly decreases to 0, and

$$\int_0^t \theta(\tau)\mathrm{d}\tau > 1, \quad \text{as } t \to +\infty;$$

$\Gamma^{(2)}$ (Strictly Increasing Structural Dissipation): there exists a critical point $T_c \geq 0$, such that $\theta(T_c) \in (2, +\infty)$, and when $t \in [0, T_c)$, $\theta(t) = 0$, $\theta(t) \in C^2[T_c, \infty)$ strictly increases to $+\infty$ and

$$\int_{T_c}^t \theta^{-1}(\tau)\mathrm{d}\tau > 1, \quad \text{as } t \to +\infty.$$

Moreover, when $t \gg T_c$,

$$d_0\frac{\theta(t)}{\Theta(t)} \leq \frac{\theta'(t)}{\theta(t)} \leq d_1\frac{\theta(t)}{\Theta(t)}, \quad |\theta''(t)| \leq d_2\theta(t)\Big(\frac{\theta(t)}{\Theta(t)}\Big)^2,$$

with $\Theta(t) := \int_{T_c}^t \theta(\tau)\mathrm{d}\tau$, and the constants $d_0, d_1, d_2 > 0$ are all independent of t.

Accordingly, we define the homogeneous high-order energy of the solution as

$$E_\sigma^\kappa(u)(t) := \|u\|^2_{\dot{H}^{|\kappa|+2\sigma}(\mathbb{R}^N)} + \|u_t\|^2_{\dot{H}^{|\kappa|}(\mathbb{R}^N)}$$

with $|\kappa| > 0$.

Remark 1.1 When $\sigma = \frac{1}{2}$, this model is frequently used in the determination of lifespan for primary or rechargeable batteries. When $\sigma = 1$, this is the Petrowsky system.

Remark 1.2 Typical $\theta(t)$ for the strictly decreasing or increasing structural dissipations are classified as

(1) $\theta(t) = \mu(1+t)^{-\gamma_0}(\log(\mathrm{e}+t))^{-\gamma_1}\cdots(\log^{[m]}(\mathrm{e}^{[m]}+t))^{-\gamma_m}$, with $\mu \in (0,2)$, nonnegative γ_i, $i = 1,\cdots,m$ and $\gamma_0 \in (0,1]$. In particular, $\log^{[0]} x = x$, $\log^{[m+1]} x = \log\log^{[m]} x$ and $\mathrm{e}^{[m+1]} = \mathrm{e}^{\mathrm{e}^{[m]}}$;

(2) $\theta(t) = \mu(1+t)^\gamma$, $\mu \in (2,+\infty)$, $\gamma \in (0,1]$.

Remark 1.3 The case $\Gamma^{(2)}$ models the typical superconductivity phenomenon occurring in certain materials generally at very low temperatures, characterized by exactly zero electrical resistance and the exclusion of the interior magnetic field (the Meissner effect), including tin and aluminium, various metallic alloys and some heavily-doped semiconductors. Generally speaking, the electrical resistivity of a metallic conductor decreases gradually as the temperature is lowered. However, in ordinary conductors such as copper and silver, impurities and other defects impose a lower limit. Even near absolute zero, a real sample of copper shows a non-zero resistance. This is why we assume $\theta(T_c) > 2$, $T_c = 0$ for this model. And the resistance of a superconductor, despite these imperfections, drops abruptly to zero when the material is cooled below its critical temperature. In this case, $T_c > 0$.

Now we denote H^s as $H^s(\mathbb{R}^N)$ $(s > 0)$ and turn to the main result.

Theorem 1.1 *For Cauchy problem* (1.1) *with a positive* $\theta(t)$ *satisfying* $\Gamma^{(0)}$, $\Gamma^{(1)}$ *or* $\Gamma^{(2)}$, *for large time* t, *we have*

$$\Gamma^{(0)} : E^\kappa_\sigma(u)(t) \lesssim t^{-\frac{|\kappa|+2\sigma}{\sigma}} \|u_0\|^2_{H^{|\kappa|+2\sigma}} + t^{-\frac{|\kappa|}{\sigma}} \|u_1\|^2_{H^{|\kappa|}}; \tag{1.2}$$

$$\Gamma^{(1)} : E^\kappa_\sigma(u)(t) \lesssim \Big(\int_0^t \theta(\tau)\mathrm{d}\tau \Big)^{-\frac{|\kappa|+2\sigma}{\sigma}} \|u_0\|^2_{H^{|\kappa|+2\sigma}} + \Big(\int_0^t \theta(\tau)\mathrm{d}\tau \Big)^{-\frac{|\kappa|}{\sigma}} \|u_1\|^2_{H^{|\kappa|}}; \tag{1.3}$$

