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Mean Curvature Flow via Convex Functions on Grassmannian Manifolds***

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Abstract Using the convex functions on Grassmannian manifolds, the authors obtain the interior estimates for the mean curvature flow of higher codimension. Confinable properties of Gauss images under the mean curvature flow have been obtained, which reveal that if the Gauss image of the initial submanifold is contained in a certain sublevel set of the v-function, then all the Gauss images of the submanifolds under the mean curvature flow are also contained in the same sublevel set of the v-function. Under such restrictions, curvature estimates in terms of v-function composed with the Gauss map can be carried out.

Keywords Mean curvature flow, Convex function, Gauss map
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1 Introduction

For a hypersurface, there are support functions which play an important role in the hypersurface investigation. This technique would also be used for a general submanifold in Euclidean space. We can define generalized support functions related to the generalized Gauss map whose image is the Grassmannian manifold. The Plücker imbedding of the Grassmannian manifold into Euclidean space gives us the "height function" w on the Grassmanian manifold.

In the case of positive "height function", we can define the function $v = w^{-1}$ on an open subset $\mathbb{U}$ in the Grassmannian manifold. Now, the key issue is the estimates of Hessian of v-function. In our previous paper [18], a quite accurate lower bound of the Hess(v) has been given. The estimates also give the corresponding convex region of the function.

In the previous work of the first author with Jost [6], the largest geodesic convex set B_{JX} in the Grassmannian manifold was found. It is interesting to note that the convex region of the v-function is just $B_{\rm JX}$. Based on it, we can define auxiliary functions which enable us to carry out the Schoen-Simon-Yau type curvature estimates and Ecker-Huisken type curvature estimates for minimal submanifolds in higher codimension (see [18]), and for submanifolds with prescribed Gauss image and mean curvature (see [17]).

Now, we continue to explore applications of those convex functions on the Grassmannian manifolds to other related problems.

We consider the deformation of a complete submanifold in $\mathbb{R}^{m+n}$ under the mean curvature flow. For codimension one case, there are many deep results given by Ecker-Huisken [4, 5, 7, 8].

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In recent years, considerable attention has been paid to higher codimensional mean curvature flow (see [1–3, 9–12]). In previous papers, we studied mean curvature flow with convex Gauss image (see [16]) and curvature estimates for minimal submanifolds (see [19]). Some results in [4] has been generalized to higher codimension. Now, the convex v-function on the Grassmannian manifold can be used in the interior estimates for mean curvature flow in higher codimension and some results in [5] can be generalized to the higher codimensional situation.

We obtain the confinable properties (see Theorem 4.1). This is an interesting feature which tells us that if the Gauss image of the initial submanifold is contained in a certain sublevel set of the v-function, then all the Gauss images of the submanifolds under the MCF are also contained in the same sublevel set of the v-function. In particular, if the initial submanifold is an entire graph, then the graphic situation is always remained under the MCF. Moreover, v-function composed with Gauss map is just the volume element. If its value is less than 2 initially, then their values are always less than 2 under the MCF.

Under such restrictions, we can carry out the curvature estimates under the MCF (see Theorems 5.1 and 5.2) in terms of $\widetilde{v} = v \circ \gamma$ with the Gauss map γ.

2 Convex Functions on Grassmannian Manifolds

Let $\mathbb{R}^{m+n}$ be an $(m+n)$-dimensional Euclidean space. All oriented n-subspaces constitute the Grassmannian manifold $\mathbf{G}_{n,m}$.

Fix $P_0 \in \mathbf{G}_{n,m}$ in the sequel, which is expressed by a unit n-vector $\epsilon_1 \wedge \cdots \wedge \epsilon_n$. For any $P \in \mathbf{G}_{n,m}$, expressed by an n-vector $e_1 \wedge \cdots \wedge e_n$, we define an important function on $\mathbf{G}_{n,m}$,

$$w \overset{\text{def.}}{=} \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_n, \epsilon_1 \wedge \cdots \wedge \epsilon_n \rangle = \det W,$$

where $W = (\langle e_i, \epsilon_j \rangle)$.

Denote

$$\mathbb{U} = \{P \in \mathbf{G}_{n,m} : w(P) > 0\}.$$

Let $\{\epsilon_{n+\alpha}\}$ be m vectors such that $\{\epsilon_i, \epsilon_{n+\alpha}\}$ form an orthornormal basis of $\mathbb{R}^{m+n}$. Then we can span arbitrary $P \in \mathbb{U}$ by n vectors f_i,

$$f_i = \epsilon_i + z_{i\alpha} \epsilon_{n+\alpha},$$

where $Z = (z_{i\alpha})$ are the local coordinates of P in $\mathbb{U}$. Here and in the sequel we use the summation convention and agree the range of indices:

$$1 \le i, j \le n, \quad 1 \le \alpha, \beta \le m.$$

The Jordan angles between P and P_0 are defined by

$$\theta_\alpha = \arccos(\lambda_\alpha),$$

where $\lambda_\alpha \ge 0$ and λ_α^2 are the eigenvalues of the symmetric matrix $W^{\mathrm{T}} W$. On $\mathbb{U}$ we can define

$$v = w^{-1}.$$

Then it is easily seen that

$$v(P) = [\det(I_n + Z Z^{\mathrm{T}})]^{\frac{1}{2}} = \prod_{\alpha=1}^{m} \sec \theta_\alpha.$$

The canonical metric on $\mathbf{G}_{n,m}$ in the local coordinates can be described as (see [14, Chapter VII])

$$g = \operatorname{tr}((I_n + ZZ^{\mathrm{T}})^{-1}\mathrm{d}Z(I_m + Z^{\mathrm{T}}Z)^{-1}\mathrm{d}Z^{\mathrm{T}}). \tag{2.1}$$

Let $E_{i\alpha}$ be the matrix with 1 in the intersection of row i and column α and 0 otherwise. Denote $g_{i\alpha,j\beta} = \langle E_{i\alpha}, E_{j\beta}\rangle$ and let $(g^{i\alpha,j\beta})$ be the inverse matrix of $(g_{i\alpha,j\beta})$. Then

$$(1+\lambda_i^2)^{\frac{1}{2}}(1+\lambda_\alpha^2)^{\frac{1}{2}}E_{i\alpha}$$

form an orthonormal basis of $T_P\mathbf{G}_{n,m}$, where $\lambda_\alpha = \tan\theta_\alpha$. Denote its dual basis in $T_P^*\mathbf{G}_{n,m}$ by $\omega_{i\alpha}$.

