

# Carleman Estimates for the Schrödinger Equation and Applications to an Inverse Problem and an Observability Inequality\*\*\*

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**Abstract** The authors prove Carleman estimates for the Schrödinger equation in Sobolev spaces of negative orders, and use these estimates to prove the uniqueness in the inverse problem of determining  $L^p$ -potentials. An  $L^2$ -level observability inequality and unique continuation results for the Schrödinger equation are also obtained.

**Keywords** Schrödinger equation, Carleman estimate, Observability inequality,  
 Inverse problem, Unique continuation

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## 1 Introduction and Main Results

In this paper, we discuss Carleman estimates and apply them to an inverse problem and an observability inequality for the Schrödinger equations. First we derive Carleman estimates for the Schrödinger equations in Sobolev spaces of negative orders.

Throughout this paper, we set

$$(Pv)(x, t) = i\partial_t v(x, t) + g(x, t)\Delta v(x, t) + \sum_{j=1}^n r_{1j}(x, t)\partial_j v(x, t) \\
 + \sum_{j=1}^n \int_0^t r_{2j}(x, t, \theta)\partial_j v(x, \theta)d\theta, \quad (x, t) \in Q = \Omega \times (-T, T),$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega \in C^3$ . Set  $\Sigma = \partial\Omega \times (-T, T)$ ,  $i = \sqrt{-1}$ ,  $\partial_t = \frac{\partial}{\partial t}$ ,  $\partial_j = \frac{\partial}{\partial x_j}$ ,  $j = 1, 2, \dots, n$ . Let  $\nabla u = (\partial_1 u, \partial_2 u, \dots, \partial_n u)$ , and  $\nu(x)$  be the unit outward normal vector to  $\partial\Omega$  at  $x$ . Moreover, let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  be a multi-index with  $\alpha_j \in \mathbb{N} \cup \{0\}$ , and let

$$\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

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Let  $p > 1$ ,  $\gamma > 0$ , and let  $W_p^\gamma(\Omega)$  and  $W_{p,0}^\gamma(\Omega)$  denote usual Sobolev spaces. Set  $H^\gamma(\Omega) = W_2^\gamma(\Omega)$  and  $H_0^\gamma(\Omega) = W_{2,0}^\gamma(\Omega)$  (see e.g., [39]). Let  $\frac{1}{p} + \frac{1}{p'} = 1$ . Identifying the dual of  $L^p(\Omega)$  with  $L^{p'}(\Omega)$ , we denote the dual of  $W_{p,0}^\gamma(\Omega)$  by  $W_{p'}^{-\gamma}(\Omega)$ . Set  $H^{-\gamma}(\Omega) = W_2^{-\gamma}(\Omega)$ .

Assume that  $r_{1j}, r_{2j}$ ,  $1 \leq j \leq n$  are complex-valued and satisfy

$$r_{1j} \in C^1(\overline{Q}), \quad r_{2j} \in C^1(\overline{\Omega} \times [-T, T] \times [-T, T]), \quad 1 \leq j \leq n. \quad (1.1)$$

Assume that  $d$  and  $g$  are real-valued and satisfy

$$\begin{cases} g \in C^2(\overline{Q}), & g > 0, & \text{on } \overline{Q}, \\ d \in C^4(\overline{\Omega}), & |\nabla d| > 0, & \text{on } \overline{\Omega}, \\ \nabla \log \left( \frac{1}{g(x,t)} \right) \cdot \nabla d(x) + |\xi \cdot \nabla d(x)|^2 > -2 \sum_{j,k=1}^n (\partial_j \partial_k d(x)) \xi_j \bar{\xi}_k, & (x,t) \in \overline{Q}, \\ & \text{for every } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n \text{ satisfying } |\xi| = 1. \end{cases} \quad (1.2)$$

First we show a Carleman estimate whose right-hand side is estimated in  $L^2(-T, T; H^{-s}(\Omega))$  with  $\frac{3}{4} < s \leq 1$ .

**Theorem 1.1** *Let  $\frac{3}{4} < s \leq 1$ ,  $0 < T_1 < T$ , and  $\Omega^0 \subset \Omega$  be a domain satisfying  $\overline{\Omega^0} \subset \Omega$ . Assume that (1.1) and (1.2) hold. For positive constants  $\lambda$  and  $\beta$ , we set  $\phi(x, t) = e^{\lambda(d(x) - \beta t^2)}$ . Then for large enough  $\lambda > 0$ , there exists a constant  $C_1 = C_1(\lambda) > 0$  such that for all large enough  $\tau > 0$ ,*

$$\tau \int_{-T}^T \|ue^{\tau\phi}\|_{H^{1-s}(\Omega)}^2 dt \leq C_1 \int_{-T}^T \|e^{\tau\phi} Pu\|_{H^{-s}(\Omega)}^2 dt \quad (1.3)$$

for all  $u \in L^2(-T, T; H^{1-s}(\Omega))$  with compact support in  $\Omega^0 \times (-T_1, T_1)$  such that  $Pu \in L^2(-T, T; H^{-s}(\Omega))$ .

