Chin. Ann. Math.
31B(3), 2010, 373–384
DOI: 10.1007/s11401-009-0009-5

Chinese Annals of Mathematics, Series B

Montel-Type Theorems in Several Complex Variables with Continuously Moving Targets**

Zhenhan TU* Shasha ZHANG*

Abstract The authors introduce a new idea related to Montel-type theorems in higher dimension and prove some Montel-type criteria for normal families of holomorphic mappings and normal holomorphic mappings of several complex variables into $P^N(\mathbb{C})$ for continuously moving hyperplanes in pointwise general position. The main results are also true for continuously moving hypersurfaces in pointwise general position. Examples are given to show the sharpness of the results.

Keywords Holomorphic mappings, Normal families, Picard-type theorems, Value distribution theory
2000 MR Subject Classification 32A19, 32H25, 32H30

1 Introduction

The study of value distribution theory may be considered to have its origin in the famous Picard's theorem.

Theorem 1.1 (see [11]) *Suppose that f is a meromorphic function on the complex plane. If f omits three mutually distinct points on the Riemann sphere, then f is a constant.*

A central result in the theory of normal families is the following Montel's theorem related to Picard's theorem.

Theorem 1.2 (see [11]) *Let F be a family of meromorphic functions on a domain D of the complex plane. Suppose that there exist three mutually distinct points w_1, w_2, w_3 on the Riemann sphere such that each f in F omits w_i $(i = 1, 2, 3)$. Then F is a normal family on D.*

In the case of higher dimension, Bloch [4], Fujimoto [7] and Green [8] established Picard-type theorem for holomorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$, the complex N-dimensional projective space. Nochka [12] extended the result in [4, 7, 8] to the case of finite intersection multiplicity. In [15], the present first author gave some normality criteria for families of holomorphic mappings of several complex variables into $P^N(\mathbb{C})$ for fixed hyperplanes related to Nochka's results. Motivated by the accomplishment of the second main theorem of value distribution theory in higher dimension for moving hyperplanes (see, e.g., [13]), Wang [18] extended Picard-type the-

Manuscript received January 13, 2009. Revised May 1, 2009. Published online February 2, 2010.
*Department of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail: Tuzhenhan@yahoo.com amushasha@163.com
**Project supported by the National Natural Science Foundation of China (No. 10971156).

orem of holomorphic mappings of $\mathbb{C}$ into $P^N(\mathbb{C})$ to the case of moving hyperplanes in pointwise general position. By starting from Fujimoto-Green's and Nochka's Picard-type theorems and using the heuristic principle obtained by Aladro and Krantz [2], Tu and Li [17] obtained some normality criteria for families of holomorphic mappings of several complex variables into $P^N(\mathbb{C})$ for moving hyperplanes in pointwise general position. Bargmann, Bonk, Hinkkanen and Martin [3] introduced a new idea related to Montel's theorem and proved a normality critrion for families of meromorphic functions omitting three continuous functions. It seems clear that the idea in these researches suggests some insights into the normal holomorphic mappings and normal family of holomorphic mappings of several complex variables into $P^N(\mathbb{C})$ for continuously moving targets in pointwise general position. This line of thought will be pursued in this note.

2 Main Results

For the general reference of this paper, see [1, 3, 15, 17].

Let $P^N(\mathbb{C})$ be the complex projective space of dimension N, and $\rho : \mathbb{C}^{N+1}\backslash\{0\} \to P^N(\mathbb{C})$ be the standard projective mapping. Let h be a holomorphic mapping from a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$. For any $a \in D$, then h has a reduced representation

$$\widetilde{h}(z) = (h_0(z), \cdots, h_N(z))$$

on some neighborhood U of a in D, i.e., $h_0(z), \cdots, h_N(z)$ are holomorphic functions on U without common zeroes such that $h(z) = \rho(\widetilde{h}(z))$ on U. Let F be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$. F is said to be a normal family on D if any sequence in F contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into $P^N(\mathbb{C})$, where a subsequence $\{f^{(p)}\} \subset F$ is said to converge uniformly on compact subsets of D to a holomorphic mapping f of D into $P^N(\mathbb{C})$ if and only if, for any $a \in D$, each $f^{(p)}(z)$ has a reduced representation

$$\widetilde{f}^{(p)}(z) = (f_0^{(p)}(z), \cdots, f_N^{(p)}(z))$$

on some fixed neighborhood U of a in D such that $\{f_i^{(p)}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function $f_i(z)$ $(i = 0, \cdots, N)$ on U with the property that

$$\widetilde{f}(z) := (f_0(z), \cdots, f_N(z))$$

is a reduced representation of $f(z)$ on U.

Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain and M be a complete complex Hermitian manifold with metric $\mathrm{d}s_M^2$. A holomorphic mapping $f(z)$ from Ω into M is said to be a normal holomorphic mapping from Ω into M if and only if there exists a positive constant c such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,

$$|\mathrm{d}s_M^2(f(z), \mathrm{d}f(z)(\xi))| \le cK_\Omega(z, \xi),$$

where $\mathrm{d}f(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by f and K_Ω denotes the Kobayashi metric for Ω. For the detailed discussion of normal holomorphic mapping, see [1].