$$\Gamma^{(2)} : E^\kappa_\sigma(u)(t) \lesssim \Big(\int_{T_c}^t \theta^{-1}(\tau)\mathrm{d}\tau \Big)^{-\frac{|\kappa|+2\sigma}{\sigma}} \|u_0\|^2_{H^{|\kappa|+2\sigma}} + \Big(\int_{T_c}^t \theta^{-1}(\tau)\mathrm{d}\tau \Big)^{-\frac{|\kappa|}{\sigma}} \|u_1\|^2_{H^{|\kappa|}}. \tag{1.4}$$

Remark 1.4 When we choose $\theta(t) = \mu(1+t)^\delta$, $|\delta| \in (0, 1]$, $\mu > 0$, for sufficiently large time, we have the following polynomial decay and log-type decay respectively:

$$\begin{aligned} |\delta| \in [0,1) : & \ E^\kappa_\sigma(u)(t) \lesssim t^{-\frac{(1-|\delta|)(|\kappa|+2\sigma)}{\sigma}} \|u_0\|^2_{H^{|\kappa|+2\sigma}} + t^{-\frac{(1-|\delta|)|\kappa|}{\sigma}} \|u_1\|^2_{H^{|\kappa|}}, \\ |\delta| = 1 : & \ E^\kappa_\sigma(u)(t) \lesssim \log^{-\frac{|\kappa|+2\sigma}{\sigma}}(t) \|u_0\|^2_{H^{|\kappa|+2\sigma}} + \log^{-\frac{|\kappa|}{\sigma}}(t) \|u_1\|^2_{H^{|\kappa|}}. \end{aligned}$$

The rest of the paper is organized as follows. Section 2 is devoted to the constant structural dissipation, while Section 3 is for strictly decreasing structural dissipation and Section 4 for strictly increasing structural dissipation. Some powerful tools from micro-local analysis and WKB analysis will be used to obtain the precise decay estimates. In the final analysis, some concluding remarks concerned with engineering applications and open problems complete this paper.

2 Constant Structural Dissipation

In this section, we deal with the constant structural dissipation, which corresponds to the constant distributed resistance of the conductors.

Denote $v(t,\xi) = \widehat{u}(t,\xi)$. After partial Fourier transformation, (1.1) becomes

$$\begin{cases} v_{tt} + \mu|\xi|^{2\sigma} v_t + |\xi|^{4\sigma} v = 0, \\ (v(0,\xi), v_t(0,\xi)) = (v_0(\xi), v_1(\xi)). \end{cases} \tag{2.1}$$

Let

$$\lambda_i(\xi) = \frac{-\mu + (-1)^{i+1}\sqrt{\mu^2 - 4}}{2} |\xi|^{2\sigma}, \quad i = 1, 2.$$

Then apply the basic principles for ordinary differential equations, we have the following explicit representations concerned with the solution:

(1) For $\mu \neq 2$,

$$\begin{aligned} v(t,\xi) &= (\lambda_1 \exp(\lambda_2 t) - \lambda_2 \exp(\lambda_1 t))(\lambda_1 - \lambda_2)^{-1} v_0(\xi) \\ &\quad + (\exp(\lambda_1 t) - \exp(\lambda_2 t))(\lambda_1 - \lambda_2)^{-1} v_1(\xi), \\ v_t(t,\xi) &= (\exp(\lambda_2 t) - \exp(\lambda_1 t))\lambda_1\lambda_2(\lambda_1 - \lambda_2)^{-1} v_0(\xi) \\ &\quad + (\lambda_1 \exp(\lambda_1 t) - \lambda_2 \exp(\lambda_2 t))(\lambda_1 - \lambda_2)^{-1} v_1(\xi); \end{aligned} \tag{2.2}$$