Carleman estimates in Sobolev space of negative orders are useful tools for dealing with inverse problems (see e.g., [12, 14]) and unique continuation (see e.g., [38]) for the solutions of partial differential equation with less regular coefficients or non-homogenous terms. For  $H^{-1}$  Carleman estimate for hyperbolic equations, we can refer to [11, 12]. For  $H^{-1}$  Carleman estimate for parabolic equations, we can refer to [14, 17]. We can refer to [32] for an  $L^2$ -energy estimate with right-hand side in a Sobolev space of negative order for a nonconservative Schrödinger equation. The estimate in [32] involves a lower-order term on the right-hand side.

From Theorem 1.1, it is easy to see the following result.

**Corollary 1.1** *Let  $\frac{3}{4} < s \leq 1$ ,  $0 < T_1 < T$ , and  $\Omega^0 \subset \Omega$  be a domain satisfying  $\overline{\Omega^0} \subset \Omega$ . Assume that (1.1) holds, and real-valued  $g$  satisfies*

$$\begin{cases} g \in C^2(\overline{Q}), & g > 0, & \text{on } \overline{Q}, \\ \nabla \log \left( \frac{1}{g(x,t)} \right) \cdot (x - x_0) > -2, & (x,t) \in \overline{Q} \end{cases} \quad (1.4)$$

for a fixed  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ . We set  $\phi_1(x, t) = e^{\lambda(|x-x_0|^2 - \beta t^2)}$  for  $\lambda, \beta > 0$ . Then for large enough  $\lambda > 0$ , there exists a constant  $C_2 = C_2(\lambda) > 0$  such that for all large enough  $\tau > 0$ ,

$$\tau \int_{-T}^T \|ue^{\tau\phi_1}\|_{H^{1-s}(\Omega)}^2 dt \leq C_2 \int_{-T}^T \|e^{\tau\phi_1} Pu\|_{H^{-s}(\Omega)}^2 dt \quad (1.5)$$

for all  $u \in L^2(-T, T; H^{1-s}(\Omega))$  with compact support in  $\Omega^0 \times (-T_1, T_1)$  such that  $Pu \in L^2(-T, T; H^{-s}(\Omega))$ .

Moreover, we can prove a Carleman estimate with less regular potential in  $x$ .

**Corollary 1.2** *Let  $n \geq 2$ ,  $\frac{3}{4} < s < 1$ ,  $0 < T_1 < T$ , and  $\Omega^0 \subset \Omega$  be a domain satisfying  $\overline{\Omega^0} \subset \Omega$ . Assume that (1.1) and (1.2) hold. Let  $0 < \gamma < 1-s$  and  $p > 1$  satisfy  $1-\gamma > n(1-\frac{1}{p})$ , and let  $\frac{1}{p} + \frac{1}{p'} = 1$ . If  $\zeta_1$  is complex-valued and  $\zeta_1 \in L^\infty(-T, T; W_{p'}^{-\gamma}(\Omega))$ , then for large enough  $\lambda > 0$ , there exists a constant  $C_3 = C_3(\lambda) > 0$  such that for all large enough  $\tau > 0$ ,*

$$\tau \|ue^{\tau\phi}\|_{L^2(-T, T; H^{1-s}(\Omega))}^2 \leq C_3 \|e^{\tau\phi}(P + \zeta_1)u\|_{L^2(-T, T; H^{-s}(\Omega))}^2 \quad (1.6)$$

for all  $u \in L^2(-T, T; H^{1-s}(\Omega))$  with compact support in  $\Omega^0 \times (-T_1, T_1)$  such that  $(P + \zeta_1)u \in L^2(-T, T; H^{-s}(\Omega))$ .

From Corollary 1.2, we can obtain a unique continuation result for the Schrödinger equation with potential in a Sobolev space in  $x$  of negative order.

We consider the Schrödinger equation:

$$(P_1 u)(x, t) \equiv i\partial_t u(x, t) + \operatorname{div}(\rho(x, t)\nabla u(x, t)) - \zeta(x, t)u(x, t), \quad (x, t) \in Q. \quad (1.7)$$

For  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$  and a constant  $\beta > 0$ , we set  $J(x, t) = |x - x_0|^2 - \beta t^2$ .