Let E be a subset of $\mathbb{C}^n$. For $z \in E$, define the linear form

$$L(z)(Z_0, \cdots, Z_N) := \sum_{j=0}^{N} a_j(z) Z_j,$$

where $a_j(z)$ $(0 \le j \le N)$ are continuous functions on E without common zeroes. Then

$$H(z) := \rho(\{(Z_0, Z_1, \cdots, Z_N) \in \mathbb{C}^{N+1}\backslash\{0\} : L(z)(Z_0, Z_1, \cdots, Z_N) = 0\})$$

is called a continuous moving hyperplane in $P^N(\mathbb{C})$ corresponding to the linear form $L(z)$ $(z \in E)$. Continuous moving hyperplanes $H_j(z)$ (where $j = 1, \cdots, q$; $q \ge N+1$) are said to be located in $P^N(\mathbb{C})$ in general position on E if there exists some point $z_0 \in E$ such that the hyperplanes $H_j(z_0)$ $(j = 1, \cdots, q)$ are located in general position. Continuous moving hyperplanes $H_j(z)$ (where $j = 1, \cdots, q$; $q \ge N+1$) are said to be located in $P^N(\mathbb{C})$ in pointwise general position on E if for any fixed point $z_0 \in E$, the hyperplanes $H_j(z_0)$ $(j = 1, \cdots, q)$ are located in general position. In the special case, when E is a domain D of $\mathbb{C}^n$ and $a_j(z)$ $(0 \le j \le N)$ are holomorphic in D, $H(z)$ is said to be a moving hyperplane in $P^N(\mathbb{C})$.

Let f be a holomorphic mapping from a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$, and $H(z)$ be a continuous moving hyperplane in $P^N(\mathbb{C})$ defined by the linear form $L(z)(Z_0, Z_1, \cdots, Z_N)$ $(z \in D)$. For any $a \in D$, let f have a reduced representation

$$\widetilde{f}(z) = (f_0(z), \cdots, f_N(z))$$

on a neighborhood U of a. We consider the continuous function

$$F(z) := L(z)(f_0(z), \cdots, f_N(z)), \quad z \in U.$$

If $F(a) \ne 0$, then $f(a) \in P^N(\mathbb{C})\backslash H(a)$. When $f(z) \in P^N(\mathbb{C})\backslash H(z)$ for all $z \in D$, we say that the holomorphic mapping f omits the continuous moving hyperplane $H(z)$ on D. Since $F(z)$ is only continuous in U, we cannot define the multiplicity for a zero of $F(z)$ in U. Therefore, we cannot define the finite intersection multiplicity for f with the continuous moving hyperplane $H(z)$ on D.

Bloch [4], Fujimoto [7] and Green [8, 9] extended the Picard's theorem and Montel's theorem to the case of higher dimension and got the following results.

Theorem 2.1 (see [4, 7, 8]) *Suppose that f is a holomorphic mapping from $\mathbb{C}^n$ into $P^N(\mathbb{C})$. If f omits $2N+1$ hyperplanes in $P^N(\mathbb{C})$ located in general position, then f is constant.*

Theorem 2.2 (see [4, 9]) *Let $\{H_j\}_{j=1}^{2N+1}$ be a set of hyperplanes in $P^N(\mathbb{C})$ located in general position. Then $P^N(\mathbb{C})\backslash \bigcup_{j=1}^{2N+1} H_j$ is complete hyperbolic and hyperbolically imbedded to $P^N(\mathbb{C})$.*

Wang [18] obtained a weak Picard-type theorem of holomorphic mappings of $\mathbb{C}$ into $P^N(\mathbb{C})$ for moving hyperplanes in pointwise general position. For example (see, e.g., [18, p. 40]), $\widetilde{f}(z_1, z_2) := (1, \mathrm{e}^z)$ (from $\mathbb{C}$ to $\mathbb{C}^2$) omits three moving hyperplanes $Z_0 = 0,\ Z_1 = 0,\ \mathrm{e}^z Z_0 + Z_1 = 0$ $(z \in \mathbb{C},\ (Z_0, Z_1) \in \mathbb{C}^2)$, where these three moving hyperplanes are located in pointwise general position on $\mathbb{C}$. However, $f(z) := \rho(\widetilde{f}(z))$ (from $\mathbb{C}$ to $P^1(\mathbb{C})$) is not a constant. Although a holomorphic mapping from $\mathbb{C}$ into $P^N(\mathbb{C})$ omitting $2N+1$ moving hyperplanes in pointwise general position may not be a constant, Tu and Li [17] extended Theorem 2.2 to the case of moving hyperplanes in pointwise general position and proved the following results.

Theorem 2.3 (see [17]) *Let F be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$, and $H_1(z), \cdots, H_{2N+1}(z)$ $(z \in D)$ be $2N+1$ moving hyperplanes in $P^N(\mathbb{C})$ located in pointwise general position on D. If each f in F omits $H_j(z)$ $(j = 1, \cdots, 2N+1)$ on D, then F is a normal family on D.*

Bargmann, Bonk, Hinkkanen and Martin [3] proved a normality criterion for families of meromorphic functions omitting three continuous functions as follows.

Theorem 2.4 (see [3]) *Let F be a family of meromorphic functions on a domain D of the complex plane. Suppose that there exist three continuous functions $w_1(z), w_2(z), w_3(z)$ on D with values in the Riemann sphere $P^1(\mathbb{C})$ such that $w_1(z), w_2(z), w_3(z)$ are three mutually distinct points for any fixed $z \in D$. If each f in F omits $w_i(z)$ $(i = 1, 2, 3)$ on D, then F is a normal family on D.*

Inspired by these developments, we prove Montel-type criteria for normal families of holomorphic mappings and normal holomorphic mappings of several complex variables into $P^N(\mathbb{C})$ for continuously moving hyperplanes in pointwise general position as follows.