(2) For $\mu = 2$,

$$\begin{aligned} v(t,\xi) &= (1 - \lambda_1 t)\exp(\lambda_1 t) v_0(\xi) + t\exp(\lambda_1 t) v_1(\xi), \\ v_t(t,\xi) &= -t\lambda_1^2 \exp(\lambda_1 t) v_0(\xi) + (1 + t\lambda_1)\exp(\lambda_1 t) v_1(\xi) \end{aligned} \tag{2.3}$$

with $\lambda_1 + \lambda_2 = -\mu|\xi|^{2\sigma},\ \lambda_1 - \lambda_2 = \sqrt{\mu^2 - 4}\,|\xi|^{2\sigma}$ and $\lambda_1\lambda_2 = |\xi|^{4\sigma}$. Actually, according to the mean value theorem, when we choose a fixed $0 < \epsilon < 1$, for $2 - \epsilon \leq \mu < 2$ and $2 < \mu \leq 2 + \epsilon$, (2.2) can be transformed into

$$\begin{aligned} v(t,\xi) &= (\exp(\lambda_1 t) - \lambda_1 t\exp(\widetilde{\lambda} t)) v_0(\xi) + t\exp(\widetilde{\lambda} t) v_1(\xi), \\ v_t(t,\xi) &= -t\lambda_1\lambda_2 \exp(\widetilde{\lambda} t) v_0(\xi) + (\exp(\lambda_2 t) + t\lambda_1 \exp(\widetilde{\lambda} t)) v_1(\xi), \end{aligned} \tag{2.4}$$

$\widetilde{\lambda}$ lies between λ_1 and λ_2. In particular, in $\mathbb{C}$, only the imaginary part changes accordingly, i.e., λ_1 and λ_2 have the same real part. $\widetilde{\lambda}$ may be not identical in (2.4); since no confusion appears, we keep the same notation. For the intervals $2 - \epsilon \leq \mu < 2$ and $2 < \mu \leq 2 + \epsilon$, we deduce from (2.4) respectively (c is independent of t and μ):

$$\begin{aligned} |\xi|^{|\alpha|}|v(t,\xi)| &\lesssim \exp(-c\mu|\xi|^{2\sigma} t)(|\xi|^{|\alpha|}|v_0(\xi)| + |\xi|^{|\alpha| - 2\sigma}|v_1(\xi)|), \\ |\xi|^{|\kappa|}|v_t(t,\xi)| &\lesssim \exp(-c\mu|\xi|^{2\sigma} t)(|\xi|^{|\kappa| + 2\sigma}|v_0(\xi)| + |\xi|^{|\kappa|}|v_1(\xi)|), \end{aligned} \tag{2.5}$$

$$\begin{aligned} |\xi|^{|\alpha|}|v(t,\xi)| &\lesssim \exp(-c(\mu - \sqrt{\mu^2 - 4})|\xi|^{2\sigma} t)(|\xi|^{|\alpha|}|v_0(\xi)| + |\xi|^{|\alpha| - 2\sigma}|v_1(\xi)|), \\ |\xi|^{|\kappa|}|v_t(t,\xi)| &\lesssim \exp(-c(\mu - \sqrt{\mu^2 - 4})|\xi|^{2\sigma} t)(|\xi|^{|\kappa| + 2\sigma}|v_0(\xi)| + |\xi|^{|\kappa|}|v_1(\xi)|). \end{aligned} \tag{2.6}$$

When $\mu = 2$, one easily finds out that the estimates are just the limit case of $\mu \to 2$ in (2.5) or (2.6). This indicates the continuity of the estimates for $2 - \epsilon \leq \mu \leq 2 + \epsilon$. For the other intervals, we apply the same procedure and reach similar conclusions. Finally, the property of exponential functions and the Parseval's formula jointly show (1.2).

3 Strictly Decreasing Structural Dissipation

In this section, we consider the strictly decreasing structural dissipation, which corresponds to the nonconstant distributed resistance of typical semiconductors. Define $D_t := -\mathrm{i}\partial_t$ and the micro-energy:

$$\mathbf{V}(t,\xi) = (v_1(t,\xi), v_2(t,\xi))^{\mathrm{T}} = (|\xi|^{2\sigma} v(t,\xi), D_t v(t,\xi))^{\mathrm{T}}.$$

We tackle this problem by virtue of the fundamental system of

$$D_t\mathbf{V} - A\mathbf{V} := D_t\mathbf{V} - \begin{pmatrix} 0 & |\xi|^{2\sigma} \\ |\xi|^{2\sigma} & \mathrm{i}\theta(t)|\xi|^{2\sigma} \end{pmatrix}\mathbf{V} = 0. \tag{3.1}$$

The characteristic roots for A are

$$\lambda_i = \frac{\mathrm{i}\theta(t)|\xi|^{2\sigma} + (-1)^{i-1}\sqrt{-\theta^2(t)+4}|\xi|^{2\sigma}}{2}, \quad i = 1, 2.$$