**Theorem 1.2** *Let  $n \geq 2$  and  $\rho$  satisfy*

$$\begin{cases} \rho \in C^2(\overline{Q}), & \rho(x, t) > 0, \\ \nabla \log\left(\frac{1}{\rho(x, t)}\right) \cdot (x - x_0) > -2, & (x, t) \in \overline{Q} \end{cases} \quad (1.8)$$

for a fixed  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ . Let  $\zeta \in L^\infty(-T, T; W_p^{-\gamma}(\Omega))$  be complex-valued and  $u \in L^2(-T, T; H^{1-s}(\Omega))$ , where  $\frac{3}{4} < s < 1$ ,  $0 < \gamma < 1-s$ ,  $p > 1$  satisfying  $1-\gamma > n(1-\frac{1}{p})$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ . If  $P_1 u = 0$  in  $Q$  and  $u = 0$  in  $\{(x, t) \in Q; J(x, t) > J(x^1, t^1)\}$  for a point  $(x^1, t^1) \in Q$ , then there exists a neighbourhood  $\mathcal{U}$  of  $(x^1, t^1)$  such that

$$u(x, t) = 0, \quad (x, t) \in \mathcal{U}.$$

Theorem 1.2 is a consequence of Corollary 1.2 by using a known argument in deriving the unique continuation by a Carleman estimate (see e.g., [9, 10]). We give the details of the proof of Theorem 1.2 in Appendix.

For unique continuation across any non-characteristic surface for the Schrödinger equation with partially analytic coefficients, we can refer to [40]. For the unique continuation of the  $H^1$ -solution to the Schrödinger equation with  $L^\infty$ -potential, we refer to [19, 20, 41], and for the unique continuation of the solution in  $L^r(\mathbb{R}; W_r^2(\mathbb{R}^n)) \cap W_r^1(\mathbb{R}; L^r(\mathbb{R}^n))$  with  $r = \frac{2(n+2)}{n+4}$  to the Schrödinger equation with  $L^{\frac{n+2}{2}}$ -potential, we refer to [23]. We note that Theorem 1.2 is a unique continuation result with potential in a Sobolev space of negative order if  $u \in L^2(-T, T; H^{1-s}(\Omega))$ .

Next we consider an inverse problem of determining an  $L^p$ -potential  $q$  locally by some suitable local observation data.

We consider the Schrödinger equation:

$$\begin{cases} i\partial_t y(x, t) + \operatorname{div}(b(x)\nabla y(x, t)) - q(x)y(x, t) = 0, & \text{in } Q, \\ y = y_\Sigma, & \text{on } \Sigma, \\ y(\cdot, 0) = y_0, & \text{in } \Omega. \end{cases} \quad (1.9)$$

First we introduce an admissible set  $\mathcal{Q}$  of unknown potentials: let

$$p > \max\{n, 2\}.$$

We set  $\mathcal{Q} = \{q \in L^p(\Omega); q(x) \geq -c_0, x \in \Omega\}$ , where  $c_0 > 0$  is a constant.

For

$$b \in \mathcal{B} \equiv \left\{ b \in C^2(\overline{\Omega}); b(x) > 0 \text{ and } \nabla \log\left(\frac{1}{b(x)}\right) \cdot (x - x_0) > -2, x \in \overline{\Omega} \right\},$$

we introduce an admissible set of boundary values:

$$\begin{aligned} \mathcal{V}_b = \{ & (y_0, y_\Sigma); \text{ the solution } y = y(q, b, y_0, y_\Sigma) \text{ of (1.9)} \\ & \text{is in } H^1((-T, T); L^2(\Omega)) \text{ for any } q \in \mathcal{Q} \}. \end{aligned}$$

**Remark 1.1** The admissible set  $\mathcal{V}_b$  is not empty. For example, if  $y_0 \in H^3(\Omega)$  and  $y_\Sigma \in H^{\frac{7}{2}, \frac{7}{4}}(\Sigma)$  satisfy the compatibility condition  $y_\Sigma(x, 0) = y_0(x)$ , then we can choose a function  $\Phi \in H^{4,2}(Q)$  (see e.g., [34, Chapter 4]) such that  $\Phi(x, t) = y_\Sigma(x, t)$ ,  $(x, t) \in \Sigma$ , and  $\Phi(x, 0) = y_0(x)$ ,  $x \in \Omega$ . By (1.9), we have

$$\begin{cases} i\partial_t(y - \Phi) + \operatorname{div}(b(x)\nabla(y - \Phi)) - q(x)(y - \Phi) = F, & \text{in } Q, \\ y - \Phi = 0, & \text{on } \Sigma, \\ (y - \Phi)(\cdot, 0) = 0, & \text{in } \Omega, \end{cases}$$

where  $F := -i\partial_t\Phi - \operatorname{div}(b(x)\nabla\Phi) + q(x)\Phi \in H^1(-T, T; L^2(\Omega))$ . Then one can verify that  $y \in L^2(-T, T; H^2(\Omega)) \cap H^1(-T, T; L^2(\Omega))$  (see e.g., Theorem 12.1 in Chapter 5 of [34]).

