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# Dirichlet Forms Associated with Linear Diffusions****

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**Abstract** One-dimensional local Dirichlet spaces associated with linear diffusions are studied. The first result is to give a representation for any 1-dim local, irreducible and regular Dirichlet space. The second result is a necessary and sufficient condition for a Dirichlet space to be regular subspace of another Dirichlet space.

**Keywords** Symmetrizing measure, Linear diffusion, Dirichlet space, Regular subspace
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## 1 Introduction

The problem we are concerned here is to characterize regular subspaces of a Dirichlet form. This problem could be traced back to an early paper of the third author (see [14]), where he proved that the killing transform of Markov processes is equivalent to the strong subordination of Dirichlet forms. The problem was to ask whether the strong subordination may be reduced to the usual subordination. This amounts to ask whether a Dirichlet form has non-trivial regular subspaces. This work is a continuation of [3], a paper written by the first and third author together with M. Fukushima, which gave a complete characterization for regular subspaces of linear Brownian motion.

Let us now introduce some background on Markov processes and Dirichlet forms. Let $X = (X_t, \mathbf{P}^x)$ be a right Markov process on a state space, which is usually assumed to be Radon, with transition semigroup $(P_t)$. It is a bit difficult to make clear what a right Markov process means, but roughly it is a right continuous process with strong Markov property. The interested readers may refer to Sharpe's book [13] for details. A $\sigma$-finite measure $m$ on $E$ is called a symmetrizing measure for $X$ or $(P_t)$ if $(P_t f, g)_m = (f, P_t g)_m$ for each non-negative measurable $f, g$ and $t > 0$. Suppose that $m$ is a symmetrizing measure for $X$. Then it associates a bilinear form

$$\mathcal{F} = \Big\{u \in L^2(E,m) :\ \lim_{t\downarrow 0} \frac{1}{t}(u - P_t u, u) < \infty\Big\},$$

$$\mathcal{E}(u,v) = \lim_{t\downarrow 0} \frac{1}{t}(u - P_t u, v), \quad u, v \in \mathcal{F},$$

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where $(\cdot,\cdot)$ means the inner product on $L^2(E,m)$. It is known that $(\mathcal{E},\mathcal{F})$ is a Dirichlet form on $L^2(E,m)$, i.e., a densely defined, closed symmetric form with Markovian property. However, it is more important to know when a Dirichlet form is associated with a nice Markov process. The regularity is a sufficient condition introduced by Fukushima. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(E,m)$ where $E$ is assumed to be a locally compact with countable base (LCCB) space, and $m$ a fully supported Radon measure. It is said to be regular on $E$ (or simply say that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(E,m)$) if $C_0(E)\cap\mathcal{F}$ is both dense in $C_0(E)$ and in $\mathcal{F}$, where $C_0(E)$ is the Banach space of continuous functions vanishing at infinity with uniform norm and the norm on $\mathcal{F}$ is defined by

$$\|u\| := \sqrt{\mathcal{E}(u,u) + \int u^2 \mathrm{d}m}\,.$$

One of the great contributions for Dirichlet forms made by Fukushima was to introduce the notion of regularity and indicate that the main novelty of it is to guarantee the existence of a unique $m$-symmetric Hunt process $X=(X_t,\mathbf{P}^x)$ associated with $(\mathcal{E},\mathcal{F})$ (see [7] for details). A Dirichlet form $(\mathcal{E}',\mathcal{F}')$, associated with a process $X'$, on $L^2(E,m)$ is called a regular subspace of $(\mathcal{E},\mathcal{F})$ if $\mathcal{F}'\subset\mathcal{F}$, $\mathcal{E}|_{\mathcal{F}'\times\mathcal{F}'}=\mathcal{E}'$ and, the most importantly, it is also regular on $E$. For convenience, we also say that $X'$ is a regular subspace of $X$.

There has been no essential progress on this issue until 2001 when Fukushima and the third author published a paper (see [6]) in which they asserts that (the Dirichlet form of) Brownian motion admits no regular subspaces. But soon after, Fang (then a Ph. D student of the third author) indicated a fatal mistake in it and we found that Brownian motion actually has regular subspaces. In a joint paper with Fukushima [3], we successfully characterized all regular subspaces for Brownian motion.

Unfortunately, the method can not be applied to general Markov processes. But it can be applied to linear diffusions. In this paper, we shall give a necessary and sufficient condition for a linear diffusion to be a regular subspace of another. Before this, a simple and clear representation of Dirichlet forms for linear diffusions is formulated and it does not appear in literature as we know. The content in this paper is a part of the first author's Ph. D. thesis (see [2]) in 2004. For the representation part, Fukushima wrote a more "elementary" proof in his recent paper (see [5]) in which the more "advanced" theorems such as Blumenthal-Getoor-McKean theorem and uniqueness theorem of symmetrizing measures are avoided.

Due to the pioneering works of Feller, Itô, etc., one-dimensional diffusion has been a mature and very interesting topic in theory of Markov processes with its simplicity and clarity. There are a lot of literatures on this topic, e.g., Itô-Mckean [10], Revuz-Yor [11], Rogers-Williams [12], among those most influential. As is well-known (see e.g., [10, Subsection 4.11]), one-dimensional irreducible diffusion is always symmetric. Thus it has no loss of generality that Dirichlet form approach is introduced to study the properties of one-dimensional diffusions. In Section 2, a regular Dirichlet form is associated with a strictly increasing continuous function. In Section 3, we establish a one-to-one correspondence between linear diffusions and Dirichlet forms constructed in Section 2. Then a necessary and sufficient condition for a linear diffusion to be a regular subspace of another is given. Finally in Section 5, we shall use theory of Dirichlet forms to prove a well-known criterion in linear diffusions and two interesting examples are presented.