Theorem 2.5 *Let F be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$, and $H_1(z), \cdots, H_{2N+1}(z)$ $(z \in D)$ be $2N+1$ continuously moving hyperplanes in $P^N(\mathbb{C})$ located in pointwise general position on D. If each f in F omits $H_j(z)$ $(j = 1, \cdots, 2N+1)$ on D, then F is a normal family on D.*

Theorem 2.6 *Let f be a holomorphic mapping from a bounded domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$, and $H_1(z), \cdots, H_{2N+1}(z)$ $(z \in \overline{D})$ be $2N+1$ continuously moving hyperplanes in $P^N(\mathbb{C})$ located in pointwise general position on $\overline{D}$. If f omits $H_j(z)$ $(j = 1, \cdots, 2N+1)$ on D, then f is a normal holomorphic mapping from D into $P^N(\mathbb{C})$.*

As an application of Theorem 2.5, we give a necessary and sufficient condition for a family to be normal as follows.

Corollary 2.1 *Let F be a family of holomorphic mappings of the ball*

$$B(R) := \{z \in \mathbb{C}^n : \|z\| < R\}$$

into $P^1(\mathbb{C})$ $(0 < R \leq +\infty)$. Then F is a normal family on $B(R)$ if and only if for each sequence $\{f_i(z)\}_{i=1}^{\infty}$ in F, there exist three continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ $(z \in B(R))$ in $P^1(\mathbb{C})$ located in pointwise general position on $B(R)$ such that for each closed ball $\overline{B(r)} := \{z \in \mathbb{C}^n : \|z\| \leq r\}$ $(0 < r < R)$, infinitely many mappings in $\{f_i(z)\}_{i=1}^{\infty}$ omit $H_j(z)$ $(j = 1, 2, 3)$ on $\overline{B(r)}$.

Here we give some examples to complement our theory in this paper.

Example 2.1 Let $\{f_k(z)\}_{k=1}^{\infty}$ be a family of holomorphic mappings of the unit disc D into $P^1(\mathbb{C})$, where $f_k(z)$ has a reduced representation $\widetilde{f_k}(z) = (kz, 1)$ on D $(k = 1, 2, \cdots)$. Then $\{f_k(z)\}_{k=1}^{\infty}$ is not normal on D. Let the continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ in $P^1(\mathbb{C})$ be defined by the linear equations $Z_1 = 0,\ xZ_0 + Z_1 = 0,\ xZ_0 + 2Z_1 = 0$ $(z \in D)$ respectively. Then $H_1(z), H_2(z), H_3(z)$ are three continuously moving hyperplanes in $P^1(\mathbb{C})$ located in general position (but not in pointwise general position) on D such that each f_k omits $H_j(z)$ $(j = 1, 2, 3)$ on D. Thus Theorem 2.5 fails for $2N + 1$ continuously moving hyperplanes located "only" in general position on D.

Example 2.2 Let $F := \{f_k(z)\}_{k=1}^{\infty}$ be a family of holomorphic mappings of $\mathbb{C}$ into $P^1(\mathbb{C})$, where $f_k(z)$ has a reduced representation $\widetilde{f_k}(z)$ on $\mathbb{C}$ as follows:

$$\widetilde{f_1}(z) = (1, 0), \quad \widetilde{f_2}(z) = (1, 1), \quad \widetilde{f_3}(z) = (0, 1) \quad \text{and} \quad \widetilde{f_k}(z) = \Big(1, \frac{z}{k}\Big), \quad k \geq 4.$$

Let the continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ in $P^1(\mathbb{C})$ be defined by the linear equations $(|z| + 1)Z_0 + Z_1 = 0,\ (|z| + 2)Z_0 + Z_1 = 0,\ (|z| + 3)Z_0 + Z_1 = 0$ $(z \in \mathbb{C})$ respectively. Then $H_1(z), H_2(z), H_3(z)$ are three continuously moving hyperplanes in $P^1(\mathbb{C})$ located in pointwise general position on $\mathbb{C}$ such that each f_k omits $H_j(z)$ $(j = 1, 2, 3)$ on $\mathbb{C}$.

Let $H(z)$ be a moving hyperplane in $P^1(\mathbb{C})$ defined by the linear equation $a_0(z)Z_0 + a_1(z)Z_1 = 0$, where $a_0(z)$ and $a_1(z)$ are holomorphic functions on $\mathbb{C}$ without common zeroes. If f_1, f_2, f_3 omit $H(z)$ on $\mathbb{C}$, then by Picard theorem $H(z)$ can be defined by the linear equation $Z_0 + c_0 Z_1 = 0$ (where c_0 is a nonzero complex number). Thus f_4 cannot omit $H(z)$ on $\mathbb{C}$. Therefore, F can omit three continuously moving hyperplanes in $P^1(\mathbb{C})$ located in pointwise general position on $\mathbb{C}$ but cannot omit any moving hyperplane in $P^1(\mathbb{C})$ on $\mathbb{C}$.