For fixed  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ , we set  $\Omega(\delta) = \{x \in \Omega; |x - x_0| > \delta\}$ , where  $\delta > 0$  is a constant.

**Theorem 1.3** *Let  $S \subset \partial\Omega$  be a relatively open subset of  $\partial\Omega$  and  $\omega_S \subset \Omega$  be a subdomain such that  $\partial\omega_S \supset S$ , and let a constant  $\delta_0 > 0$  satisfy*

$$\overline{\Omega(\delta_0)} \subset (\Omega \cup S). \quad (1.10)$$

*Assume that  $b \in \mathcal{B}$ ,  $(y_0, y_\Sigma) \in \mathcal{V}_b$ ,  $q_1 \in \mathcal{Q}$ , and*

$$|y_0(x)| > c > 0, \quad x \in \overline{\Omega(\delta_0)}, \quad (1.11)$$

$$\partial_x^\alpha \partial_t y(q_1, b, y_0, y_\Sigma) \in C(\overline{\Omega(\delta_0)} \times (-T, T)), \quad |\alpha| = 2. \quad (1.12)$$

*For  $q_2 \in \mathcal{Q}$ , if*

$$y(q_2, b, y_0, y_\Sigma)(x, t) = y(q_1, b, y_0, y_\Sigma)(x, t), \quad (x, t) \in \omega_S \times (-T, T), \quad (1.13)$$

*then*

$$q_2(x) = q_1(x), \quad x \in \Omega(\delta_0).$$

We can prove also the stability, but in this paper we omit the details.

The uniqueness in determining the  $L^\infty$ -potential in the Schrödinger equation was firstly proved by Bukhgeim [5]. Afterwards, the stability was proved by Baudouin and Puel [2]. In those two papers, Carleman estimates in  $L^2$ -spaces are used. We further refer to [1] which proved the stability in determining  $L^\infty(\Omega)$ -potential in the Schrödinger equation with discontinuous principal term, and refer to [36] which established a new Carleman estimate and proved the Lipschitz stability in determining potentials in  $L^\infty(\Omega)$ . In this paper, we use an  $H^{-1}$ -Carleman estimate which leads to the uniqueness in  $L^p(\Omega)$  with  $p > \max\{n, 2\}$ . The method of solving inverse problems by Carleman estimates was firstly introduced by Bukhgeim and Klivanov [6] (see also [25, 26]). As for works on the stability and the uniqueness for inverse problems for hyperbolic or parabolic equations by Carleman estimates, see [3, 4, 13, 15, 16, 18, 20, 24, 28, 43] and the references therein.

Finally, we derive an  $L^2$ -level observability inequality for the Schrödinger equations:

$$\begin{cases} i\partial_t \eta(x, t) + \operatorname{div}(b(x)\nabla \eta(x, t)) - q(x)\eta(x, t) = 0, & \text{in } Q = \Omega \times (-T, T), \\ \eta(x, t) = 0, & \text{on } \Sigma = \partial\Omega \times (-T, T), \\ \eta(\cdot, 0) = \eta_0, & \text{in } \Omega. \end{cases} \quad (1.14)$$

Now we present an  $L^2$ -level observability inequality for the Schrödinger equation.

**Theorem 1.4** *Let  $\omega \subset \Omega$  be a neighbourhood of  $\partial\Omega$  satisfying  $\partial\omega \supset \partial\Omega$ . Assume that*

$$b \in C^2(\overline{\Omega}), \quad b > 0, \quad \text{on } \overline{\Omega}, \quad \nabla \log \left( \frac{1}{b(x)} \right) \cdot (x - x_0) > -2, \quad x \in \overline{\Omega}$$

*and  $q \in L^p(\Omega)$  with  $p > \max\{n, 2\}$ ,  $q(x) \geq -c_0$ ,  $x \in \Omega$  with some constant  $c_0 > 0$ . Then for every  $T > 0$ , there exists a constant  $C_4 > 0$  such that*

$$\|\eta_0\|_{L^2(\Omega)}^2 \leq C_4 \int_{-T}^T \int_{\omega} |\eta|^2 dx dt \quad (1.15)$$

*for every solution  $\eta \in L^2(Q)$  to (1.14) with  $\eta_0 \in L^2(\Omega)$ .*