A lengthy computation yields (see [18])

$$\begin{aligned}\operatorname{Hess}(v)_P = &\sum_{\substack{m+1\le i\le n\\ \alpha}} v\omega_{i\alpha}^2 + \sum_\alpha (1+\lambda_\alpha^2)v\omega_{\alpha\alpha}^2 + v^{-1}\mathrm{d}v\otimes\mathrm{d}v\\ &+ \sum_{\alpha<\beta}\Big[(1+\lambda_\alpha\lambda_\beta)v\Big(\frac{\sqrt{2}}{2}(\omega_{\alpha\beta}+\omega_{\beta\alpha})\Big)^2\\ &+ (1-\lambda_\alpha\lambda_\beta)v\Big(\frac{\sqrt{2}}{2}(\omega_{\alpha\beta}-\omega_{\beta\alpha})\Big)^2\Big].\end{aligned} \tag{2.2}$$

Define

$$B_{\mathrm{JX}}(P_0) = \Big\{P\in\mathbb{U} : \text{sum of any two Jordan angles between } P \text{ and } P_0 < \frac{\pi}{2}\Big\}.$$

This is a geodesic convex set, larger than the geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ and centered at P_0. This was found in a previous work of Jost-Xin [6]. For any real number a, let $\mathbb{V}_a = \{P\in\mathbf{G}_{n,m}, v(P) < a\}$. From [6, Theorem 3.2], we know that

$$\mathbb{V}_2 \subset B_{\mathrm{JX}} \quad \text{and} \quad \overline{\mathbb{V}}_2 \cap \overline{B}_{\mathrm{JX}} \ne \emptyset.$$

$\operatorname{Hess}(v)_P$ is positive definite if and only if $\theta_\alpha+\theta_\beta < \frac{\pi}{2}$ for arbitrary $\alpha\ne\beta$, i.e., $P\in B_{\mathrm{JX}}(P_0)$.

From (2.2), it is easy to get an estimate

$$\operatorname{Hess}(v) \ge v(2-v)g + v^{-1}\mathrm{d}v\otimes\mathrm{d}v, \quad \text{on } \overline{\mathbb{V}}_2.$$

For later applications, the above estimate is not accurate enough. Using the radial compensation technique, the estimate could be refined.

Theorem 2.1 (see [18]) *v is a convex function on $B_{\mathrm{JX}}(P_0) \subset \mathbb{U} \subset \mathbf{G}_{n,m}$, and*

$$\operatorname{Hess}(v) \ge v(2-v)g + \Big(\frac{v-1}{pv(v^{\frac{2}{p}}-1)} + \frac{p+1}{pv}\Big)\mathrm{d}v\otimes\mathrm{d}v$$

on $\overline{\mathbb{V}}_2$, where g is the metric tensor on $\mathbf{G}_{n,m}$ and $p = \min(n,m)$.

Remark 2.1 For any $a\le 2$, the sub-level set $\mathbb{V}_a$ is a convex set in $\mathbf{G}_{n,m}$.

Remark 2.2 The sectional curvature varies in $[0,2]$ under the canonical Riemannian metric on $\mathbf{G}_{n,m}$. By the standard Hessian comparison theorem, we have

$$\operatorname{Hess}(\rho) \ge \sqrt{2}\cot(\sqrt{2}\,\rho)(g - \mathrm{d}\rho\otimes\mathrm{d}\rho),$$

where ρ is the distance function from a fixed point in $\mathbf{G}_{n,m}$.

3 Evolution Equations

Let M be a complete n-submanifold in $\mathbb{R}^{m+n}$. Consider the deformation of M under the mean curvature flow, i.e., there exists a one-parameter family $F_t = F(\,\cdot\,,t)$ of immersions $F_t : M \to \mathbb{R}^{m+n}$ with corresponding images $M_t = F_t(M)$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}F(x,t) &= H(x,t), \quad x \in M, \\ F(x,0) &= F(x), \end{aligned} \tag{3.1}$$

where $H(x,t)$ is the mean curvature vector of M_t at $F(x,t)$.

From equation (3.1), it is easily known that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)|F|^2 = -2n. \tag{3.2}$$

Let B denote the second fundamental form of M_t in $\mathbb{R}^{m+n}$. It satisfies the evolution equation.

Lemma 3.1 (see [16, Lemma 3.1])

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)|B|^2 \le -2|\nabla|B||^2 + 3|B|^4. \tag{3.3}$$

The Gauss map $\gamma : M \to \mathbf{G}_{n,m}$ is defined by

$$\gamma(x) = T_xM \in \mathbf{G}_{n,m}$$

via the parallel translation in $\mathbb{R}^{m+n}$ for all $x \in M$. The Gauss map under the MCF satisfies the following relation.

Proposition 3.1 (see [12])

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \tau(\gamma(t)), \tag{3.4}$$

where $\tau(\gamma(t))$ is the tension fields of the Gauss map $\gamma(t)$ from M_t.