The system of the corresponding eigenvalues is

$$\Phi(t,\xi) := \begin{pmatrix} 1 & 1 \\ \lambda_1(t,\xi)|\xi|^{-2\sigma} & \lambda_2(t,\xi)|\xi|^{-2\sigma} \end{pmatrix}$$

with $\|\Phi(t,\xi)\| = 1$. Moreover,

$$\det \Phi(t,\xi) = \frac{-\sqrt{-\theta^2(t)|\xi|^{4\sigma} + 4|\xi|^{4\sigma}}}{|\xi|^{2\sigma}} = -\sqrt{-\theta^2(t)+4}.$$

From the monotonicity of θ and $\theta(0) \in (0,2)$, we conclude that $|\det \Phi(t,\xi)| \sim 1$. This fact indicates that $\Phi(t,\xi)$ is invertible and its inverse is

$$\Phi^{-1}(t,\xi) = -\frac{1}{\sqrt{-\theta^2(t)+4}} \begin{pmatrix} \lambda_2(t,\xi)|\xi|^{-2\sigma} & -1 \\ -\lambda_1(t,\xi)|\xi|^{-2\sigma} & 1 \end{pmatrix}$$

with $\|\Phi^{-1}(t,\xi)\| \sim 1$. Let $\mathbf{V}(t,\xi) = \Phi(t,\xi)\mathbf{V}_1(t,\xi)$. Then

$$D_t\mathbf{V}_1 := \mathscr{D}\mathbf{V}_1 - B\mathbf{V}_1 = \Phi^{-1}A\Phi\mathbf{V}_1 - \Phi^{-1}D_t\Phi\mathbf{V}_1.$$

And the concrete forms are

$$\begin{aligned}
\Phi^{-1}A\Phi &= \mathrm{diag}\Big(\frac{\sqrt{-\theta^2(t)+4}|\xi|^{2\sigma} + \mathrm{i}\theta(t)|\xi|^{2\sigma}}{2}, \frac{-\sqrt{-\theta^2(t)+4}|\xi|^{2\sigma} + \mathrm{i}\theta(t)|\xi|^{2\sigma}}{2}\Big), \\
\Phi^{-1}D_t\Phi &= (\det\Phi(t,\xi)|\xi|^{2\sigma})^{-1}\begin{pmatrix} -D_t\lambda_1(t,\xi) & -D_t\lambda_2(t,\xi) \\ D_t\lambda_1(t,\xi) & D_t\lambda_2(t,\xi) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \frac{\theta'(t)}{\sqrt{-\theta^2(t)+4}} - \frac{\theta(t)\theta'(t)}{\mathrm{i}(-\theta^2(t)+4)} & \frac{\theta'(t)}{\sqrt{-\theta^2(t)+4}} + \frac{\theta(t)\theta'(t)}{\mathrm{i}(-\theta^2(t)+4)} \\ -\frac{\theta'(t)}{\sqrt{-\theta^2(t)+4}} + \frac{\theta(t)\theta'(t)}{\mathrm{i}(-\theta^2(t)+4)} & -\frac{\theta'(t)}{\sqrt{-\theta^2(t)+4}} - \frac{\theta(t)\theta'(t)}{\mathrm{i}(-\theta^2(t)+4)} \end{pmatrix}.
\end{aligned}$$

The fundamental solution for this equation is $E = E_1\mathbf{Q}$ with

$$\begin{cases}
E_1(t,s,\xi)^{(11)} = \exp\Big(\frac{1}{2}\mathrm{i}\int_s^t \sqrt{-\theta^2(t)+4}|\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big), \\
E_1(t,s,\xi)^{(22)} = \exp\Big(-\frac{1}{2}\mathrm{i}\int_s^t \sqrt{-\theta^2(t)+4}|\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big), \\
E_1(t,s,\xi)^{(12)} = E_1(t,s,\xi)^{(21)} = 0.
\end{cases}$$

And $\mathbf{Q}$ satisfies

$$D_t\mathbf{Q} + E_1(t,s,\xi)^{-1}B(t,\xi)E_1(t,s,\xi)\mathbf{Q} = 0, \quad \mathbf{Q}(s,s,\xi) = I.$$

In fact, the method of successive approximation enables us to construct the fundamental solution of the system $D_tE(t,s,\xi) = A(t,\xi)E(t,s,\xi)$, $E(s,s,\xi) = I$. More precisely, $E(t,s,\xi)$ is given in the form of matrizant representation:

$$E(t,s,\xi) = I + \sum_{k=1}^{\infty}\mathrm{i}^k\int_s^t A(t_1,\xi)\int_s^{t_1} A(t_2,\xi)\cdots\int_s^{t_{k-1}} A(t_k,\xi)\mathrm{d}t_k\cdots\mathrm{d}t_1.$$

Actually, we have the following result.