## 2 Dirichlet Forms on Intervals

Let $I$ be an interval or a connected subset of $\mathbb{R}$ with two end-points

$$-\infty \le a_1 < a_2 \le +\infty,$$

which may or may not be in $I$, and $I^\circ$ its interior. We write it as $I = \langle a_1, a_2 \rangle$. Denote by $\mathbf{S}(I)$ the totality of strictly increasing continuous functions on $I$. Let $\mathtt{s} \in \mathbf{S}(I)$. Let $m$ and $k$ be two Radon measures on $I$ with $\operatorname{supp}(m) = I$. Define a symmetric form $(\mathcal{E}^{(\mathtt{s},m,k)}, \mathcal{F}^{(\mathtt{s},m,k)})$ as follows:

$$\begin{aligned}
\mathcal{F}^{(\mathtt{s},m,k)} &= \Big\{u \in L^2(I, m+k) : u \ll \mathtt{s} \text{ and } \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}} \in L^2(I, \mathrm{d}\mathtt{s})\Big\}, \\
\mathcal{E}^{(\mathtt{s},m,k)}(u,v) &= \int_I \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}}\frac{\mathrm{d}v}{\mathrm{d}\mathtt{s}}\mathrm{d}\mathtt{s} + \int_I u(x)v(x)k(\mathrm{d}x) \quad \text{for } u, v \in \mathcal{F}^{(\mathtt{s},m,k)}.
\end{aligned}$$

It follows from [4] that $\mathcal{F}^{(\mathtt{s},m,k)}$ is the closure of the algebra generated by $\mathtt{s}$ with respect to the norm $\sqrt{\mathcal{E}^{(\mathtt{s},m,k)}(\,\cdot\,,\,\cdot\,) + (\,\cdot\,,\,\cdot\,)_m}$. As in [7, Example 1.2.2], if $I = \langle a_1, a_2 \rangle$, we call $a_1$ an $\mathtt{s}$-regular boundary if $a_1 \notin I$, $\mathtt{s}(a_1+) > -\infty$ and $m((a_1, c)) + k((a_1, c)) < \infty$ for some $c \in I$. When $\mathtt{s}(x) = x$, $a_1$ is just a regular boundary as in [7]. The regularity of $a_2$ is defined similarly. Define also

$$\begin{aligned}
\mathcal{F}_0^{(\mathtt{s},m,k)} &= \{u \in \mathcal{F}^{(\mathtt{s},m,k)} : u(a_i) = 0 \text{ if } a_i \text{ is regular boundary}\}, \\
\mathcal{E}_0^{(\mathtt{s},m,k)}(u,v) &= \mathcal{E}^{(\mathtt{s},m,k)}(u,v) \quad \text{for } u, v \in \mathcal{F}_0^{(\mathtt{s},m,k)}.
\end{aligned}$$

When $k = 0$, we write it as $(\mathcal{E}_0^{(\mathtt{s},m)}, \mathcal{F}_0^{(\mathtt{s},m)})$ for simplicity. The next lemma asserts that a Dirichlet form is built by this way.

**Lemma 2.1** *The form* $(\mathcal{E}_0^{(\mathtt{s},m,k)}, \mathcal{F}_0^{(\mathtt{s},m,k)})$ *is a local irreducible Dirichlet space on* $L^2(I; m)$ *regular on* $I$ *and it is strong local if and only if* $k = 0$.

**Proof** We only prove the first statement. The second is clear. Let

$$J = \mathtt{s}(I) = \langle \mathtt{s}(a_1), \mathtt{s}(a_2) \rangle$$

and define a Dirichlet space $(\mathcal{E}, \mathcal{F}^R)$ on $L^2(J, m \circ \mathtt{s}^{-1})$ as follows:

$$\begin{aligned}
\mathcal{F}^R &:= \{u \in L^2(J, (m+k)\circ\mathtt{s}^{-1}) : u \text{ is absolutely continuous and } u' \in L^2(J)\}, \\
\mathcal{E}(u,v) &:= \int_J u'(x)v'(x)\mathrm{d}x + \int_J u(x)v(x)(k \circ \mathtt{s}^{-1})(\mathrm{d}x) \quad \text{for } u, v \in \mathcal{F}.
\end{aligned}$$

Define further

$$\mathcal{F} := \{u \in \mathcal{F}^R : u(\mathtt{s}(a_i)) = 0 \text{ if } \mathtt{s}(a_i) \text{ is regular}, \ i = 1, 2\}.$$

As proved in [7, Example 1.2.2], $(\mathcal{E},\mathcal{F})$ is regular. Note that in [7, Example 1.2.2] the interval is assumed to be open, but the conclusion is certainly true when the interval is closed at any end.