For  $L^2$ -level observability inequality for the Schrödinger equation with  $b \equiv 1$  and  $q = 0$  in (1.14), we refer [35, 37] where methods based on multiplier techniques were used to prove the observability inequalities. In this paper, we derive an  $L^2$ -level observability inequality in Theorem 1.4 for the Schrödinger equation with a variable principal term and an  $L^p$ -potential. Our method is based on the  $H^{-1}$  Carleman estimate (see Theorem 1.1). As for the method of applying Carleman estimate to derive observability inequalities, see [22, 27]. As for observability inequalities by Carleman estimates, see further [7, 8, 17, 21, 28–31, 33].

This paper is composed of five sections. In Section 2, we prove Theorem 1.1 and Corollary 1.2. In Section 3 and Section 4, we prove Theorem 1.3 and Theorem 1.4 respectively. A proof of a Carleman estimate in  $L^2$  space is given in Appendix. We also present the proof of Theorem 1.2 in Appendix.

## 2 Proofs of Carleman Estimates

For  $\delta > 0$ , we set  $\Omega_\delta = \{x \in \mathbb{R}^n; \operatorname{dist}(x, \Omega) < \delta\}$ ,  $Q^\delta = \Omega_\delta \times (-T - \delta, T + \delta)$ . Let

$$(P_0 u)(x, t) = i\partial_t u(x, t) + g(x, t)\Delta u(x, t), \quad (x, t) \in Q$$

where  $g \in C^2(\overline{Q})$ .

Henceforth,  $C$  and  $C_j$  denote generic constants which are dependent on  $\Omega$ ,  $T$  and  $\lambda$ , but independent of  $\tau$ . The numbering in  $C_j$  can be independent in the different sections and the appendix. By  $\bar{c}$ , we denote the complex conjugate of  $c \in \mathbb{C}$ , while we note that  $\overline{\Omega}$  means the closure of a domain  $\Omega$ .

We will prove Carleman estimates with right-hand sides in  $L^2$  and negative-order Sobolev spaces as well.

First we present a Carleman estimate in  $L^2$  space.

**Proposition 2.1** *Let  $\phi(x, t) = e^{\lambda(d(x) - \beta t^2)}$ , where  $\beta > 0$ ,  $\lambda > 0$ . Assume that  $g$  and  $d$  satisfy (1.2). Then there exists a constant  $\lambda_0 > 0$  such that for arbitrary  $\lambda \geq \lambda_0$ , we can choose  $\tau_0 > 0$  satisfying: there exists a constant  $C(\tau_0, \lambda_0) > 0$  such that*

$$\begin{aligned} & \tau \int_Q |\nabla v|^2 e^{2\tau\phi} dx dt + \tau^3 \int_Q |v|^2 e^{2\tau\phi} dx dt \\ & \leq C \int_Q |P_0 v|^2 e^{2\tau\phi} dx dt + C\tau \int_{-T}^T \int_{\partial\Omega} \left| \frac{\partial v}{\partial \nu} \right|^2 e^{2\tau\phi} (\nabla d \cdot \nu) d\Sigma \end{aligned} \quad (2.1)$$

for all  $\tau > \tau_0$  and all  $v$  satisfying

$$\begin{cases} P_0 v \in L^2(Q), & v \in H^1(Q) \cap L^2(-T, T; H_0^1(\Omega)), \\ \frac{\partial v}{\partial \nu} \in L^2(-T, T; L^2(\partial\Omega)), & v(\cdot, -T) = v(\cdot, T) = 0. \end{cases} \quad (2.2)$$

We give the proof of Proposition 2.1 in Appendix.

Next we show Carleman estimates in Sobolev spaces of negative orders by an argument similar to [12].

We set

$$\begin{aligned} P_\phi w &= i\partial_t w + g\Delta w - 2\tau g \nabla \phi \cdot \nabla w + (\tau^2 |\nabla \phi|^2 - \tau \Delta \phi) gw - i\tau(\partial_t \phi)w \\ &+ \sum_{j=1}^n [r_{1j} \partial_j w - \tau r_{1j} (\partial_j \phi)w] + \sum_{j=1}^n \int_0^t [r_{2j} \partial_j w - \tau r_{2j} (\partial_j \phi)w] d\theta. \end{aligned}$$

Let

$$\begin{aligned} \xi' &= (\xi_0, \xi_1, \xi_2, \dots, \xi_n), \quad \xi = (\xi_1, \xi_2, \dots, \xi_n), \\ \langle \xi \rangle &= \sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2 + 1}. \end{aligned}$$