Lemma 3.1 *For $k \in \mathbb{N}_+$, it holds that*

$$\Big\| \int_s^t A(t_1,\xi) \int_s^{t_1} A(t_2,\xi) \cdots \int_s^{t_{k-1}} A(t_k,\xi) \mathrm{d}t_k \cdots \mathrm{d}t_1 \Big\| \leqslant \frac{1}{k!} \Big(\int_s^t \|A(r,\xi)\| \mathrm{d}r \Big)^k.$$

As a matter of fact,

$$\begin{aligned} &\int_s^t \|A(t_1,\xi)\| \int_s^{t_1} \|A(t_2,\xi)\| \mathrm{d}t_2 \mathrm{d}t_1 \\ &= \int_s^t \frac{\partial}{\partial t_1} \Big(\int_s^{t_1} \|A(t_2,\xi)\| \mathrm{d}t_2 \Big) \Big(\int_s^{t_1} \|A(t_2,\xi)\| \mathrm{d}t_2 \Big) \mathrm{d}t_1 \\ &= \int_s^t \frac{1}{2} \frac{\partial}{\partial t_1} \Big(\int_s^{t_1} \|A(t_2,\xi)\| \mathrm{d}t_2 \Big)^2 \mathrm{d}t_1 = \frac{1}{2} \Big(\int_s^t \|A(r,\xi)\| \mathrm{d}r \Big)^2. \end{aligned}$$

By the induction method the statement follows immediately.

It is obvious that $\|\mathbf{Q}(t,s,\xi)\| \le C$, which leads to $E(t,s,\xi) \le \exp\big(-\frac{1}{2} \int_s^t \theta(\tau)|\xi|^{2\sigma} \mathrm{d}\tau \big)$. We have

$$\|\mathbf{V}(t,\xi)\| \lesssim \exp\Big(-\frac{1}{2} \int_s^t \theta(\tau)|\xi|^{2\sigma} \mathrm{d}\tau \Big) \|\mathbf{V}(s,\xi)\|. \tag{3.2}$$

The energy estimate (1.3) follows immediately.

Remark 3.1 In this case, the division of the phase space is of no significance, and one step of diagonalization is sufficient. But for the strictly increasing structural dissipation, the situation becomes more complex, and we need a further step of diagolization based on the theory of normal forms.

4 Strictly Increasing Structural Dissipation

In this section, we deal with the strictly increasing structural dissipation, which corresponds to the nonconstant distributed resistance of typical superconductors or metals. First and foremost, we introduce some useful tools from micro-local analysis. We choose sufficiently large numbers N and T_0, which will be determined later, and define the following three zones:

$$\begin{aligned} Z_L(N) &= \{(t,\xi) \in [0,\infty) \times \{\xi \in \mathbb{R}^N \setminus \mathbf{0}\} : \Theta(t)|\xi|^{2\sigma} \le N\}, \\ Z_M(N,T_0) &= \{(t,\xi) \in (0,T_0] \times \{\xi \in \mathbb{R}^N \setminus \mathbf{0}\} : \Theta(t)|\xi|^{2\sigma} \ge N\}, \\ Z_H(N,T_0) &= \{(t,\xi) \in [T_0,\infty) \times \{\xi \in \mathbb{R}^N \setminus \mathbf{0}\} : \Theta(t)|\xi|^{2\sigma} \ge N\}. \end{aligned}$$

The separating lines are defined by $\Theta(t_\xi)|\xi|^{2\sigma} = N$ and $t = T_0$. In these zones, as usual, we define the micro-energy as

$$\mathbf{V}(t,\xi) = (v_1(t,\xi), v_2(t,\xi))^{\mathrm{T}} = (|\xi|^{2\sigma} v(t,\xi), D_t v(t,\xi))^{\mathrm{T}},$$

and in $Z_H(N,T_0)$ the classes of symbols as

$$\begin{aligned} S_k\{m_1,m_2,m_3\} = \Big\{ & a(t,|\xi|) \in C^k([T_0,\infty); C^\infty(\mathbb{R}^N \setminus \mathbf{0})) : \\ & |D_t^\ell D_\xi^\alpha a(t,|\xi|)| \le C_{k,\alpha} |\xi|^{m_1-|\alpha|} \theta^{m_2}(t) \Big(\frac{\theta(t)}{\Theta(t)} \Big)^{m_3+\ell} \\ & \text{for all multi-indices } \alpha \text{ and non-negative integers } \ell \le k \Big\}. \end{aligned}$$