Then it is easy to see that $(\mathcal{E}_0^{(\mathtt{s},m,k)},\mathcal{F}_0^{(\mathtt{s},m,k)})$ is a state-space transform of $(\mathcal{E},\mathcal{F})$ induced by the function $\mathtt{s}^{-1}$. It shows that $(\mathcal{E}_0^{(\mathtt{s},m,k)},\mathcal{F}_0^{(\mathtt{s},m,k)})$ is a Dirichlet form on $L^2(I,m)$ by [3, Lemma 3.1]. The regularity follows from the fact that $u\circ\mathtt{s}^{-1}\in\mathcal{F}_0^{(\mathtt{s},m,k)}\cap C_0(I)$ whenever $u\in\mathcal{F}\cap C_0(J)$. The local property of $(\mathcal{F}_0^{(\mathtt{s},m,k)},\mathcal{E}_0^{(\mathtt{s},m,k)})$ is obvious.

## 3 Representation of One-Dimensional Local Dirichlet Space

We shall give a short introduction to linear diffusions, which may be found in many classical textbooks, say [10, 11]. Let $I$ be an interval or a connected subset of $\mathbb{R}$ and $I^\circ$ its interior.

**Definition 3.1** *A linear diffusion $X=(X_t,\mathbf{P}^x)$ is simply a diffusion (may have finite life time) on a linear interval $I$. A linear diffusion $X$ is called irreducible if for any $x,y\in I$, $\mathbf{P}^x(T_y<\infty)>0$, where $T_y$ denotes the hitting time of $y$.*

The irreducibility defined here implies the regularity in [11, 12]. The reason which we use irreducibility is that $I$ is the state space of $X$, while in [11, 12], $I$ may contain a trap, thus not a real state space. Another thing which needs to be noted is that a diffusion defined this way is allowed being "killed" inside $I$, while in some literature it is not allowed. A diffusion not allowed being killed inside $I$ is called locally conservative.

For any irreducible diffusion $X$ on $I$, there exists an irreducible locally conservative diffusion $X'$ on $I$, through the well-known Ikeda-Nagasawa-Watanabe piecing together procedure, such that $X$ is obtained by killing $X'$ at a rate given by a positive continuous additive functional (or simply, a PCAF). In this case we say that $X'$ is a resurrected process of $X$ and $X$ is a subprocess of $X'$.

As [11, VII(3.2)] or [11, (46.12)], a locally conservative regular diffusion $X$ on $I$ admits a so-called scale function, namely, there exists a continuous, strictly increasing function $\mathtt{s}$ on $I$ such that for any $a,b,x\in I$ with $a<b$ and $a\le x\le b$,

$$\mathbf{P}^x(T_b<T_a)=\frac{\mathtt{s}(x)-\mathtt{s}(a)}{\mathtt{s}(b)-\mathtt{s}(a)}. \tag{3.1}$$

The function $\mathtt{s}$ is unique up to a linear transformation. This function $\mathtt{s}$ is called a scale function of $X$. A diffusion with scale function $\mathtt{s}(x)=x$ is said to be on natural scale. It is easy to check that if $\mathtt{s}$ is a scale function of $X$, then $\mathtt{s}(X)$ is a diffusion on $\mathtt{s}(I)$ in natural scale. A Brownian motion on $I$ is a diffusion on $I$ which moves like Brownian motion inside $I$ and is reflected at any end-point which is finite and in $I$ and get absorbed at any end point which is finite but not in $I$. Clearly Brownian motion on $I$ is clearly in natural scale. Thus Blumenthal-Getoor-Mckean's theorem (see [1, Theorem 5.5.1]) implies that a diffusion on $I$ on natural scale is identical in law with a time change of Brownian motion on $I$.

More precisely, let $X$ be a locally conservative regular diffusion on natural scale. Then there exists a measure $\xi$ on $\mathbb{R}$, fully supported on $I$, such that $X$ is equivalent in law to $(B_{\tau_t})$ where

$B = (B_t)$ is Brownian motion on $I$ and $\tau = (\tau_t)$ is the continuous inverse of the PCAF $A = (A_t)$ of $B$ with Revuz measure $\xi$. The measure $\xi$ is called the speed measure of $X$. The basic fact is that an irreducible linear diffusion is determined uniquely by its scale function and speed measure. Since an irreducible linear diffusion on $I$ is essentially a time change plus state-space transform from Brownian motion on $I$, it is also symmetrizable. In this sense, it is reasonable to study linear diffusions using theory of Dirichlet forms.

Fixing an interval $I$ and for a given fully-supported Radon measure $m$ on $I$, we shall consider in this section the representation of a local, irreducible and regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(I, m)$ in terms of the scale function of the associated diffusion. The following representation theorem is actually a generalized form of Theorem 2.1 in [3]. However the proof is rather different.

**Theorem 3.1** *Let $I = \langle a_1, a_2 \rangle$ be any interval and $m$ a Radon measure on $I$ with*

$$\mathrm{supp}\,(m) = I.$$

*If $(\mathcal{E}, \mathcal{F})$ is a local irreducible regular Dirichlet space on $L^2(I, m)$, then*

$$(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_0^{(\mathtt{s},m,k)}, \mathcal{F}_0^{(\mathtt{s},m,k)}),$$

*where $k$ is a Radon measure on $I$ and $\mathtt{s} \in \mathbf{S}(I)$. Furthermore, $\mathtt{s}$ is a scale function for $(X_t, \mathbf{P}^x)$ which is the diffusion associated with $(\mathcal{E}, \mathcal{F})$.*

**Proof** First of all, for a linear diffusion, any singleton has positive capacity. Hence by [7, Theorem 4.6.6] the linear diffusion associated with the irreducible $(\mathcal{E}, \mathcal{F})$ is irreducible. In addition $(\mathcal{E}, \mathcal{F})$ is strong local if and only if $(X_t, \mathbf{P}^x)$ is locally conservative.