Let $h : \mathbb{V} \to \mathbb{R}$ be a smooth function defined on an open subset $\mathbb{V} \subset G_{n,m}$ and denote $\widetilde{h} = h \circ \gamma$. Then

$$\frac{\mathrm{d}\widetilde{h}}{\mathrm{d}t} = \frac{\mathrm{d}(h\circ\gamma)}{\mathrm{d}t} = \mathrm{d}h(\tau(\gamma)).$$

On the other hand, by the composition formula,

$$\Delta\widetilde{h} = \Delta(h\circ\gamma) = \mathrm{Hess}(h)(\gamma_* e_i, \gamma_* e_i)\circ\gamma + \mathrm{d}h(\tau(\gamma)),$$

where $\{e_i\}$ is a local orthonormal frame field on M_t. Then we derive

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)\widetilde{h} = -\mathrm{Hess}(h)(\gamma_* e_i, \gamma_* e_i)\circ\gamma. \tag{3.5}$$

4 Confinable Properties

Now, we consider the convex Gauss image situation which is preserved under the flow, so called confinable property.

Let $r : \mathbb{R}^{n+m} \times \mathbb{R} \to \mathbb{R}$ be a smooth, nonnegative function, such that for any $R > 0$,

$$\overline{M}_{t,R} = \{x \in M_t : r(x,t) \le R^2\}$$

is compact.

Lemma 4.1 *Assume that r satisfies $(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta)r\ge 0$. Let $R>0$, such that $\gamma(\overline{M}_{0,R})\subset\mathbb{V}\subset\mathbf{G}_{n,m}$. Define $\varphi=R^2-r$ and φ_+ denotes the positive part of φ. $h:\mathbb{V}\to\mathbb{R}$ is a smooth positive function such that*

$$\mathrm{Hess}(h)\ge Ch^{-1}\mathrm{d}h\otimes\mathrm{d}h \tag{4.1}$$

with $C\ge\frac{3}{2}$. Then we have the estimate

$$\widetilde{h}\varphi_+^2\le\sup_{\overline{M}_{0,R}}\widetilde{h}\varphi_+^2,$$

where $\widetilde{h}=h\circ\gamma$.

Proof Denote $\eta=\varphi_+^2$. Then at an arbitrary interior point of the support of φ_+, we have

$$\eta'\le 0,\quad \eta^{-1}(\eta')^2=4,\quad\text{and}\quad \eta''=2, \tag{4.2}$$

where $'$ denotes differentiation with respect to r. By (4.1) and (3.5), we have

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\widetilde{h}\le -C\widetilde{h}^{-1}|\nabla\widetilde{h}|^2 \tag{4.3}$$

and moreover

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(\widetilde{h}\eta)&=\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\widetilde{h}\cdot\eta+\widetilde{h}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\eta-2\nabla\widetilde{h}\cdot\nabla\eta\\&\le -C\widetilde{h}^{-1}|\nabla\widetilde{h}|^2\eta+\widetilde{h}\Big(\eta'\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)r-\eta''|\nabla r|^2\Big)-2\nabla\widetilde{h}\cdot\nabla\eta\\&\le -C\widetilde{h}^{-1}|\nabla\widetilde{h}|^2\eta-2\widetilde{h}|\nabla r|^2-2\nabla\widetilde{h}\cdot\nabla\eta.\end{aligned} \tag{4.4}$$

Observe

$$\begin{aligned}-2\nabla\widetilde{h}\cdot\nabla\eta&=(2C-2)\nabla\widetilde{h}\cdot\nabla\eta-2C\nabla\widetilde{h}\cdot\nabla\eta\\&=(2C-2)\eta^{-1}(\nabla(\widetilde{h}\eta)-\widetilde{h}\nabla\eta)\cdot\nabla\eta-2C\nabla\widetilde{h}\cdot\nabla\eta\\&\le(2C-2)\eta^{-1}\nabla\eta\cdot\nabla(\widetilde{h}\eta)-(2C-2)\widetilde{h}\eta^{-1}|\nabla\eta|^2+C\widetilde{h}^{-1}|\nabla\widetilde{h}|^2\eta+C\widetilde{h}\eta^{-1}|\nabla\eta|^2\\&=(2C-2)\eta^{-1}\nabla\eta\cdot\nabla(\widetilde{h}\eta)+C\widetilde{h}^{-1}|\nabla\widetilde{h}|^2\eta+(8-4C)\widetilde{h}|\nabla r|^2.\end{aligned} \tag{4.5}$$

Here (4.2) has been used. Substituting (4.5) into (4.4) gives

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(\widetilde{h}\eta)\le(2C-2)\eta^{-1}\nabla\eta\cdot\nabla(\widetilde{h}\eta)+(6-4C)\widetilde{h}|\nabla r|^2, \tag{4.6}$$

on the support of φ_+. The weak parabolic maximal principle then implies the result.

Lemma 4.2 *Assume that r satisfies $(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta)r\ge 0$. If $\gamma(M_t)\subset\mathbb{V}$ for arbitrary $t\in[0,T]$ $(T>0)$, and $h:\mathbb{V}\to\mathbb{R}$ is a smooth positive function satisfying* (4.1) *with $C\ge 1$, then for arbitrary $a\ge 0$, the following estimate holds:*

$$\sup_{M_t}\widetilde{h}(1+r)^{-a}\le\sup_{M_0}\widetilde{h}(1+r)^{-a}. \tag{4.7}$$

Proof By $(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta)r\ge 0$,

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(1+r)^{-a}&=-a(1+r)^{-a-1}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)r-a(a+1)(1+r)^{-a-2}|\nabla r|^2\\&\le -a(a+1)(1+r)^{-a-2}|\nabla r|^2.\end{aligned} \tag{4.8}$$

In conjunction with (4.3), we have

$$\begin{aligned}&\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)[\widetilde{h}(1+r)^{-a}]\\ &\le -C\widetilde{h}^{-1}(1+r)^{-a}|\nabla\widetilde{h}|^2-a(a+1)\widetilde{h}(1+r)^{-a-2}|\nabla r|^2-2\nabla\widetilde{h}\cdot\nabla(1+r)^{-a}\\ &= -C\widetilde{h}^{-1}(1+r)^{-a}|\nabla\widetilde{h}|^2-a(a+1)\widetilde{h}(1+r)^{-a-2}|\nabla r|^2+2a\nabla\widetilde{h}\cdot(1+r)^{-a-1}\nabla r.\end{aligned}\tag{4.9}$$

$C\ge 1$ implies $Ca(a+1)\ge a^2$. Then by Young's inequality,

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)[\widetilde{h}(1+r)^{-a}]\le 0.$$

Hence (4.7) follows from the maximal principle for parabolic equations on complete manifolds (see [4]).