For  $\vartheta \in \mathbb{R}$  and  $f \in L^2(\Omega)$ , we define

$$\wedge_\tau^s f = \begin{cases} F^{-1}[(\langle \xi \rangle^s + \tau^s) Ff], & s \geq 0, \\ F^{-1}\left[\left(\frac{1}{\langle \xi \rangle^{-s} + \tau^{-s}}\right) Ff\right], & s < 0, \end{cases}$$

where  $F$  denotes the Fourier transform in  $\mathbb{R}^n$  which is defined by

$$(Ff)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad f \in L^2(\Omega),$$

and  $F^{-1}$  denotes its inverse transform. By choosing a sufficiently small  $\delta_1 > 0$ , we extend  $d$  into  $\mathbb{R}^n$ ,  $g$ ,  $r_{1j}$ ,  $1 \leq j \leq n$  into  $\mathbb{R}^n \times (-T, T)$  and  $r_{2j}$ ,  $1 \leq j \leq n$  into  $\mathbb{R}^n \times (-T, T) \times (-T, T)$ , so that they are supported in  $\Omega_{3\delta_1}$ ,  $\Omega_{3\delta_1} \times (-T, T)$ ,  $\Omega_{3\delta_1} \times (-T, T)$  and  $\Omega_{3\delta_1} \times (-T, T) \times (-T, T)$  respectively, and satisfy

$$\begin{cases} g \in C^1(\mathbb{R}^n \times [-T, T]), \quad \sup_{-T \leq t \leq T} \|g(\cdot, t)\|_{\text{Lip}(\mathbb{R}^n)} + \sup_{-T \leq t \leq T} \|\nabla g(\cdot, t)\|_{\text{Lip}(\mathbb{R}^n)} < +\infty, \\ r_{1j} \in C(\mathbb{R}^n \times [-T, T]), \quad \sup_{-T \leq t \leq T} \|r_{1j}(\cdot, t)\|_{\text{Lip}(\mathbb{R}^n)} < +\infty, \quad 1 \leq j \leq n, \\ r_{2j} \in C(\mathbb{R}^n \times (-T, T) \times (-T, T)), \quad \sup_{-T \leq t, \theta \leq T} \|r_{2j}(\cdot, t, \theta)\|_{\text{Lip}(\mathbb{R}^n)} < +\infty, \quad 1 \leq j \leq n \end{cases} \quad (2.3)$$

and

$$\begin{cases} g > 0, \quad \text{on } \overline{Q^{2\delta_1}}, \\ d \in C^4(\mathbb{R}^n), \quad |\nabla d| > 0, \quad \text{on } \overline{\Omega_{2\delta_1}}, \\ \nabla \log \left( \frac{1}{g(x, t)} \right) \cdot \nabla d(x) + |\xi \cdot \nabla d(x)|^2 > -2 \sum_{j,k=1}^n (\partial_j \partial_k d(x)) \xi_j \bar{\xi}_k, \quad (x, t) \in \overline{Q^{2\delta_1}}, \\ \text{for every } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n \text{ satisfying } |\xi| = 1. \end{cases} \quad (2.4)$$

Here and henceforth, we use the same notations to denote the extension of the coefficients, and we set

$$\|f\|_{\text{Lip}(\mathbb{R}^n)} = \sup_{\substack{x^1, x^2 \in \mathbb{R}^n \\ x^1 \neq x^2}} \frac{|f(x^1) - f(x^2)|}{|x^1 - x^2|}$$

provided that the right-hand side is finite.

In order to prove our Carleman estimates, we need a commutator estimate as follows.

**Lemma 2.1** *Let  $\frac{3}{4} < s \leq 1$ ,  $g$ ,  $r_{1j}$  and  $r_{2j}$  satisfy (2.3). Then there exists a constant  $C_1 > 0$  such that the following commutator estimate holds:*

$$\int_{-T}^T \|\wedge_\tau^{-s} P_\phi v - P_\phi \wedge_\tau^{-s} v\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq C_1 \tau^{4-4s} \|v\|_{L^2(Q)}^2 + C_1 \tau^{2-2s} \int_{-T}^T \|\nabla(\wedge_\tau^{-s} v)\|_{L^2(\mathbb{R}^n)}^2 dt$$

for all  $v \in C_0^2(Q)$  and  $\tau > 1$ .