We shall first assume that $(\mathcal{E}, \mathcal{F})$ is strongly local. Let $\mathtt{s}$ be a scale function of $X = (X_t, \mathbf{P}^x)$ associated with $(\mathcal{E}, \mathcal{F})$, and $Y = (Y_t, \mathbf{Q}^x)$ $(x \in I)$ be the diffusion associated with Dirichlet space $(\mathcal{F}_0^{(\mathtt{s},m)}, \mathcal{E}_0^{(\mathtt{s},m)})$. Mimicking the proof of Theorem 2.1(i) in [3], $\mathtt{s}$ is the scale function of $Y$. Then $X$ and $Y$ have the same scale function and thus the same hitting distributions. It follows from Blumenthal-Getoor-Mckean Theorem (see [1, Theorem V.5.1]) that there exists a strictly increasing continuous additive functional $A_t$ of $X$ such that $(Y_t, \mathbf{Q}^x)$, $x \in I$ and $(\widetilde{X}_t, \mathbf{P}^x)$ $(x \in I)$ are equivalent, where $\widetilde{X}_t = X_{\tau_t}$, and $(\tau_t)$ is the inverse of $(A_t)$.

Note that $(\widetilde{X}_t, \mathbf{P}^x)$ $(x \in I)$, which is irreducible, is $\xi$-symmetric, where $\xi$ is the Revuz measure of $A$ with respect to $m$, and also $m$-symmetric since it is equivalent to $(Y_t, \mathbf{Q}^x)$ $(x \in I)$. By the uniqueness theorem of symmetrizing measure presented in [15], $\xi$ is a multiple of $m$ or $A_t = ct$ for some positive constant $c$. It shows that $\widetilde{X}_t = X_{\frac{t}{c}}$. Therefore

$$\mathcal{F} = \mathcal{F}_0^{(\mathtt{s},m)}, \quad \mathcal{E} = c \cdot \mathcal{E}_0^{(\mathtt{s},m)}$$

by [7, (1.3.15) and (1.3.17)].

However, scale functions of a linear diffusion could differ by a linear transform. When the scale function is properly chosen, the constant $c$ above could be 1 (and shall be taken to be 1 in the sequel). For example, $\mathtt{s}' = \frac{\mathtt{s}}{c} \in \mathbf{S}(I)$ is also a scale function for $(X_t, \mathbf{P}^x)$, and we have

$$\mathcal{F} = \mathcal{F}_0^{(\mathtt{s}',m)}, \quad \mathcal{E} = \mathcal{E}_0^{(\mathtt{s}',m)}.$$

In general, when $(\mathcal{E},\mathcal{F})$ is local, we have the following Beurling-Deny decomposition by [7, Theorem 3.2.1]

$$\mathcal{E}(u,v)=\mathcal{E}^{(c)}(u,v)+\int_I u(x)v(x)k(\mathrm{d}x),\quad u,v\in\mathcal{F}\cap C_0(I),$$

where $\mathcal{E}^c$ is the strongly local part of $\mathcal{E}$. Define a new symmetric form $(\mathcal{E}',\mathcal{F}')$ on $L^2(I,m+k)$:

$$\mathcal{F}'=\mathcal{F},\quad \mathcal{E}'=\mathcal{E}^{(c)}.$$

Then $(\mathcal{E}',\mathcal{F}')$ is a strongly local irreducible regular Dirichlet space on $L^2(I,m+k)$. By the conclusion in the first part, it follows that

$$\mathcal{E}^{(c)}=\mathcal{E}_0^{(\mathtt{s},m)},\quad \mathcal{F}=\mathcal{F}'=\mathcal{F}_0^{(\mathtt{s},m+k)}=\mathcal{F}_0^{(\mathtt{s},m,k)}.$$

The proof is completed.

**Remark 3.1** After reading the representation result above, Fukushima provides a more intrinsic proof in his recent paper (see [5]). Here "intrinsic" means a proof similar to the one in [3] which only uses the profound theory developed for one-dimensional diffusion presented in classical books (see [8, 10]).

## 4 Regular Subspaces

Let $(\mathcal{E}',\mathcal{F}')$ and $(\mathcal{E},\mathcal{F})$ be two irreducible regular Dirichlet spaces on $L^2(I,m)$. The space $(\mathcal{E}',\mathcal{F}')$ is called a regular subspace of $(\mathcal{E},\mathcal{F})$ if $\mathcal{F}'\subset\mathcal{F}$ and $\mathcal{E}(u,v)=\mathcal{E}'(u,v)$ for any $u,v\in\mathcal{F}'$. All non-trivial regular subspaces of linear Brownian motion is characterized clearly in [3]. In this section, we shall further give a necessary and sufficient condition for $(\mathcal{E}',\mathcal{F}')$ to be a regular Dirichlet subspace of $(\mathcal{E},\mathcal{F})$, which extends the result in [3].