Our considerations are based on the following properties of the symbols:

(1) $S_k\{m_1, m_2, m_3\} \subset S_k\{m_1 + 2\sigma\ell, m_2 + \ell, m_3 - \ell\}$ for $\ell \in \mathbb{N}$;

(2) if $a \in S_k\{m_1, m_2, m_3\}$ and $b \in S_k\{k_1, k_2, k_3\}$, then $ab \in S_k\{m_1 + k_1, m_2 + k_2, m_3 + k_3\}$;

(3) if $a \in S_k\{m_1, m_2, m_3\}$, then $D_t^\ell a \in S_{k-\ell}\{m_1, m_2, m_3 + \ell\}$ and $D_\xi^\alpha a \in S_k\{m_1 - |\alpha|, m_2, m_3\}$ for $\ell \le k$;

(4) if $a(t, |\xi|) \in S_0\{-2\sigma, -1, 2\}$, then $\int_{t_\xi}^t |a(\tau, |\xi|)|\mathrm{d}\tau \le C$ for all $(t, \xi) \in Z_H(N, T_0)$.

4.1 Consideration in $Z_L(N)$

We tackle this problem by virtue of the fundamental system of (3.1). Apply the transformation $\mathbf{V} = M\mathbf{V}_1$, $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then

$$D_t\mathbf{V}_1 - \mathscr{D}\mathbf{V}_1 + B\mathbf{V}_1 := D_t\mathbf{V}_1 - \begin{pmatrix} |\xi|^{2\sigma} + \frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma} & 0 \\ 0 & -|\xi|^{2\sigma} + \frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma} \end{pmatrix} \mathbf{V}_1 - \frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{V}_1 = 0.$$

The fundamental solution for this equation is $E = E_1\mathbf{Q}$ with

$$\begin{cases} E_1(t, s, \xi)^{(11)} = \exp\Big(\mathrm{i}\displaystyle\int_s^t |\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big), \\ E_1(t, s, \xi)^{(22)} = \exp\Big(-\mathrm{i}\displaystyle\int_s^t |\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big), \\ E_1(t, s, \xi)^{(12)} = E_1(t, s, \xi)^{(21)} = 0. \end{cases}$$

And $\mathbf{Q}$ satisfies

$$D_t\mathbf{Q} + E_1(t, s, \xi)^{-1}B(t, \xi)E_1(t, s, \xi)\mathbf{Q} = 0.$$

It is obvious that

$$\|E_1(t, s, \xi)\| \le \exp\Big(-\frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big).$$

Consequently, applying the definition of $Z_L(N)$, we have $\|\mathbf{Q}(t, s, \xi)\| \le C$. This leads to

$$\|\mathbf{V}(t, \xi)\| \lesssim \exp\Big(-\frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big)\|\mathbf{V}(s, \xi)\|. \tag{4.1}$$

4.2 Consideration in $Z_M(N, T_0)$

Change the characteristic roots in Section 3 into

$$\lambda_j = \frac{\mathrm{i}\theta(t)|\xi|^{2\sigma} + (-1)^{j-1}\mathrm{i}\sqrt{\theta^2(t) - 4}|\xi|^{2\sigma}}{2}, \quad j = 1, 2.$$

Applying the same procedure as in the treatment of noneffective dissipation, we get

$$D_t\mathbf{V}_1 - \mathscr{D}\mathbf{V}_1 + B\mathbf{V}_1 := D_t\mathbf{V}_1 - \Phi^{-1}A\Phi\mathbf{V}_1 + \Phi^{-1}D_t\Phi\mathbf{V}_1 = 0, \tag{4.2}$$

with

$$\Phi^{-1}A\Phi = \mathrm{diag}\Big(\frac{\mathrm{i}\sqrt{\theta^2(t) - 4}|\xi|^{2\sigma} + \mathrm{i}\theta(t)|\xi|^{2\sigma}}{2}, \frac{-\mathrm{i}\sqrt{\theta^2(t) - 4}|\xi|^{2\sigma} + \mathrm{i}\theta(t)|\xi|^{2\sigma}}{2}\Big)$$