Example 2.3 Let D be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ with the Bergman metric and let $P^1(\mathbb{C})$ carry the Fubini-Study metric. Let f be a holomorphic mapping of D into $P^1(\mathbb{C})$, where $f(z)$ has a reduced representation $\widetilde{f}(z) := (1, \mathrm{e}^{\frac{1}{(z-1)^2}})$ on D. Let the continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ in $P^1(\mathbb{C})$ be defined by the linear equations

$$Z_0 = 0, \quad Z_1 = 0, \quad Z_0 - \mathrm{e}^{-\frac{2}{|z-1|^2}} Z_1 = 0, \quad z \in \overline{D}$$

respectively. Then $H_1(z), H_2(z), H_3(z)$ are three continuously moving hyperplanes in $P^1(\mathbb{C})$ located in pointwise general position on D and in general position on $\overline{D}$ such that f omits $H_k(z)$ $(k = 1, 2, 3)$ on D.

But f is not normal on D. In fact, by the definition, f is normal on D if and only if

$$\frac{|h'(z)|}{1 + |h(z)|^2} \leq C_0 \frac{1}{1 - |z|^2}, \quad z \in D$$

for some finite constant C_0, where $h(z) := \mathrm{e}^{\frac{1}{(z-1)^2}}$. However, take

$$z(t) := 1 - \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t\sqrt{-1}, \quad 0 < t < \frac{\sqrt{2}}{2}.$$

Then $z(t) \in D$ $\big(0 < t < \frac{\sqrt{2}}{2}\big)$. Since

$$(1 - |z(t)|^2)\frac{|h'(z(t))|}{1 + |h(z(t))|^2} = \frac{\sqrt{2} - t}{t^2} \to +\infty, \quad t \to 0^+,$$

we have

$$\sup\left\{(1-|z|^2)\frac{|h'(z)|}{1+|h(z)|^2} : z\in D\right\}=+\infty.$$

So f is not normal on D. Therefore, Theorem 2.6 does not hold for $2N+1$ continuously moving hyperplanes located in pointwise general position "only" on D or "only" in general position on $\overline{D}$.

Now we give an outline of our proofs of Theorems 2.5 and 2.6. The main results in this paper are evolved by Theorems 2.3 and 2.4. Generally speaking, the proof on proving normality is achieved by using the so-called Zaclman's lemma (the reparametrization lemma), Hurwitz's theorem, and Picard-type theorem as follows: if the family is not normal, then one can use Zaclman's lemma to produce a "nonconstant" holomorphic map on $\mathbb{C}$ which also satisfies a property by Hurwitz's theorem, and then it leads a contradiction to the Picard-type theorem. Since a holomorphic mapping from $\mathbb{C}$ into $P^N(\mathbb{C})$ omitting $2N+1$ moving hyperplanes in pointwise general position may not be a constant, there are some gaps for one to use the idea in proving the normality with moving targets. In the proof of Theorem 2.3 (with moving hyperplanes), the key idea (see [17]) is as follows: when one applies Zaclman's lemma to produce a nonconstant holomorphic map h from $\mathbb{C}$ into $P^N(\mathbb{C})$, h can actually omit a set of fixed hyperplanes by Hurwitz's theorem. Hence, the Picard-type theorems with fixed hyperplanes could be applied to get the normality with moving hyperplanes. But in the case of continuously moving hyperplanes in Theorems 2.5 and 2.6, some related sequences of functions are continuous but not holomorphic, and then we need some new approach to overcome the difficulty. Here we will employ an argument in [3] and introduce the topological degree of a continuous function to finish our proofs of Theorem 2.5.

Theorem 2.6 is only a counterpart of Theorem 2.5 for normal holomorphic mapping. By [1, Proposition 1.14], a holomorphic mapping f from a bounded domain Ω in $\mathbb{C}^n$ into $P^N(\mathbb{C})$ is a normal holomorphic mapping if and only if for every sequence of holomorphic mappings $\varphi_j(z)$ from the unit disc U in $\mathbb{C}$ into Ω, the sequence $\{f\circ\varphi_j(z)\}_{j=1}^{\infty}$ from U into $P^N(\mathbb{C})$ is a normal family on U. So in many cases, a criterion for normal family will trivially imply a counterpart for normal mapping. However, when f omits $2N+1$ continuously moving hyperplanes, the sequence $\{f\circ\varphi_j(z)\}_{j=1}^{\infty}$ may not omit these continuously moving hyperplanes. Therefore, Theorem 2.5 does not tell whether $\{f\circ\varphi_j(z)\}_{j=1}^{\infty}$ is a normal family. Although Theorem 2.6 cannot come directly from Theorem 2.5, the argument in proving Theorem 2.5 will be modified to give the proof of Theorem 2.6 in this paper.

3 Proofs of the Main Results

To prove our results, we need some preparations.

Lemma 3.1 *Let F be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$. The family F is not normal on D if and only if there exist a compact set $K\subset D$ and sequences $\{f_i\}\subset F$, $\{p_i\}\subset K$, $\{r_i\}$ with $r_i>0$ and $r_i\to 0^+$ and $\{u_i\}\subset\mathbb{C}^n$ Euclidean unit vectors such* *that*

$$g_i(\xi) := f_i(p_i + r_i u_i \xi),$$

where $\xi \in \mathbb{C}$ *satisfies* $p_i + r_i u_i \xi \in D$, *converges uniformly on compact subsets of* $\mathbb{C}$ *to a nonconstant holomorphic mapping* g *of* $\mathbb{C}$ *into* $P^N(\mathbb{C})$.