Using the representation in Section 3, we have

$$(\mathcal{E},\mathcal{F})=(\mathcal{E}_0^{(\mathtt{s}_1,m,k_1)},\mathcal{F}_0^{(\mathtt{s}_1,m,k_1)}),$$
$$(\mathcal{E}',\mathcal{F}')=(\mathcal{E}_0^{(\mathtt{s}_2,m,k_2)},\mathcal{F}_0^{(\mathtt{s}_2,m,k_2)}),$$

where $\mathtt{s}_1,\mathtt{s}_2\in\mathbf{S}(I)$ and $k_1$, $k_2$ are two Radon measures on $I$. Now comes our main result.

**Theorem 4.1** *Let $(\mathcal{E}',\mathcal{F}')$ and $(\mathcal{E},\mathcal{F})$ be two local irreducible regular Dirichlet spaces on $L^2(I,m)$. Then $(\mathcal{E}',\mathcal{F}')$ is a regular subspace of $(\mathcal{E},\mathcal{F})$ if and only if*

(1) $k_1=k_2$,

(2) $\mathrm{d}\mathtt{s}_2$ *is absolutely continuous with respect to* $\mathrm{d}\mathtt{s}_1$ *and the density* $\frac{\mathrm{d}\mathtt{s}_2}{\mathrm{d}\mathtt{s}_1}$ *is either* 1 *or* 0 *a.e.* $\mathrm{d}\mathtt{s}_1$*. More precisely, if $A$ is the set of point $x$ such that $\frac{\mathrm{d}\mathtt{s}_2}{\mathrm{d}\mathtt{s}_1}(x)$ exists and equals* 0 *or* 1*, then* $\int_{A^c}\mathrm{d}\mathtt{s}_1=0$.

**Proof** It suffices to prove the theorem for the case that both $(\mathcal{E}',\mathcal{F}')$ and $(\mathcal{E},\mathcal{F})$ are strongly local. Assume $\mathcal{F}'\subseteq\mathcal{F}$ and let $(X_t,\mathbf{P}_x)$ and $(X'_t,\mathbf{P}'_x)$ be the diffusion processes associated with $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}',\mathcal{F}')$, respectively. For any $a<c<x_0<d<b$, define

$$u^{x_0}_{\{c,d\}}(x):=\mathbf{P}'_x(T_{x_0}<T_{\{c,d\}}).$$

We have $u^{x_0}_{\{c,d\}}(x) \in \mathcal{F}' \subseteq \mathcal{F}$, which shows that $u^{x_0}_{\{c,d\}}(x)$ is absolutely continuous with respect to $\mathtt{s}_1$, while $u^{x_0}_{\{c,d\}}$ is a linear transformation of $\mathtt{s}_2$ on $(c, x_0)$. It follows that $\mathrm{d}\mathtt{s}_2$ is absolutely continuous with respect to $\mathrm{d}\mathtt{s}_1$ on $(c, x_0)$. Similarly, it is also true on $(x_0, d)$. Taking $(c, d) \uparrow (a, b)$, it follows that $\mathrm{d}\mathtt{s}_2$ is absolutely continuous with respect to $\mathrm{d}\mathtt{s}_1$. Let $f := \frac{\mathrm{d}\mathtt{s}_2}{\mathrm{d}\mathtt{s}_1}$. Then we have

$$\mathcal{E}'(u,v) = \int_I \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}_2}\frac{\mathrm{d}v}{\mathrm{d}\mathtt{s}_2}\mathrm{d}\mathtt{s}_2,$$
$$\mathcal{E}(u,v) = \int_I \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}_1}\frac{\mathrm{d}v}{\mathrm{d}\mathtt{s}_1}\mathrm{d}\mathtt{s}_1 = \int_I \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}_2}\frac{\mathrm{d}v}{\mathrm{d}\mathtt{s}_2}f^2\mathrm{d}\mathtt{s}_1 = \int_I \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}_2}\frac{\mathrm{d}v}{\mathrm{d}\mathtt{s}_2}f\mathrm{d}\mathtt{s}_2$$

for any $u, v \in \mathcal{F}'$. It follows that $f\mathrm{d}\mathtt{s}_1 = f^2\mathrm{d}\mathtt{s}_1$ and that either $f = 0$ or $f = 1$ a.e. with respect to $\mathrm{d}\mathtt{s}_1$. Since $\mathtt{s}_1$ and $\mathtt{s}_2$ are continuous and strictly increasing, $f$ has the property that for any $x, y \in I$ with $x < y$,

$$\int_x^y 1_{\{f=1\}}\mathrm{d}\mathtt{s}_1 > 0. \tag{4.1}$$

The converse is obvious from the above discussion.

Let now

$$(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_0^{(\mathtt{s},m,k)}, \mathcal{F}_0^{(\mathtt{s},m,k)})$$

be a local irreducible regular Dirichlet spaces on $L^2(I, m)$. Take a Borel set $A$ having property that for any $x, y \in I$ with $x < y$,

$$\int_x^y 1_{A^c}\mathrm{d}\mathtt{s} > 0. \tag{4.2}$$

Define $\mathrm{d}\mathtt{s}_0 = 1_{A^c} \cdot \mathrm{d}\mathtt{s}$. Then $\mathtt{s}_0 \in \mathbf{S}(I)$ and $(\mathcal{E}_0^{(\mathtt{s}_0,m,k)}, \mathcal{F}_0^{(\mathtt{s}_0,m,k)})$ is a regular subspace of $(\mathcal{E}, \mathcal{F})$. It is easy to check that

$$\mathcal{F}_0^{(\mathtt{s}_0,m,k)} = \Big\{u \in \mathcal{F} : \ \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}} = 0 \text{ a.e. with respect to } \mathrm{d}\mathtt{s} \text{ on } A\Big\}.$$

Hence we have a corollary.