and

$$\Phi^{-1}D_t\Phi = (\det\Phi(t,\xi)|\xi|^{2\sigma})^{-1}\begin{pmatrix} -D_t\lambda_1(t,\xi) & -D_t\lambda_2(t,\xi) \\ D_t\lambda_1(t,\xi) & D_t\lambda_2(t,\xi)\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} \frac{\theta'(t)}{\mathrm{i}\sqrt{\theta^2(t)-4}} + \frac{\theta(t)\theta'(t)}{\mathrm{i}(\theta^2(t)-4)} & \frac{\theta'(t)}{\mathrm{i}\sqrt{\theta^2(t)-4}} - \frac{\theta(t)\theta'(t)}{\mathrm{i}(\theta^2(t)-4)} \\ -\frac{\theta'(t)}{\mathrm{i}\sqrt{\theta^2(t)-4}} - \frac{\theta(t)\theta'(t)}{\mathrm{i}(\theta^2(t)-4)} & -\frac{\theta'(t)}{\mathrm{i}\sqrt{\theta^2(t)-4}} + \frac{\theta(t)\theta'(t)}{\mathrm{i}(\theta^2(t)-4)}\end{pmatrix}.$$

After coordination, we have

$$D_t\mathbf{V}_1 := \mathscr{D}_1\mathbf{V}_1 - B_1\mathbf{V}_1 = \begin{pmatrix}\frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma} & 0 \\ 0 & \frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma}\end{pmatrix}\mathbf{V}_1 - B_1\mathbf{V}_1.$$

Applying the same discussion for fundamental systems, we have

$$\|\mathbf{V}(t,\xi)\| \lesssim \exp\Big(\frac{1}{2}\int_s^t\sqrt{\theta^2(\tau)-4}|\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_s^t\theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big)\|\mathbf{V}(s,\xi)\|. \tag{4.3}$$

4.3 Consideration in $Z_H(N,T_0)$

Note that in (4.2), $\mathscr{D}\in S_2\{2\sigma,1,0\}$ and $B\in S_1\{0,0,1\}$. To carry out a further step of diagonalization, we follow the procedure of asymptotic theory of ordinary differential equations. Namely, we look for a matrix $N_1(t,\xi) := I + N^{(1)}(t,\xi)$, where $B^{(0)} := B$, $F^{(0)} := \operatorname{diag} B^{(0)}$,

$$N_{qr}^{(1)} := \frac{B_{qr}^{(0)}}{\tau_q-\tau_r},\ q\neq r \text{ and } N_{qq}^{(1)} := 0,\ \tau_k=\lambda_k,\ k=1,2, \text{ are the characteristic roots,}$$
$$B^{(1)} := (D_t-\mathscr{D}+B)(I+N^{(1)}) - (I+N^{(1)})(D_t-\mathscr{D}+F^{(0)}).$$

According to the properties of symbols, we have $N^{(1)}\in S_1\{-2\sigma,-1,1\}$ and $F^{(0)}\in S_1\{0,0,1\}$. As for $B^{(1)}$, we obtain the following relation:

$$B^{(1)} = B + [N^{(1)},\mathscr{D}] - F^{(0)} + D_tN^{(1)} + BN^{(1)} - N^{(1)}F^{(0)}.$$

The construction principle implies that the sum of the first three terms vanishes, hence $B^{(1)}\in S_0\{-2\sigma,-1,2\}$. Finally, let

$$R_1 := N_1^{-1}B^{(1)} = N_1^{-1}((D_t-\mathscr{D}+B)(I+N^{(1)}) - (I+N^{(1)})(D_t-\mathscr{D}+F^{(0)})).$$

This definition means $R_1 = N_1^{-1}B^{(1)}\in S_0\{-2\sigma,-1,2\}$. Actually, due to the definition of symbols, $N^{(1)}\in S_1\{-2\sigma,-1,1\}$ indicates $|N_{qr}^{(1)}|\leqslant\frac{C}{N}$. Consequently, a sufficiently large N assures $\|N_1-I\|<\frac{1}{2}$ in $Z_H(N,T_0)$, which implies the invertibility of N_1. As a result, we have the following system after the second step of diagonalization:

$$(D_t-\mathscr{D}+B)N_1 = N_1(D_t-\mathscr{D}+F^{(0)}+R_1), \quad \text{where } R_1\in S_0\{-2\sigma,-1,2\}.$$