Theorem 4.1 *If the initial submanifold is an entire graph over* $\mathbb{R}^n$, *i.e.,* $M_0=\operatorname{graph} f_0$, *where* $f_0=(f_0^1,\cdots,f_0^m)$, $f_0^\alpha=f_0^\alpha(x^1,\cdots,x^n)$, *and*

$$\Delta_{f_0}<2,$$

where

$$\Delta_f(x)=\Big[\det\Big(\delta_{ij}+\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\alpha}{\partial x^j}(x)\Big)\Big]^{\frac12},$$

then the submanifolds under the MCF are still entire graphs over the same hyperplane, i.e., $M_t=\operatorname{graph} f_t$, *and*

$$\Delta_{f_t}<2.$$

Moreover, if $(2-\Delta_{f_0})^{-1}$ *has growth*

$$(2-\Delta_{f_0})^{-1}(x)\le C_0(|x|^2+1)^a,$$

where C_0,a *are both positive constants, then the growth of* $(2-\Delta_{f_t})^{-1}$ *can be controlled by*

$$(2-\Delta_{f_t})^{-1}\le 2C_0(|x|^2+2nt+1)^a.$$

Proof Define $h=v^{\frac32}(2-v)^{-\frac32}$. Then on $\{P: v(P)<2\}$, we have (see [18, inequality (4.6)])

$$\operatorname{Hess}(h)=h'\operatorname{Hess}(v)+h''\mathrm{d}v\otimes\mathrm{d}v\ge 3hg+\frac32 h^{-1}\mathrm{d}h\otimes\mathrm{d}h.\tag{4.10}$$

Define $r(x,t)=|F|^2+2nt$. Then $(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta)r=0$. Hence, the estimate in Lemma 4.1 holds. For arbitrary $x_0\in M_{t_0}$, choose $R>0$, such that $r(x_0,t_0)<R^2$. Then $\varphi_+(x_0,t_0)>0$ and Lemma 4.1 implies

$$\widetilde{h}(x_0,t_0)\le\frac{1}{\varphi_+(x_0,t_0)}\sup_{\overline{M}_{0,R}}\widetilde{h}\varphi_+^2<+\infty.\tag{4.11}$$

Noting that $\widetilde{h}\to+\infty$ when $v\to 2_-$, we have $v(x_0,t_0)<2$ and the first result follows.

For $\mathbf{x}\in\mathbb{R}^n$, it is not difficult to see that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(|\mathbf{x}|^2+2nt)\ge 0.$$

Now, we define $r=|\mathbf{x}|^2+2nt$ and the second assertion easily follows from Lemma 4.2.

Choose

$$h = \sec^2(\sqrt{2}\,\rho)$$

and by the similar argument, we can improve the previous result of the first author [16] as follows.

Theorem 4.2 *If the Gauss image of the initial complete submanifold M_0 is contained in an open geodesic ball of the radius $R_0 \le \frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{m,n}$, then the Gauss images of all the submanifolds under the MCF are also contained in the same geodesic ball. Moreover, if*

$$\Big(\frac{\sqrt{2}}{4}\pi - \rho\Big)^{-1} \le C_0(|F|^2 + 1)^a, \quad \text{on } M_0,$$

where ρ denotes the distance function on $\mathbf{G}_{n,m}$ from the center of the geodesic ball, and C_0, a are both positive constants. Then

$$\Big(\frac{\sqrt{2}}{4}\pi - \rho\Big)^{-1} \le 2C_0(|F|^2 + 2nt + 1)^a$$

for arbitrary $a \ge 0$.

Let $\gamma : M \to \mathbb{R}^4$ be a surface. Let $\pi_1 : \mathbf{G}_{2,2} \to S^2$ be the projection of $\mathbf{G}_{2,2}$ into its first factor, and π_2 be the projection into the second factor. Define $\gamma_i = \pi_i \circ \gamma$. We also obtain that if the partial Gauss image of an initial surface M in $\mathbb{R}^4$ is contained in a hemisphere, then the partial Gauss images of all the surfaces under MCF are in the same hemisphere.

5 Curvature Estimates

Let $h : \mathbb{V} \to \mathbb{R}$ be a smooth function defined on an open subset $\mathbb{V} \subset \mathbf{G}_{n,m}$, and $h \ge 1$. Suppose that $\mathrm{Hess}(h)$ is nonnegative definite on $\mathbb{V}$ and has the estimate

$$\mathrm{Hess}(h) \ge 3hg + \frac{3}{2}h^{-1}\mathrm{d}h \otimes \mathrm{d}h, \tag{5.1}$$

where g is the metric tensor on $\mathbf{G}_{n,m}$. r is a smooth, non-negative function on $\mathbb{R}^{n+m} \times \mathbb{R}$ satisfying

$$\Big|\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)r\Big| \le C(n) \quad \text{and} \quad |\nabla r|^2 \le C(n)r. \tag{5.2}$$

Theorem 5.1 *Let $R > 0$ and $T > 0$ be such that for any $x \in \overline{M}_{t,R}$, where $t \in [0,T]$, we have $\gamma(x) \in \mathbb{V}$. Then for any $t \in [0,T]$ and $\theta \in [0,1)$, we have the estimate*

$$\sup_{x \in \overline{M}_{t,\theta R}} |B|^2 \le C(n)(1-\theta^2)^{-2}(t^{-1} + R^{-2}) \sup_{x \in \overline{M}_{s,R}, s\in[0,t]} \widetilde{h}^2, \tag{5.3}$$

where $\widetilde{h} = h \circ \gamma$.