For the proof of Lemma 3.1, see [2, Theorem 3.1].

Lemma 3.2 *Let* f *be a holomorphic mapping of a hyperbolic domain* D *in* $\mathbb{C}^n$ *into* $P^N(\mathbb{C})$. *Then* f *is not normal on* D *if and only if there exist* $\{p_i\} \subset D$, $\{r_i\}$ *with* $r_i > 0$ *and* $r_i \to 0^+$ *and* $\{u_i\} \subset \mathbb{C}^n$ *Euclidean unit vectors such that*

$$g_i(\xi) := f_i(p_i + r_i u_i \xi), \quad \xi \in \mathbb{C},$$

where $\lim_{i\to\infty} \frac{r_i}{d(p_i, \mathbb{C}^n \setminus D)} = 0$ (*where* $d(p,q)$ *is the Euclidean distance between* p *and* q *in* $\mathbb{C}^n$), *converges uniformly on compact subsets of* $\mathbb{C}$ *to a nonconstant holomorphic mapping* g *of* $\mathbb{C}$ *into* $P^N(\mathbb{C})$.

Remark 3.1 Lemma 3.2 is only a version of Lemma 3.1 for normal holomorphic mapping. The proof of Lemma 3.2 is a slight modification of that of Theorem 3.1 in [2]. Here we correct a mistake in [2, Corollary 3.1] (see [10, Theorem 6.4]).

Moreover, we need some facts about the topological degree of a continuous map. See [5, Chapter 1] for a detailed discussion. In the special case which we need, [5, Theorem 3.1, p. 16] reduces to the following (see [3, Theorem 4.4]).

Lemma 3.3 *Let* M *be the set of all triples* (f, U, y), *where* U *is a bounded open subset of* $\mathbb{C}$, $f : \overline{U} \to \mathbb{C}$ *is a continuous function, and* $y \in \mathbb{C} \setminus f(\partial U)$. *Then there exists exactly one integer-valued function* $d : M \to \mathrm{Z}$ *which satisfies the following conditions:*

(1) $d(f, U, y) = 1$ *for* $f(z) \equiv z$ *and* $y \in U$;

(2) $d(f, U, y) = d(f, U_1, y) + d(f, U_2, y)$ *whenever* U_1 *and* U_2 *are disjoint open subsets of* U *such that* $y \notin f(\overline{U} \setminus (U_1 \cup U_2))$;

(3) $d(f(t, \cdot), U, y(t))$ *is independent of* t *whenever* $f : [0,1] \times \overline{U} \to \mathbb{C}$ *and* $y : [0,1] \to \mathbb{C}$ *are continuous such that* $y(t) \notin f(t, \partial U)$ *for every* $t \in [0,1]$.

The function $d : M \to \mathrm{Z}$ *is called the local degree or topological degree, and furthermore satisfies*

(4) $d(f, U, y) \neq 0$ *implies* $\overline{U} \cap f^{-1}(y) \neq \emptyset$;

(5) $d(g, U, y) = d(f, U, y)$ *whenever* $g : \overline{U} \to \mathbb{C}$ *is continuous such that* $|g(z) - f(z)| <$ $\mathrm{dist}(y, f(\partial U))$ *for each* $z \in \overline{U}$.

Proof of Theorem 2.5 If F is not a normal family on D, then, by Lemma 3.1, there exist a compact set $K \subset D$ and sequences $\{f_k\} \subset F$, $\{p_k\} \subset K$, $\{r_k\}$ with $r_k > 0$ and $r_k \to 0^+$ and $\{u_k\} \subset \mathbb{C}^n$ Euclidean unit vectors such that

$$g_k(\xi) := f_k(p_k + r_k u_k \xi),$$

where $\xi \in \mathbb{C}$ satisfies $p_k + r_k u_k \xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a "nonconstant" holomorphic mapping g of $\mathbb{C}$ into $P^N(\mathbb{C})$. We will prove that g must be a constant holomorphic mapping of $\mathbb{C}$ into $P^N(\mathbb{C})$, and then get a contradiction.

Let the linear form

$$L_i(z)(Z) := a_{i0}(z)Z_0 + a_{i1}(z)Z_1 + \cdots + a_{iN}(z)Z_N$$

define the continuously moving hyperplane $H_i(z)$ $(z \in D)$ in $P^N(\mathbb{C})$, where $Z = (Z_0, \cdots, Z_N) \in \mathbb{C}^{N+1}$, $a_{ij}(z)$ $(j = 0, 1, \cdots, N)$ are continuous functions on D without common zeroes, and $i = 1, \cdots, 2N + 1$. Since K is a compact subset of D, without loss of generality, we assume that $\{p_k\}$ $(\subset K)$ converges to p_0 $(\in K)$.

Let g have a reduced representation

$$\widetilde{g}(\xi) = (g_0(\xi), g_1(\xi), \cdots, g_N(\xi))$$

on $\mathbb{C}$. We consider the entire function

$$G_i(\xi) := a_{i0}(p_0)g_0(\xi) + a_{i1}(p_0)g_1(\xi) + \cdots + a_{iN}(p_0)g_N(\xi)$$

on $\mathbb{C}$ for a fixed i $(i = 1, 2, \cdots, 2N + 1)$.

(i) If $G_i(\xi) \equiv 0$ on $\mathbb{C}$, then g intersects $H_i(p_0)$ on $\mathbb{C}$ with multiplicity ∞.