**Corollary 4.1** *For any Borel set $A$ satisfying* (4.2)*,*

$$\mathcal{F}^A = \Big\{u \in \mathcal{F} : \ \frac{\mathrm{d}u}{\mathrm{d}\mathtt{s}} = 0 \textit{ a.e. with respect to } \mathrm{d}\mathtt{s} \textit{ on } A\Big\} \tag{4.3}$$

*is a regular subspace of $(\mathcal{E}, \mathcal{F})$. Conversely, any regular subspace of $(\mathcal{E}, \mathcal{F})$ is induced by such a set.*

## 5 Examples

In this section, we give two interesting examples. The first example is a local irreducible and regular Dirichlet space which takes the Dirichlet space $(H^1([0, 1]), \frac{1}{2}\mathbf{D})$ of reflected Brownian motion on $[0, 1]$ as a proper regular subspace.

**Example 5.1** Let $c(x)$ be the standard Cantor function on $[0,1]$ and let $\mathtt{s}(x) := x + c(x)$. Take $m$ to be the Lebesgue measure on $[0,1]$. Then the Dirichlet space $(H^1([0,1]), \frac{1}{2}\mathbf{D})$, corresponding to Brownian motion on $[0,1]$, is a regular subspace of $(\mathcal{F}^{(\mathtt{s},m)}, \frac{1}{2}\mathcal{E}^{(\mathtt{s},m)})$ by Theorem 4.1, and $H^1([0,1])$ is properly contained in $\mathcal{F}^{(\mathtt{s},m)}$.

The second example shows that 1-dim Brownian motion, which is certainly conservative, has a non-conservative regular subspace. For this we state a criterion for irreducible one-dimensional diffusions to be recurrent and conservative. The result is essentially classical (see [12, Subsection VII.3]). We include a new proof here using the power of Dirichlet forms. Let

$$(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_0^{(\mathtt{s},m,k)}, \mathcal{F}_0^{(\mathtt{s},m,k)})$$

be a local, irreducible and regular Dirichlet space on $L^2(I, m)$, where $k$ is a Radon measure on $I$ and $\mathtt{s} \in \mathbf{S}(I)$, and $X = (X_t, P_x)$ the associated diffusion. In this case, it is either recurrent or transient. We call the left endpoint $a$ of $I$ is of

(1) the first class if $a$ is finite and $a \in I$;
(2) the second class if $a \notin I$ and $\mathtt{s}(a+) = -\infty$;
(3) the third class if $a \notin I$ and $\mathtt{s}(a+) > -\infty$.

We call $a$ is dissipative if $a$ is of the third class and

$$\int_{a+} (\mathtt{s}(x) - \mathtt{s}(a)) m(\mathrm{d}x) < \infty. \tag{5.1}$$

If $a$ is not dissipative, we call it conservative. The dissipativeness and conservativeness for the right endpoint may be defined similarly. Fix a point $c > a$, and define $M(x) := m((x, c))$ for $a < x < c$.

**Lemma 5.1** *The left end-point $a$ is dissipative if and only if $a$ is of the third class and*

$$\int_{a+} M(x) \mathrm{d}\mathtt{s}(x) < \infty. \tag{5.2}$$

*If $a$ is dissipative,*

$$\lim_{x \downarrow a} M(x)(\mathtt{s}(x) - \mathtt{s}(a+)) = 0. \tag{5.3}$$

*Similar conclusions hold for the right end-point.*

**Proof** We may assume that $\mathtt{s}(a+) = 0$ without loss of generality. Let us prove that (5.2) implies (5.3) first. Suppose $\overline{\lim\limits_{x\downarrow a}} M(x)\mathtt{s}(x) = p > 0$. Then there exists a sequence $\{r_n\} \le c$ such that $r_n \downarrow a$, $M(r_n)\mathtt{s}(r_n) > \frac{p}{2}$ and $\mathtt{s}(r_{n+1}) < \frac{1}{2}\mathtt{s}(r_n)$. Therefore,

$$\int_a^c M(x)\mathrm{d}\mathtt{s}(x) \ge \sum_n \int_{r_{n+1}}^{r_n} M(x)\mathrm{d}\mathtt{s}(x) \ge \sum_n M(r_n)(\mathtt{s}(r_n) - \mathtt{s}(r_{n+1})) \ge \sum_n \frac{p}{4} = \infty,$$

which contradicts (5.2).

Using integration by parts, we have

$$\int_a^c M(x)\mathrm{d}\mathtt{s}(x) = \lim_{a' \downarrow a} \int_{a'}^c M(x)\mathrm{d}\mathtt{s}(x)$$

$$= \lim_{a'\downarrow a}\Big(M(c)\mathtt{s}(c) - M(a')\mathtt{s}(a') - \int_{a'}^{c}\mathtt{s}(x)\mathrm{d}M(x)\Big)$$
$$= M(c)\mathtt{s}(c) - \lim_{a'\downarrow a} M(a')\mathtt{s}(a') + \int_{a}^{c}\mathtt{s}(x)m(\mathrm{d}x)$$
$$\le M(c)\mathtt{s}(c) + \int_{a}^{c}\mathtt{s}(x)m(\mathrm{d}x).$$

Thus (5.1) implies (5.2). Conversely assuming (5.2), we have (5.3). Then the inequality above is an equality and (5.1) holds. The proof is completed.