The proof of Theorem 5.1 will be given later. At first, we will see several applications of it. Let $r = |\mathbf{x}|^2$ for $\mathbf{x} \in \mathbb{R}^n$. Then

$$\Big|\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)r\Big| = \Big|2x^i\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)x^i - 2|\nabla x^i|^2\Big| \le 2n,$$
$$|\nabla r|^2 = |2x^i \nabla x^i|^2 = 4(x^i)^2|\nabla x^i|^2 \le 4r.$$

Hence Theorem 5.1 yields the following corollary.

Corollary 5.1 *Let $R > 0$ and $T > 0$ be such that for any $t \in [0, T]$, $M_t \cap ((B_R \subset \mathbb{R}^n) \times \mathbb{R}^m)$ is a graph over B_R, i.e., $M_t \cap ((B_R \subset \mathbb{R}^n) \times \mathbb{R}^m) = \{(x, f_t(x)) : x \in B_R\}$, and $\Delta_{f_t} < 2$. Then the following estimate holds for arbitrary $t \in [0, T]$ and $\theta \in [0, 1)$:*

$$\sup_{(x,f_t(x))\in K(t,\theta R)} |B|^2 \le C(n)(1-\theta^2)^{-2}(t^{-1}+R^{-2}) \sup_{s\in[0,t]} \sup_{(x,f_s(x))\in K(s,R)} (2-\Delta_{f_s})^{-3},$$

where

$$K(s, R) = \{(x, f_s(x)) : x \in B_R\}.$$

Combining Corollary 5.1 and Theorem 4.1 yields the following corollary.

Corollary 5.2 *If the initial submanifold is an entire graph over $\mathbb{R}^n$, i.e., $M_0 = \operatorname{graph} f_0$, and $\Delta_{f_0} < 2$, $(2-\Delta_{f_0})^{-1} = o(|x|^{2a})$, then we have the estimate*

$$\sup_{(x,f_t(x))\in K(t,\theta R)} |B|^2 \le C(n)(1-\theta^2)^{-2}(t^{-1}+R^{-2})(R^2+2nt+1)^{3a},$$

where $\theta \in [0, 1)$ and the denotation of $K(\,\cdot\,,\,\cdot\,)$ is similar to that in Corollary 5.1.

Similarly, if

$$r = |\mathbf{x}|^2 + 2nt,$$

then it is easy to check that r satisfies (5.2). Applying Theorems 5.1 and 4.2, we have the following corollary.

Corollary 5.3 *Let $R > 0$ and $T > 0$ be such that for any $t \in [0, T]$, if $x \in M_t$ satisfies $|F|^2 + 2nt \le R^2$, then $\gamma(x)$ lies in an open geodesic ball centered at a fixed point P_0 of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$. Then the following estimate holds for arbitrary $t \in [0, T]$ and $\theta \in [0, 1)$:*

$$\sup_{x\in K(t,\theta R)} |B|^2 \le C(n)(1-\theta^2)^{-2}t^{-1} \sup_{0\le s\le t} \sup_{x\in K(s,R)} \Big(\frac{\sqrt{2}}{4}\pi - \rho\Big)^{-3},$$

where

$$K(s, R) = \{x \in M_s : |F|^2 + 2ns \le R^2\}.$$

Corollary 5.4 *If the Gauss image of the initial complete submanifold M_0 is contained in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$, and $(\frac{\sqrt{2}}{4}\pi - \rho)^{-1}$ has growth*

$$\Big(\frac{\sqrt{2}}{4}\pi - \rho\Big)^{-1} = o(|F|^{2a}),$$

then we have the estimate

$$\sup_{x\in K(t,\theta R)} |B|^2 \le C(n)(1-\theta^2)^{-2}t^{-1}(R^2+1)^{3a},$$

where $\theta \in [0, 1)$ and the denotation of $K(\,\cdot\,,\,\cdot\,)$ is similar to that in Corollary 5.3.

Remark 5.1 When $x \in K(t, \theta R)$,

$$2nt \le |F|^2 + 2nt \le \theta^2 R^2 \le R^2,$$

so

$$R^{-2} \le \frac{1}{2n} t^{-1}.$$

Hence in the process of applying Theorem 5.1 to Corollary 5.3, $t^{-1} + R^{-2}$ could be replaced by t^{-1}.

Proof of Theorem 5.1 Let $\varphi = \varphi(\widetilde{h})$ be a smooth nonnegative function of $\widetilde{h}$ to be determined later, and $'$ denotes the derivative with respect to $\widetilde{h}$. Then from (3.3), (3.5) and (5.1), we have

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)|B|^2\varphi &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)|B|^2\cdot\varphi + |B|^2\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\varphi - 2\nabla|B|^2\cdot\nabla\varphi \\ &\le (-2|\nabla|B||^2+3|B|^4)\varphi + |B|^2\Big(\varphi'\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\widetilde{h} - \varphi''|\nabla\widetilde{h}|^2\Big) - 2\nabla|B|^2\cdot\nabla\varphi \\ &\le (-2|\nabla|B||^2+3|B|^4)\varphi - |B|^2\varphi'\Big(3\widetilde{h}|B|^2 + \frac{3}{2}\widetilde{h}^{-1}|\nabla\widetilde{h}|^2\Big) \\ &\quad - |B|^2\varphi''|\nabla\widetilde{h}|^2 - 2\nabla|B|^2\cdot\nabla\varphi. \end{aligned} \tag{5.4}$$