(ii) If $G_i(\xi) \not\equiv 0$ on $\mathbb{C}$, we will prove $G_i(\xi) \neq 0$ everywhere on $\mathbb{C}$.

In fact, suppose $G_i(\xi) \not\equiv 0$ on $\mathbb{C}$ and $G_i(\xi_0) = 0$ for some $\xi_0 \in \mathbb{C}$. Choose $r > 0$ such that ξ_0 is the only zero point of $G_i(\xi)$ on $E := \{\xi \in \mathbb{C}; |\xi - \xi_0| \leq r\}$. Let k_0 be a positive integer such that $g_k(\xi) := f_k(p_k + r_k u_k \xi)$ $(k \geq k_0)$ are well-defined on E. Thus, by the assumption, $\{g_k(\xi)\}_{k=k_0}^{\infty}$ converges uniformly to g on E.

Therefore, by the definition of convergence, ξ_0 has a compact neighborhood (again denoted by E) such that $g_k(\xi) := f_k(p_k + r_k u_k \xi)$ $(k \geq k_0)$ has a reduced representation

$$\widetilde{g}_k(\xi) = (g_{k0}(\xi), g_{k1}(\xi), \cdots, g_{kN}(\xi))$$

on E and $\widetilde{g}_k(\xi)$ converges to $\widetilde{g}(\xi)$ uniformly on E as $k \to \infty$. Therefore, the continuous function

$$G_{ik}(\xi) := a_{i0}(p_k + r_k u_k \xi)g_{k0}(\xi) + \cdots + a_{iN}(p_k + r_k u_k \xi)g_{kN}(\xi)$$

converges to the holomorphic function $G_i(\xi)$ uniformly on E as $k \to \infty$.

Since $G_i(\xi)$ is holomorphic on E, the topological degree $d(G_i, E, 0)$ is the winding number of $G_i(\partial E)$ about 0 (see [5, p. 30]). Hence the argument principle implies that $d(G_i, E, 0)$ is the number of times G_i assuming the value 0 on E. Therefore, we have $d(G_i, E, 0) \neq 0$. By the conclusion (5) of Lemma 3.3, we have

$$d(G_{ik}, E, 0) = d(G_i, E, 0) (\neq 0)$$

for sufficiently large k. By the conclusion (4) of Lemma 3.3, we get $G_{ik}(\xi)$ must take on the value 0 on E for sufficiently large k. Since each f_k omits $H_i(z)$ on D by the assumption of Theorem 2.5, this is impossible.

Combining (i) and (ii), we have that g intersects $H_i(p_0)$ on $\mathbb{C}$ with multiplicity ∞ ($i = 1, 2, \cdots, 2N+1$). Since $H_1(z), \cdots, H_{2N+1}(z)$ ($z \in D$) are $2N+1$ continuously moving hyperplanes in $P^N(\mathbb{C})$ located in pointwise general position, $H_1(p_0), \cdots, H_{2N+1}(p_0)$ are $2N+1$ hyperplanes in $P^N(\mathbb{C})$ located in general position. By Nochka's Picard-type theorem in [12], g must be a constant mapping of $\mathbb{C}$ into $P^N(\mathbb{C})$. The proof of Theorem 2.5 is completed.

Proof of Theorem 2.6 If f is not normal on D, then, by Lemma 3.2, there exist sequences $\{p_k\} \subset D$, $\{r_k\}$ with $r_k > 0$ and $r_k \to 0^+$ and $\{u_k\} \subset \mathbb{C}^n$ Euclidean unit vectors, with $\lim\limits_{i\to\infty} \frac{r_i}{d(p_i, \mathbb{C}^n \setminus D)} = 0$, such that

$$g_k(\xi) := f(p_k + r_k u_k \xi),$$

where $\xi \in \mathbb{C}$ satisfies $p_k + r_k u_k \xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a "nonconstant" holomorphic mapping g of $\mathbb{C}$ into $P^N(\mathbb{C})$. We will prove that g must be a constant holomorphic mapping of $\mathbb{C}$ into $P^N(\mathbb{C})$, and then get a contradiction.

In fact, since $\overline{D}$ is compact, without loss of generality, we assume that $\{p_k\}$ ($\subset D$) converges to p_0 ($\in \overline{D}$). By the same reasoning as that in the proof of Theorem 2.5, we can get that g intersects $H_i(p_0)$ on $\mathbb{C}$ with multiplicity ∞. Since $H_1(z), \cdots, H_{2N+1}(z)$ ($z \in \overline{D}$) are $2N+1$ continuously moving hyperplanes in $P^N(\mathbb{C})$ located in pointwise general position, $H_1(p_0), \cdots, H_{2N+1}(p_0)$ are $2N+1$ hyperplanes in $P^N(\mathbb{C})$ located in general position. By Nochka's Picard-type theorem in [12], g must be a constant mapping of $\mathbb{C}$ into $P^N(\mathbb{C})$. This proves Theorem 2.6.

In order to prove Corollary 2.1, we need the following lemma.