**Proof** It is clear that

$$\wedge_\tau^{-s} i \partial_t v - i \partial_t \wedge_\tau^{-s} v = 0,$$

and

$$\wedge_\tau^{-s} g(x, t) \partial_j^2 v - g(x, t) \partial_j^2 \wedge_\tau^{-s} v = \wedge_\tau^{-s} \{ \partial_j [(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s} \partial_j v)] + [\wedge_0^s \partial_j g - (\partial_j g) \wedge_0^s] (\wedge_\tau^{-s} \partial_j v) \}$$

for  $1 \leq j \leq n$ . By the Parseval equality and the definition of  $\wedge_\tau^{-s}$ , we obtain

$$\int_{-T}^T \|\wedge_\tau^{-s} \partial_j [(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s} \partial_j v)]\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq \int_{-T}^T \|(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s} \partial_j v)\|_{H^{1-s}(\mathbb{R}^n)}^2 dt$$

for  $1 \leq j \leq n$ . By an argument in Chapter 3 of [42], we can prove an estimate of a commutator of a pseudo-differential operator

$$\int_{-T}^T \|(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s} \partial_j v)\|_{H^{1-s}(\mathbb{R}^n)}^2 dt \leq \sup_{-T \leq t \leq T} \|g(\cdot, t)\|_{\text{Lip}(\mathbb{R}^n)}^2 \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_\tau^{-s} \partial_j v|^2 dx dt$$

for  $1 \leq j \leq n$ . Thus we obtain

$$\int_{-T}^T \|\wedge_{\tau}^{-s} \partial_j [(g \wedge_0^s - \wedge_0^s g)(\wedge_{\tau}^{-s} \partial_j v)]\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq C_2 \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt \quad (2.5)$$

for  $1 \leq j \leq n$ . On the other hand, by the Parseval equality and an estimate of a commutator of a pseudo-differential operator (see e.g., [42, Subsection 3.6]), we obtain

$$\begin{aligned} & \int_{-T}^T \|[\wedge_0^s \partial_j g - (\partial_j g) \wedge_0^s](\wedge_{\tau}^{-s} \partial_j v)\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & \leq \sup_{-T \leq t \leq T} \|\nabla g(\cdot, t)\|_{\text{Lip}(\mathbb{R}^n)}^2 \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt \end{aligned} \quad (2.6)$$

for  $1 \leq j \leq n$ . By (2.5) and (2.6), we have

$$\int_{-T}^T \|\wedge_{\tau}^{-s} g(\cdot, t) \partial_j^2 v - g(\cdot, t) \partial_j^2 \wedge_{\tau}^{-s} v\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq C_3 \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt$$

for  $1 \leq j \leq n$ . By the Parseval equality and an estimate of a commutator of a pseudo-differential operator (see e.g., [42, Subsection 3.6]), we obtain

$$\begin{aligned} & \int_{-T}^T \|\wedge_{\tau}^{-s} \tau^2 (\partial_j \phi)^2 g v - \tau^2 (\partial_j \phi)^2 g \wedge_{\tau}^{-s} v\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & = \int_{-T}^T \|\wedge_{\tau}^{-s} \tau [(\partial_j \phi)^2 g \wedge_0^s - \wedge_0^s (\partial_j \phi)^2 g] \wedge_{\tau}^{-s} \tau v\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & \leq C_3 \tau^{4-4s} \int_Q |v|^2 dx dt \end{aligned}$$

for  $1 \leq j \leq n$ . Similarly,

$$\begin{aligned} & \int_{-T}^T \|\wedge_{\tau}^{-s} 2\tau (\partial_j \phi) g \partial_j v - 2\tau (\partial_j \phi) g \partial_j \wedge_{\tau}^{-s} v\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & \leq C_4 \tau^{2-2s} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt, \\ & \int_{-T}^T \|\wedge_{\tau}^{-s} [i\tau (\partial_t \phi) - \tau (\nabla \phi) g] v - [i\tau (\partial_t \phi) - \tau (\nabla \phi) g] \wedge_{\tau}^{-s} v\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & \leq C_5 \tau^{2-2s} \int_Q |v|^2 dx dt, \\ & \int_{-T}^T \left\| \wedge_{\tau}^{-s} \int_0^t r_{2j}(\cdot, t, \theta) \partial_j v(\cdot, \theta) d\theta - \int_0^t r_{2j}(\cdot, t, \theta) \partial_j \wedge_{\tau}^{-s} v(\cdot, \theta) d\theta \right\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & = \int_{-T}^T \left\| \int_0^t (\wedge_{\tau}^{-s} r_{2j}(\cdot, t, \theta) \partial_j v(\cdot, \theta) - r_{2j}(\cdot, t, \theta) \partial_j \wedge_{\tau}^{-s} v(\cdot, \theta)) d\theta \right\|_{L^2(\Omega_{2\delta_1})}^2 dt \\ & \leq C_6 \int_{-T}^T \int_{-T}^T \|\wedge_{\tau}^{-s} r_{2j}(\cdot, t, \theta) \partial_j v(\cdot, \theta) - r_{2j}(\cdot, t, \theta) \partial_j \wedge_{\tau}^{-s} v(\cdot, \theta)\|_{L^2(\Omega_{2\delta_1})}^2 d\theta dt \\ & \leq 2TC_7 \sup_{-T \leq t, \theta \leq T} \|r_{2j}(\cdot, t, \theta)\|_{\text{Lip}(\mathbb{R}^n)} \int_{-T}^T \|\wedge_{\tau}^{-s} \partial_j v(\cdot, \theta)\|_{L^2(\Omega_{2\delta_1})}^2 d\theta \\ & \leq C_8 \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt, \end{aligned}$$