The last term can be estimated by

$$\begin{aligned}-2\nabla|B|^2\cdot\nabla\varphi &= -\nabla|B|^2\cdot\nabla\varphi - \nabla|B|^2\cdot\nabla\varphi \\ &= -\varphi^{-1}(\nabla(|B|^2\varphi) - |B|^2\nabla\varphi)\cdot\nabla\varphi - 2|B|\nabla|B|\cdot\nabla\varphi \\ &\le -\varphi^{-1}\nabla\varphi\cdot\nabla(|B|^2\varphi) + |B|^2\varphi^{-1}|\nabla\varphi|^2 + 2|\nabla|B||^2\varphi + \frac{1}{2}|B|^2\varphi^{-1}|\nabla\varphi|^2 \\ &= -\varphi^{-1}\nabla\varphi\cdot\nabla(|B|^2\varphi) + 2|\nabla|B||^2\varphi + \frac{3}{2}|B|^2\varphi^{-1}|\nabla\varphi|^2. \end{aligned} \tag{5.5}$$

Substituting (5.5) into (5.4) gives

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)|B|^2\varphi &\le -(3\varphi'\widetilde{h} - 3\varphi)|B|^4 - \Big(\frac{3}{2}\varphi'\widetilde{h}^{-1} + \varphi'' - \frac{3}{2}\varphi^{-1}(\varphi')^2\Big)|B|^2|\nabla\widetilde{h}|^2 \\ &\quad - \varphi^{-1}\nabla\varphi\cdot\nabla(|B|^2\varphi). \end{aligned} \tag{5.6}$$

Now we let $\varphi(\widetilde{h}) = \frac{\widetilde{h}}{1-k\widetilde{h}}$, $k \ge 0$ to be chosen. Then

$$3\varphi'\widetilde{h} - 3\varphi = 3k\varphi^2, \tag{5.7}$$

$$\frac{3}{2}\varphi'\widetilde{h}^{-1} + \varphi'' - \frac{3}{2}\varphi^{-1}(\varphi')^2 = \frac{k}{2\widetilde{h}(1-k\widetilde{h})^2}\varphi, \tag{5.8}$$

$$\varphi^{-1}\nabla\varphi = \frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}. \tag{5.9}$$

Substituting these identities into (5.6), we derive for $g = |B|^2\varphi$ the inequality

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)g \le -3kg^2 - \frac{k}{2\widetilde{h}(1-k\widetilde{h})^2}|\nabla\widetilde{h}|^2 g - \frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot\nabla g. \tag{5.10}$$

As in Lemma 4.1, we define $\eta = (R^2 - r)_+^2$. Then on the support of η,

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\eta &= -2(R^2-r)\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)r - 2|\nabla r|^2 \\ &\le 2C(n)R^2 - 2|\nabla r|^2\end{aligned}$$

and

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)g\eta &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)g\cdot\eta + g\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\eta - 2\nabla g\cdot\nabla\eta \\ &\le -3kg^2\eta - \frac{k}{2\widetilde{h}(1-k\widetilde{h})^2}|\nabla\widetilde{h}|^2 g\eta - \frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot\nabla g\cdot\eta \\ &\quad + 2C(n)R^2 g - 2g|\nabla r|^2 - 2\nabla g\cdot\nabla\eta, \end{aligned} \tag{5.11}$$

where

$$-2\nabla g \cdot \nabla \eta = -2\eta^{-1}\nabla\eta \cdot \nabla(g\eta) + 2g\eta^{-1}|\nabla\eta|^2 \\ = -2\eta^{-1}\nabla\eta \cdot \nabla(g\eta) + 8g|\nabla r|^2 \tag{5.12}$$

and

$$-\frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot\nabla g\cdot\eta = -\frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot\nabla(g\eta) + \frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot g\nabla\eta \\ \le -\frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot\nabla(g\eta) + \frac{k}{2\widetilde{h}(1-k\widetilde{h})^2}|\nabla\widetilde{h}|^2 g\eta + \frac{1}{2k\widetilde{h}}g\eta^{-1}|\nabla\eta|^2 \\ = -\frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\cdot\nabla(g\eta) + \frac{k}{2\widetilde{h}(1-k\widetilde{h})^2}|\nabla\widetilde{h}|^2 g\eta + \frac{2}{k\widetilde{h}}g|\nabla r|^2. \tag{5.13}$$

Substituting (5.12) and (5.13) into (5.11) gives

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)g\eta \le -3kg^2\eta - \Big(2\eta^{-1}\nabla\eta + \frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\Big)\cdot\nabla(g\eta) \\ + C(n)\Big[\Big(1+\frac{1}{k\widetilde{h}}\Big)r + R^2\Big]g. \tag{5.14}$$

Furthermore,

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)(tg\eta) \le -3ktg^2\eta - \Big(2\eta^{-1}\nabla\eta + \frac{1}{\widetilde{h}(1-k\widetilde{h})}\nabla\widetilde{h}\Big)\cdot\nabla(tg\eta) \\ + C(n)\Big[\Big(1+\frac{1}{k\widetilde{h}}\Big)r + R^2\Big]tg + g\eta. \tag{5.15}$$

Denote

$$m(T) = \sup_{0\le t\le T}\sup_{\overline{M}_{t,R}} tg\eta = t_0 g(x_0,t_0)\eta(x_0,t_0).$$

Then $t_0 > 0$, $r(x_0,t_0) < R^2$ and hence

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)(tg\eta) \ge 0, \quad \nabla(tg\eta) = 0$$

at (x_0,t_0). (5.15) implies

$$3kt_0g^2\eta \le C(n)\Big[\Big(1+\frac{1}{k\widetilde{h}}\Big)r + R^2\Big]t_0 g + g\eta.$$

Multiplying by $\frac{t_0\eta}{3k}$ yields

$$m(T)^2 \le \frac{C(n)}{3k}\Big(1+\frac{1}{k\widetilde{h}}\Big)R^2t_0^2g\eta + \frac{t_0g\eta^2}{3k} \\ \le \frac{C(n)}{3k}\Big(\Big(1+\frac{1}{k\widetilde{h}}\Big)R^2T + \eta\Big)m(T).$$