Next, we shall prove necessary and sufficient conditions for one-dimensional diffusion to be transient, recurrent or conservative. For the definition of recurrence and conservativeness of Dirichlet forms and their equivalence to that of corresponding Markov processes, we refer to [7, Subsections 1.6 and 4.5].

**Theorem 5.1** *The Dirichlet space* $(\mathcal{E},\mathcal{F})$ *or* $X$ *is*

(1) *recurrent if and only if* $k=0$ *and both endpoints are of the first class or the second class*;

(2) *conservative if and only if* $k=0$ *and both endpoints are conservative.*

**Proof** (1) is easy to be proved. In fact, if $k=0$, we can construct $(X_t,P_x)$ $(x\in I)$ from Brownian motion on $J=\langle \mathtt{s}(a),\mathtt{s}(b)\rangle$ by a time change and a transform of state space, where $\langle\, ,\, \rangle$ means that the endpoints may be open or closed. When both endpoints are of the first or the second class, the endpoints of $J$ are closed or equal to infinity. But time changes and the transform of state space does not change recurrence and transience, it proves (1) when $k=0$. If $k\neq 0$, for any $u_n\in\mathcal{F}\cap C_c(I)$ with $u_n\uparrow 1$, $\mathcal{E}(u_n,u_n)\ge (u_n,u_n)_k$ and $(u_n,u_n)_k$ is increasing. It shows that $\mathcal{E}(u_n,u_n)\to 0$ does not hold.

(2) We have shown that if $k\neq 0$, $(\mathcal{E},\mathcal{F})$ is transient. Now we shall prove that in this case it is not even conservative. For any sequence $\{u_n\}\subset\mathcal{F}$ with $0\le u_n\le 1$, $u_n\uparrow 1$ $m$-a.e., take three points $a'$, $b'$, $c'$ with $a<a'<c'<b'<b$, $k((a',b'))>0$ and

$$\lim_{n\to\infty}u_n(a')=\lim_{n\to\infty}u_n(b')=\lim_{n\to\infty}u_n(c')=1.$$

Define a function $v\in\mathcal{F}$ as follows:

$$v(x)=\frac{\mathtt{s}(x)-\mathtt{s}(a')}{\mathtt{s}(c')-\mathtt{s}(a')},\qquad \text{if } a'\le x\le c',$$
$$=\frac{\mathtt{s}(x)-\mathtt{s}(b')}{\mathtt{s}(c')-\mathtt{s}(b')},\qquad \text{if } c'\le x\le b',$$

$$\lim_{n\to\infty}\mathcal{E}(u_n,v)=\lim_{n\to\infty}(u_n,v)_k + C\int_{a'}^{b'}\frac{\mathrm{d}u_n}{\mathrm{d}\mathtt{s}}\frac{\mathrm{d}v}{\mathrm{d}\mathtt{s}}\mathrm{d}\mathtt{s}$$
$$=(1,v)_k + C\lim_{n\to\infty}\Big(\frac{u_n(c')-u_n(a')}{\mathtt{s}(c')-\mathtt{s}(a')}+\frac{u_n(c')-u_n(b')}{\mathtt{s}(b')-\mathtt{s}(c')}\Big)$$
$$=\int_{a'}^{c'}v(x)k(\mathrm{d}x)+0>0,$$

since $v>0$ on $(a',b')$ and $k((a',b'))>0$. It shows that $(\mathcal{E},\mathcal{F})$ is not conservative.

Assumed that $a$ is dissipative and $b$ is an endpoint of the first class. Let $\mathtt{s}$ be a scale function with $\mathtt{s}(a) = 0$. By Lemma 4.1, we have $\mathtt{s} \in L^1(I, m) \cap \mathcal{F}$. For any function $u_n$ with $0 \le u_n \le 1$ and $u_n \uparrow 1$ $m$-a.e., take a point $b'$ such that $u_n(b') \uparrow 1$ and let $v(x) := \mathtt{s}(x)$ if $a \le x < b'$, $v(x) = v(b')$ if $b' \le x < b$. We have

$$\mathcal{E}(u_n, \mathtt{s}) = \int_a^{b'} \frac{\mathrm{d}u_n}{\mathrm{d}\mathtt{s}} \frac{\mathrm{d}\mathtt{s}}{\mathrm{d}\mathtt{s}} \mathrm{d}\mathtt{s} = u_n(b') \uparrow 1.$$

It proves that $(\mathcal{E}, \mathcal{F})$ is not conservative.