Lemma 3.4 *Let h be a holomorphic mapping of the ball $B(R) := \{z \in \mathbb{C}^n : \|z\| < R\}$ into $P^1(\mathbb{C})$ ($0 < R \le +\infty$). Then there exist three continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ ($z \in B(R)$) in $P^1(\mathbb{C})$ located in pointwise general position on $B(R)$ such that h omits $H_j(z)$ ($j = 1, 2, 3$) on $B(R)$.*

Proof Let h have a reduced representation

$$\widetilde{h}(z) = (h_0(z), h_1(z))$$

on the ball $B(R)$. Define

$$U(z) := \frac{1}{|h_0(z)|^2 + |h_1(z)|^2} \begin{pmatrix} h_0(z) & h_1(z) \\ -\overline{h_1(z)} & \overline{h_0(z)} \end{pmatrix}, \quad z \in B(R).$$

Then $U(z)$ is a unitary matrix for each $z \in B(R)$. Obviously we have

$$\widetilde{h}(z) = (|h_0(z)|^2 + |h_1(z)|^2)(1, 0)U(z), \quad z \in B(R).$$

Take

$$(a_k(z), b_k(z)) := (1, k)U(z), \quad z \in B(R)$$

for $k = 1, 2, 3$. Then $(a_k(z), b_k(z))$ is continuous on $B(R)$ and $(a_k(z), b_k(z)) \ne (0, 0)$ everywhere on $B(R)$ ($k = 1, 2, 3$). Let the linear form

$$L_k(z)(Z_0, Z_1) := \overline{a_k(z)} Z_0 + \overline{b_k(z)} Z_1$$

define the continuously moving hyperplane $H_k(z)$ $(z \in B(R))$ in $P^1(\mathbb{C})$ for $k = 1, 2, 3$.

Since

$$\begin{pmatrix} a_1(z) & b_1(z) \\ a_2(z) & b_2(z) \\ a_3(z) & b_3(z) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} U(z), \quad z \in B(R),$$

we have that $H_1(z), H_2(z), H_3(z)$ $(z \in B(R))$ are three continuously moving hyperplanes in $P^1(\mathbb{C})$ located in pointwise general position on $B(R)$ and

$$L_k(z)(h_0(z), h_1(z)) = \overline{a_k(z)}h_0(z) + \overline{b_k(z)}h_1(z) = |h_0(z)|^2 + |h_1(z)|^2$$

is nowhere zero on $B(R)$ $(k = 1, 2, 3)$. Therefore, h omits $H_k(z)$ $(k = 1, 2, 3)$ on $B(R)$. This proves Lemma 3.4.

Remark 3.2 (1) Let f be a meromorphic function on the unit disc of $\mathbb{C}$ with

$$\limsup_{r \to 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = \infty,$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. Then f misses at most two values in $P^1(\mathbb{C})$ (see [14, Theorem VII. 18]). Therefore, the example and Picard theorem imply that the assumption "three continuously moving hyperplanes" in Lemma 3.4 cannot be replaced by "three fixed hyperplanes". But it is not clear whether the assumption "three continuously moving hyperplanes" in Lemma 3.4 can be replaced by "three moving hyperplanes".

(2) Let g be a holomorphic mapping of the ball $B(R) := \{z \in \mathbb{C}^n : \|z\| < R\}$ into $P^N(\mathbb{C})$ $(0 < R \leq +\infty)$. In the special case which g has a reduced representation $\widetilde{g}(z) = (g_0(z), \cdots, g_N(z))$ on the ball $B(R)$ with $g_0(z) \neq 0$ everywhere on $B(R)$. Take

$$e_k := (0, \cdots, 0, 1, 0, \cdots, 0),$$

where e_k has a 1 in the kth coordinate, and 0s elsewhere $(k = 2, \cdots, N + 1)$. Then

$$\{\widetilde{g}(z), e_2, \cdots, e_{N+1}\}$$

is a basis of $\mathbb{C}^{N+1}$ for each $z \in B(R)$ and, by Gram-Schmidt orthogonalization process, we get a unitary $(N + 1) \times (N + 1)$ matrix $U(z)$ for each $z \in B(R)$. Therefore, by the method in the proof of Lemma 3.4, we can find $2N+1$ continuously moving hyperplanes $H_1(z), \cdots, H_{2N+1}(z)$ $(z \in B(R))$ in $P^N(\mathbb{C})$ located in pointwise general position on $B(R)$ such that g omits $H_j(z)$ $(j = 1, \cdots, 2N + 1)$ on $B(R)$. But for a general holomorphic mapping g of $B(R)$ into $P^N(\mathbb{C})$ with $N > 1$, it is not clear how to find such a unitary $(N + 1) \times (N + 1)$ matrix $U(z)$ for each $z \in B(R)$.

Proof of Corollary 2.1 Suppose that F satisfies the condition. For a given sequence $\{f_i(z)\}_{i=1}^{\infty}$ in F and a point $z_0 \in B(R)$, then by the assumption, there exist three continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ $(z \in B(R))$ in $P^1(\mathbb{C})$ located in pointwise general position on $B(R)$ such that some subsequence $\{f_{i_k}(z)\}_{k=1}^{\infty}$ omits $H_j(z)$ $(j = 1, 2, 3)$ on $B(r_0)$, where $|z_0| < r_0 < R$ (note $z_0 \in B(r_0)$). By Theorem 2.5, $\{f_{i_k}(z)\}_{k=1}^{\infty}$ has a subsequence which converges uniformly on compact subsets of $B(r_0)$ to a holomorphic mapping of $B(r_0)$ into $P^1(\mathbb{C})$, i.e., there exists a neighborhood $B(r_0)$ of z_0 in $B(R)$ such that $\{f_i(z)\}_{i=1}^{\infty}$ has a subsequence which converges uniformly on compact subsets of $B(r_0)$ to a holomorphic mapping of $B(r_0)$ into $P^1(\mathbb{C})$. Thus, by the usual diagonal argument, $\{f_i(z)\}_{i=1}^{\infty}$ has a subsequence which converges uniformly on compact subsets of $B(R)$ to a holomorphic mapping of $B(R)$ into $P^1(\mathbb{C})$. Therefore, F is normal on $B(R)$.