By $\eta = (R^2 - r)_+^2 \le R^4$, we arrive at

$$g\eta T \le m(T) \le \frac{C(n)}{3k}\Big(\Big(1+\frac{1}{k\widetilde{h}}\Big)R^2T + R^4\Big)$$

in $\overline{M}_{T,R}$. Now let

$$k = \frac{1}{2} \inf_{\substack{x \in \overline{M}_{t,R} \\ t \in [0,T]}} \widetilde{h}^{-1}. \tag{5.16}$$

Since $\varphi = \frac{\widetilde{h}}{1-k\widetilde{h}} \geq \frac{1}{1-k} \geq 1$ (by $\widetilde{h} \geq 1$) and $\eta \geq (1-\theta^2)^2 R^4$ in $\overline{M}_{T,\theta R}$, we have

$$\sup_{x \in \overline{M}_{T,\theta R}} |B|^2 \leq C(n)(1-\theta^2)^{-2}(T^{-1}+R^{-2}) \sup_{t \in [0,T]} \sup_{x \in \overline{M}_{t,R}} \widetilde{h}^2, \tag{5.17}$$

and finally (5.3) follows from replacing T by t and replacing t by s in (5.17).

Substituting $\varphi = \widetilde{h}$ into (5.6) gives

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)|B|^2 \widetilde{h} \leq -\widetilde{h}^{-1} \nabla \widetilde{h} \cdot \nabla(|B|^2 \widetilde{h}).$$

Using the parabolic maximum principle for complete manifolds in [4], we have

Corollary 5.5 *Let M be a complete n-submanifold in $\mathbb{R}^{m+n}$ with bounded curvature. Then*

$$\sup_{M_t} |B|^2 \widetilde{h} \leq \sup_{M_0} |B|^2 \widetilde{h}.$$

Remark 5.2 When $\mathbb{V}$ is a geodesic ball of radius $\rho_0 < \frac{\sqrt{2}}{4}\pi$, we can choose $h = \sec^2(\sqrt{2}\,\rho)$. So the above estimate is an improvement of [16, Theorem 4.2].

Furthermore, we can give the a priori estimates for $|\nabla^m B|^2$ by induction.

Theorem 5.2 *The denotation and assumption are similar to those in Theorem* 5.1*. Then for arbitrary $m \geq 0$, $\theta \in [0,1)$ and $t \in [0,T]$, we have the estimate*

$$\sup_{x \in \overline{M}_{t,\theta R}} |\nabla^m B|^2 \leq c_m (R^{-2} + t^{-1})^{m+1},$$

where $c_m = c_m\Big(\theta, n, \sup_{\substack{\overline{M}_{s,R} \\ s \in [0,t]}} \widetilde{h}\Big)$.

Proof We proceed by induction on m. The case $m = 0$ has been established by Theorem 5.1. Now we suppose the inequality holds for $0 \leq k \leq m-1$. Denote $\psi(t) = (R^{-2} + t^{-1})^{-1} = \frac{R^2 t}{R^2 + t}$. We will estimate the upper bound of $\psi^{m+1} |\nabla^m B|^2$ on $\overline{M}_{T,\theta R}$ for fixed $\theta \in [0,1)$.

By computing, we have

$$\begin{aligned} \Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)\psi^{m+1}|\nabla^m B|^2 \leq & -2\psi^{m+1}|\nabla^{m+1}B|^2 + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\psi^{m+1}\Big)|\nabla^m B|^2 \\ & + C(m,n)\psi^{m+1} \sum_{\substack{i \leq j \leq k \\ i+j+k=m}} |\nabla^i B||\nabla^j B||\nabla^k B||\nabla^m B|. \end{aligned} \tag{5.18}$$

By the inductive assumption, we get

$$\sup_{x \in \overline{M}_{t, \frac{1+\theta}{2}R}} \psi^{k+1} |\nabla^k B|^2 \leq c_k$$

for every $0 \leq k \leq m-1$ and $t \in [0,T]$, where

$$c_k = c_k\Big(\theta, n, \sup_{\substack{x \in \overline{M}_{t,R} \\ t \in [0,T]}} \widetilde{h}\Big)$$

(note that c_k depends on $\frac{1+\theta}{2}$, which only depends on $\theta \in [0,1)$), which implies $|\nabla^i B| \le c_i^{\frac{1}{2}} \psi^{-\frac{i+1}{2}}$, $|\nabla^j B| \le c_j^{\frac{1}{2}} \psi^{-\frac{j+1}{2}}$. Moreover,

$$\begin{aligned} \psi^{m+1} \sum_{\substack{i \le j \le k \\ i+j+k=m}} |\nabla^i B||\nabla^j B||\nabla^k B||\nabla^m B| &\le C \sum_{\substack{i \le j \le k \\ i+j+k=m}} \psi^{\frac{k+m}{2}} |\nabla^k B||\nabla^m B| \\ &\le C \sum_{k \le m} \psi^k |\nabla^k B|^2. \end{aligned} \tag{5.19}$$

On the other hand, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \psi^{m+1} = (m+1)\psi^m \frac{R^4}{(R^2+t)^2} \le (m+1)\psi^m. \tag{5.20}$$

Substituting (5.19) and (5.20) into (5.18) gives

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) \psi^{m+1} |\nabla^m B|^2 \le -2\psi^{m+1} |\nabla^{m+1} B|^2 + C \sum_{k \le m} \psi^k |\nabla^k B|^2 \tag{5.21}$$

on $\overline{M}_{t, \frac{1+\theta}{2} R}$ for arbitrary $t \in [0,T]$, where $C = C\Big(\theta, n, \sup_{\substack{x \in \overline{M}_{t,R} \\ t \in [0,T]}} \widetilde{h}\Big)$. Now we define $f = \psi^{m+1} |\nabla^m B|^2 (\Lambda + \psi^m |\nabla^{m-1} B|^2)$, where $\Lambda > 0$ to be chosen later. By computation, we have