Finally, let us prove that the converse is true. Assume without loss of generality that $a$ is conservative and $b$ is an endpoint of the first class. For any $c \in (a, b]$, define

$$M^c(x) = m((x, c]), \quad M_c(x) = m((c, x]).$$

The conservativeness guarantees that

$$\int_{a'}^{c} M^c(x) \mathrm{d}\mathtt{s}(x) \to \infty,$$

as $a' \downarrow a$ for any $c$. Take any points $c, d$ with $a < c < d \le b$. Assume at first that $m$ does not charge singleton or $M$ is continuous. Then there exists uniquely $e \in (c, d)$ such that $m(c, e) = m(e, d)$. Define a function

$$w_{c,d}(x) := \begin{cases} M_c(x), & x \in (c, e], \\ M^c(x), & x \in (e, d), \\ 0, & \text{elsewhere.} \end{cases}$$

Obviously $w_{c,d} \ge 0$ is continuous and it may be written as

$$w_{c,d}(x) = \int_a^x (1(c, e] - 1(e, d)) \mathrm{d}m.$$

The conservativeness implies that, starting from $d$, we may choose $c$ such that

$$\int_a^b w_{c,d}(x) \mathrm{d}\mathtt{s}(x) \ge 1.$$

In general, $M$ may not be continuous. But for any $d \in I$, by a delicate analysis, we can still find a point $c \in (a, d)$ and a right continuous non-negative function $w$ on $I$ satisfying

(w1) $w$ is continuous at $c$ and $d$;

(w2) $w = 0$ on $(a, c]$ and $[d, b]$;

(w3) there exists a simple function $w'$ supported on $[c, d]$ with $|w'| \le 1$ such that $w(x) = \int_a^x w' \mathrm{d}m$;

(w4) $\int_a^b w(x) \mathrm{d}\mathtt{s}(x) \ge 1$.

Now we start from any point $a_1 \in (a, b]$. There exists an $a_2 \in (a, a_1)$ such that a function $w_{a_2,a_1}$ satisfying the four conditions above. Hence we have a sequence $\{a_n\}$ which decreases strictly to $a$ such that

$$\int_{a_{n+1}}^{a_n} w_{a_{n+1},a_n} \mathrm{d}\mathtt{s} \ge 1$$

for any $n$.

Define now

$$u_n := \frac{1}{q_n}\int_a^x w_{a_{n+1},a_n}\mathrm{d}\mathtt{s}, \quad x\in(a,b]$$

with

$$q_n = \int_{a_{n+1}}^{a-n} w_{a_{n+1},a_n}\mathrm{d}\mathtt{s}.$$

It is not hard to verify that $u_n\in\mathcal{F}$ and $u_n\uparrow 1$ on $(a,b]$. To check the conservativeness of $(\mathcal{E},\mathcal{F})$, choose any $v\in\mathcal{F}\cap L^1(I,m)$. Then using integration by parts and by the condition (w2) above, we have

$$\mathcal{E}(u_n,v)=\frac{1}{q_n}\int_a^b w_{a_{n+1},a_n}\mathrm{d}v=\frac{1}{q_n}\int_a^b v\mathrm{d}w_{a_{n+1},a_n}=\frac{1}{q_n}\int_a^b v\cdot w'_{a_{n+1},a_n}\mathrm{d}m,$$

where $w'_{a_{n+1},a_n}$ is chosen to satisfy (w3) above. Note that $v\in L^1(I,m)$ and $q_n\geq 1$, $|w'_{a_{n+1},a_n}|\leq 1$. Therefore

$$\sum n\Big|\int_a^b v\frac{1}{q_n}w'_{a_{n+1},a_n}\mathrm{d}m\Big|\leq\int_a^b|v|\sum_n\Big|\frac{w'_{a_{n+1},a_n}}{q_n}\Big|\mathrm{d}m\leq\int_a^b|v|\mathrm{d}m<\infty.$$

It follows that $\mathcal{E}(u_n,v)\to 0$. That completes the proof.

We now give an example which illustrates that the Dirichlet space $(\frac{1}{2}\mathbf{D},H_0^1(\mathbb{R}))$ of Brownian motion on the real line $\mathbb{R}$ has non-conservative regular subspaces, comparing an example in [3] which shows that Brownian motion has transient regular subspaces.

**Example 5.2** Define a local irreducible and regular Dirichlet space $(\mathcal{E}_0^{(\mathtt{s},m)},\mathcal{F}_0^{(\mathtt{s},m)})$ on $L^2(\mathbb{R},m)$, where $m$ is the usual Lebesgue measure, by giving a scale function

$$\mathtt{s}(x)=\int_0^x 1_G(y)\mathrm{d}y,\quad x\in\mathbb{R},$$

where

$$G=\bigcup_{r_n\in Q}\Big(r_n-\frac{1}{2^{n+1}},r_n+\frac{1}{2^{n+1}}\Big), \tag{5.4}$$

where $Q$ is the set of positive rational numbers. Since $Q$ is dense, $\mathtt{s}$ is strictly increasing. We choose an order on $Q$ as follows: if $a,b\in Q$, taking $a=\frac{q_1}{p_1}$, $b=\frac{q_2}{p_2}$ to be the simplest form, we define

$$a\prec b\Leftrightarrow \text{either } p_1+q_1<p_2+q_2 \text{ or } p_1+q_1=p_2+q_2 \text{ and } q_1<q_2.$$

Then the order $\prec$ makes $Q$ a sequence $\{r_n\}$ in (5.4). Clearly, we have $r_n\leq n$. Thus

$$\int_0^\infty x\mathrm{d}\mathtt{s}(x)\leq\sum_n\int_{(r_n-\frac{1}{2^{n+1}},r_n+\frac{1}{2^{n+1}})}x\mathrm{d}x=\sum_n\frac{r_n}{2^n}\leq\sum_n\frac{n}{2^n}<\infty.$$

This shows that the right endpoint is dissipative. Therefore, the associated process is not conservative.

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