Now we consider the system

$$(D_t-\mathscr{D}+F^{(0)}+R_1)\mathbf{V}_2 = 0.$$

After coordination, we have

$$\begin{aligned}
D_t\mathbf{V}_2 &:= \mathscr{D}_1\mathbf{V}_2 + B_1\mathbf{V}_2 - R_1\mathbf{V}_2\\
&= \begin{pmatrix} -\frac{\theta(t)\theta'(t)}{2\mathrm{i}(\theta^2(t)-4)} + \frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma} & 0\\ 0 & -\frac{\theta(t)\theta'(t)}{2\mathrm{i}(\theta^2(t)-4)} + \frac{\mathrm{i}}{2}\theta(t)|\xi|^{2\sigma}\end{pmatrix}\mathbf{V}_2\\
&\quad + \begin{pmatrix} -\frac{\theta'(t)}{2\mathrm{i}\sqrt{\theta^2(t)-4}} + \frac{\mathrm{i}}{2}\sqrt{\theta^2(t)-4}|\xi|^{2\sigma} & 0\\ 0 & \frac{\theta'(t)}{2\mathrm{i}\sqrt{\theta^2(t)-4}} - \frac{\mathrm{i}}{2}\sqrt{\theta^2(t)-4}|\xi|^{2\sigma}\end{pmatrix}\mathbf{V}_2 - R_1\mathbf{V}_2.
\end{aligned}$$

The fundamental solution $E_2(t,s,\xi)$ for $D_t\mathbf{V}_2 = \mathscr{D}_1\mathbf{V}_2$ is of the form:

$$\begin{cases}
E_2(t,s,\xi)^{(11)} = \exp\Big(-\dfrac{1}{2}\displaystyle\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau + \log\Big(\dfrac{\theta^2(s)-4}{\theta^2(t)-4}\Big)^{\frac{1}{4}}\Big),\\
E_2(t,s,\xi)^{(22)} = \exp\Big(-\dfrac{1}{2}\displaystyle\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau + \log\Big(\dfrac{\theta^2(s)-4}{\theta^2(t)-4}\Big)^{\frac{1}{4}}\Big),\\
E_2(t,s,\xi)^{(12)} = E_1(t,s,\xi)^{(21)} = 0.
\end{cases}$$

It is clear that

$$\|E_2(t,s,\xi)\| \lesssim \sqrt{\frac{\theta(s)}{\theta(t)}}\exp\Big(-\frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big),$$

and for the remainder term, we have

$$\begin{aligned}
\exp\Big(\int_s^t \|B_1 - R_1\|\mathrm{d}\tau\Big) &\lesssim \exp\Big(\frac{1}{2}\int_s^t \sqrt{\theta^2(\tau)-4}\,|\xi|^{2\sigma}\mathrm{d}\tau\Big)\exp\Big(\int_s^t \frac{\theta'(\tau)}{2\sqrt{\theta^2(\tau)-4}}\mathrm{d}\tau\Big)\\
&\lesssim \sqrt{\frac{\theta(t)}{\theta(s)}}\exp\Big(\frac{1}{2}\int_s^t \sqrt{\theta^2(\tau)-4}\,|\xi|^{2\sigma}\mathrm{d}\tau\Big).
\end{aligned}$$

These lead to the following estimate for large time:

$$\|\mathbf{V}(t,\xi)\| \lesssim \exp\Big(\frac{1}{2}\int_s^t \sqrt{\theta^2(\tau)-4}|\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_s^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big)\|\mathbf{V}(s,\xi)\|. \tag{4.4}$$

Summarizing (4.1), (4.3) and (4.4), we have the following lemma.

Lemma 4.1 *For the strictly increasing structural dissipation of the Cauchy problem* (1.1), *we have*

$$\|\mathbf{V}(t,\xi)\| \lesssim \exp\Big(\frac{1}{2}\int_{T_c}^t \sqrt{\theta^2(\tau)-4}\,|\xi|^{2\sigma}\mathrm{d}\tau - \frac{1}{2}\int_{T_c}^t \theta(\tau)|\xi|^{2\sigma}\mathrm{d}\tau\Big)\|\mathbf{V}(T_c,\xi)\|. \tag{4.5}$$

For sufficiently large t, we notice the fact that $\frac{1}{2}(\sqrt{\theta^2(t)-4} - \theta(t)) \le \theta^{-1}(t)$. The energy estimate (1.4) follows immediately.

5 Concluding Remarks

For the over-damping and under-damping coefficients, many problems still remain open (see [1]). In the viscous damped systems, it is also very interesting to tackle the self-adjoint positive definite operator with compact resolvent by the application of spectral theory (see [2]). As for the engineering applications of this kind of models, in 1986, the discovery of a family of cuprate-perovskite ceramic materials known as high-temperature superconductors, with critical temperatures in excess of 90 kelvin, spurred renewed interest and research in superconductivity for several reasons. Nowadays, new materials with even higher critical temperature have been discovered and more commercial applications are feasible, for instance, mag-lev trains in Shanghai, etc. For further information in this aspect, please refer to [6].

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