Suppose that F is normal on $B(R)$. For a given sequence $\{f_i(z)\}_{i=1}^{\infty}$ in F, then there exists a subsequence $\{f_{i_k}(z)\}_{k=1}^{\infty}$ which converges uniformly on compact subsets of $B(R)$ to a holomorphic mapping h of $B(R)$ into $P^1(\mathbb{C})$. By Lemma 3.4, there exist three continuously moving hyperplanes $H_1(z), H_2(z), H_3(z)$ $(z \in B(R))$ in $P^1(\mathbb{C})$ located in pointwise general position on $B(R)$ such that h omits $H_j(z)$ $(j = 1, 2, 3)$ on $B(R)$. For a given closed ball $\overline{B(r_0)}$ $(0 < r_0 < R)$, since $\{f_{i_k}(z)\}_{k=1}^{\infty}$ converges uniformly on $\overline{B(r_0)}$ to h, we have $\{f_{i_k}(z)\}_{k=1}^{\infty}$ omits $H_j(z)$ $(j = 1, 2, 3)$ on $\overline{B(r_0)}$ for sufficiently large k. Therefore, there exists k_0 such that $\{f_{i_k}(z)\}_{k=k_0}^{\infty}$ omits $H_j(z)$ $(j = 1, 2, 3)$ on $\overline{B(r_0)}$.

The proof of Corollary 2.1 is completed.

4 Concluding Remark

In this paper, we have restricted our attention to the continuously moving hyperplanes. In fact, it is not difficult to generalize Theorems 2.5 and 2.6 to the case of continuously moving hypersurfaces in $P^N(\mathbb{C})$ located in pointwise general position by Eremenko's Picard-type theorem in [6] and the argument of this paper (see [16]). We omit these considerations here.

Acknowledgements The first author wishes to thank the Department of Mathematics, HKUST and Fields Institute for kind hospitality and support while part of the work on this paper took place.

References

[1] Aladro, G., Applications of the Kobayashi metric to normal functions of several complex variables, *Utilitas Math.*, **31**, 1987, 13–24.

[2] Aladro, G. and Krantz S. G., A criterion for normality in $\mathbb{C}^n$, *J. Math. Anal. Appl.*, **161**, 1991, 1–8.

[3] Bargmann, D., Bonk, M., Hinkkanen A., et al., Families of meromorphic functions avoiding continuous functions, *J. d'Anal. Math.*, **79**, 1999, 379–387.

[4] Bloch, A., Sur les systèmes de fonctions holomorphes à variétés linéaires lacunaires, *Ann. Sci. École Norm. Sup.*, **43**(3), 1926, 309–362.

[5] Deimling, K., Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[6] Eremenko, A., A Picard type theorem for holomorphic curves, *Period. Math. Hung.*, **38**, 1999, 39–42.

[7] Fujimoto, H., Extensions of the big Picard's theorem, *Tôhoku Math J.*, **24**, 1972, 415–422.

[8] Green, M., Holomorphic maps into complex projective space omitting hyperplanes, *Trans. Amer. Math. Soc.*, **169**, 1972, 89–103.

[9] Green, M., The hyperbolicity of the complement of $2n + 1$ hyperplanes in general positonn in P_n and related results, *Proc. Amer. Math. Soc.*, **66**, 1977, 109–113.

[10] Hahn, K. T., Asymptotic behavior of normal mappings of several complex variables, *Canad. J. Math.*, **36**, 1984, 718–746.

[11] Montel, P., Sur les familles de fonction analytiques qui admettent des valeurs exceptionnelles dans un domaine, *Ann. Sci. École Norm. Sup.*, **29**(3), 1912, 487–535.

[12] Nochka, E., On the theory of meromorphic functions, *Soviet Math. Dokl.*, **27**, 1983, 377–381.

[13] Ru, M. and Stoll, W., The second main theorem for moving targets, *J. Geom. Anal.*, **1**, 1991, 99–138.

[14] Tsuji, M., Potential Theory in Modern Function Theory, Maruzen, Tokyo, 1959.

[15] Tu, Z. H., Normality criteria for families of holomorphic mappings of several complex variables into $P^n(\mathbb{C})$, *Proc. Amer. Math. Soc.*, **127**, 1999, 1039–1049.

[16] Tu, Z. H. and Cao, H. Z., Normal criterion for families of holomorphic maps of several complex variables into $P^n(\mathbb{C})$ with moving hypersurfaces, *Acta Math. Sci.*, **29B**(1), 2009, 169–175.

[17] Tu, Z. H. and Li, P. L., Normal families of meromorphic mappings of several complex variables into $P^n(\mathbb{C})$ for moving targets, *Sci. China Ser. A*, **48**(supp.), 2005, 355–364.

[18] Wang, T. Y. Julie, A generalization of Picard's theorem with moving targets, *Complex Variables*, **44**, 2001, 39–45.