$$\begin{aligned} \Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) f \le& -2\psi^{m+1} |\nabla^{m+1} B|^2 (\Lambda + \psi^m |\nabla^{m-1} B|^2) + C \sum_{k \le m} \psi^k |\nabla^k B|^2 (\Lambda + \psi^m |\nabla^{m-1} B|^2) \\ & - 2\psi^{2m+1} |\nabla^m B|^4 + C \sum_{k \le m-1} \psi^k |\nabla^k B|^2 \psi^{m+1} |\nabla^m B|^2 \\ & - 2\psi^{2m+1} \nabla |\nabla^m B|^2 \cdot \nabla |\nabla^{m-1} B|^2, \end{aligned} \tag{5.22}$$

where the last term can be estimated by

$$\begin{aligned} & - 2\psi^{2m+1} \nabla |\nabla^m B|^2 \cdot \nabla |\nabla^{m-1} B|^2 \\ =& -8\psi^{2m+1} |\nabla^m B| \nabla |\nabla^m B| \cdot |\nabla^{m-1} B| \nabla |\nabla^{m-1} B| \\ \le& \, 2\psi^{m+1} |\nabla^{m+1} B|^2 (\Lambda + \psi^m |\nabla^{m-1} B|^2) + 8\psi^{2m+1} |\nabla^m B|^4 \frac{\psi^m |\nabla^{m-1} B|^2}{\Lambda + \psi^m |\nabla^{m-1} B|^2} \\ \le& \, 2\psi^{m+1} |\nabla^{m+1} B|^2 (\Lambda + \psi^m |\nabla^{m-1} B|^2) + \frac{8c_{m-1}}{\Lambda + c_{m-1}} \psi^{2m+1} |\nabla^m B|^4. \end{aligned} \tag{5.23}$$

Hence we derive

$$\begin{aligned} \Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) f \le& -\Big(2 - \frac{8c_{m-1}}{\Lambda + c_{m-1}}\Big) \psi^{-1} (\psi^{m+1} |\nabla^m B|^2)^2 \\ & + C\psi^{-1} \Big(\sum_{k \le m} \psi^{k+1} |\nabla^k B|^2 (\Lambda + \psi^m |\nabla^{m-1} B|^2) \\ & + \sum_{k \le m-1} \psi^{k+1} |\nabla^k B|^2 \psi^{m+1} |\nabla^m B|^2 \Big). \end{aligned} \tag{5.24}$$

Now let $\Lambda = 7c_{m-1} + 1$. Then

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) f \le -\psi^{-1} (\Lambda + \psi^m |\nabla^{m-1} B|^2)^{-2} f^2 + C\psi^{-1}(1+f).$$

By Young's inequality, we have

$$\begin{aligned} Cf &\le \frac{1}{2}(\Lambda+\psi^m|\nabla^{m-1}B|^2)^{-2}f^2+\frac{1}{2}C^2(\Lambda+\psi^m|\nabla^{m-1}B|^2)^2 \\ &\le \frac{1}{2}(\Lambda+\psi^m|\nabla^{m-1}B|^2)^{-2}f^2+\frac{1}{2}C^2(8c_{m-1}+1)^2. \end{aligned}$$

Hence we have

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)f \le -\psi^{-1}(\delta f^2 - C), \tag{5.25}$$

where

$$\delta = \frac{(C(8c_{m-1}+1)^2-1)^2}{2(8c_{m-1}+1)^2} > 0$$

and C is a positive constant depending on n, m and $\sup\limits_{\substack{\overline{M}_{t,R} \\ t\in[0,T]}} \widetilde{h}$.

Now let $\varphi = (\frac{1+\theta}{2}R)^2 - r$ and $\eta = (\varphi_+)^2$. Then η is a nonnegative function which vanishes outside $\overline{M}_{t,\frac{1+\theta}{2}R}$. Similarly to (5.11), we can derive

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)f\eta \le \psi^{-1}(\delta f^2 - C)\eta + C(n)R^2 f - 2\eta^{-1}\nabla\eta\cdot\nabla(f\eta) \tag{5.26}$$

on $\overline{M}_{t,\frac{1+\theta}{2}R}$. Denote $m(T) = \max\limits_{0\le t\le T}\max\limits_{x\in\overline{M}_{t,\frac{1+\theta}{2}R}} f\eta = f\eta(x_0,t_0)$. We have

$$f^2\eta \le \frac{1}{\delta}(C\eta + C(n)R^2 f\psi).$$

Multiplying by η and using $\eta\le R^4$, $\psi\le R^2$, we have

$$\begin{aligned} f^2\eta^2 &\le \frac{1}{\delta}(C\eta^2 + C(n)R^2 f\eta\psi) \le \frac{1}{\delta}(CR^8 + C(n)R^4 f\eta) \\ &\le \frac{1}{\delta}\Big(CR^8 + \frac{\delta}{2}f^2\eta^2 + \frac{C(n)^2R^8}{2\delta}\Big), \end{aligned}$$

i.e., $m(T)^2 = f^2\eta^2 \le CR^8$, and

$$\sup_{0\le t\le T}\sup_{x\in\overline{M}_{t,\frac{1+\theta}{2}R}} f\eta \le CR^4,$$

where $C = C\Big(\theta, n, m, \sup\limits_{\substack{\overline{M}_{t,R} \\ t\in[0,T]}} \widetilde{h}\Big)$.

Finally, since $\eta = ((\frac{1+\theta}{2}R)^2 - (\theta R)^2)^2 = \frac{1+2\theta-3\theta^2}{4}R^4$ on $\overline{M}_{T,R}$ and $\Lambda + \psi^m|\nabla^{m-1}B| \ge 7c_{m-1}+1$, we have

$$\sup_{x\in\overline{M}_{T,\theta R}} \psi^{m+1}|\nabla^m B| \le c_m\Big(\theta, n, \sup_{\substack{x\in\overline{M}_{t,R} \\ t\in[0,T]}} \widetilde{h}\Big). \tag{5.27}$$

Then the conclusion follows from replacing T by t and replacing t by s.

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