$$\begin{aligned}
& \int_{-T}^T \left\| \wedge_{\tau}^{-s} \int_0^t \tau r_{2j}(\cdot, t, \theta) (\partial_j \phi(\cdot, \theta)) v(\cdot, \theta) d\theta \right. \\
& \left. - \int_0^t \tau r_{2j}(\cdot, t, \theta) (\partial_j \phi(\cdot, \theta)) \wedge_{\tau}^{-s} v(\cdot, \theta) d\theta \right\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq C_9 \tau^{2-2s} \int_Q |v|^2 dx dt, \\
& \int_{-T}^T \left\| \wedge_{\tau}^{-s} r_{1j} \partial_j v - r_{1j} \partial_j \wedge_{\tau}^{-s} v \right\|_{L^2(\Omega_{2\delta_1})}^2 dt \\
& \leq \sup_{-T \leq t \leq T} \|r_{1j}(\cdot, t)\|_{\text{Lip}(\mathbb{R}^n)} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt \leq C_{10} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt, \\
& \int_{-T}^T \left\| \wedge_{\tau}^{-s} \tau r_{1j} (\partial_j \phi) v d\tau - \tau r_{1j} (\partial_j \phi) \wedge_{\tau}^{-s} v d\tau \right\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq C_{10} \tau^{2-2s} \int_Q |v|^2 dx dt
\end{aligned}$$

for  $j = 1, 2, \dots, n$ . Consequently we have

$$\int_{-T}^T \left\| \wedge_{\tau}^{-s} P_{\phi} v - P_{\phi} \wedge_{\tau}^{-s} v \right\|_{L^2(\Omega_{2\delta_1})}^2 dt \leq C_{11} \tau^{4-4s} \int_Q |v|^2 dx dt + C_{11} \tau^{2-2s} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} \partial_j v|^2 dx dt.$$

Thus the proof of Lemma 2.1 is completed.

**Lemma 2.2** *Let  $K(x, y; \tau)$  be the Schwartz kernel of the pseudo-differential operator  $\wedge_{\tau}^{-s}$  with  $\tau > 1$ . Then*

$$|\partial_x^{\alpha} K(x, y; \tau)| \leq C_{12}(\mu) \tau^{-2s} |x - y|^{-2n-3}$$

*provided that  $|\alpha| \leq 1$  and  $|x - y| \geq \mu > 0$ .*

**Proof** The Schwartz kernel  $K(x, y; \tau)$  is the oscillatory integral

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} (\langle \xi \rangle^s + \tau^s)^{-1} d\xi &= -\frac{1}{|x - y|^2} \int_{\mathbb{R}^n} (\Delta_{\xi} e^{i(x-y) \cdot \xi}) (\langle \xi \rangle^s + \tau^s)^{-1} d\xi \\
&= -\frac{1}{|x - y|^2} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \Delta_{\xi} (\langle \xi \rangle^s + \tau^s)^{-1} d\xi \\
&\quad \vdots \\
&= \frac{(-1)^{\ell}}{|x - y|^{2\ell}} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \Delta_{\xi}^{\ell} (\langle \xi \rangle^s + \tau^s)^{-1} d\xi, \quad \ell \in \mathbb{N}.
\end{aligned}$$

We have integrated by parts in the above equalities. Observing that

$$|\Delta_{\xi}^{\ell} (\langle \xi \rangle^s + \tau^s)^{-1}| \leq C_{13}(\ell) (\langle \xi \rangle^s + \tau^s)^{-2} \langle \xi \rangle^{-2\ell+3s}, \quad \ell = 1, 2, \dots$$

and choosing  $\ell = n + 2$ , we complete the proof of Lemma 2.2.

**Lemma 2.3** *Let  $\frac{3}{4} < s \leq 1$ , and  $g, r_{1j}, r_{2j}$ ,  $1 \leq j \leq n$  satisfy (2.3). Moreover, assume that  $g$  and  $d$  satisfy (2.4). Then for large enough  $\lambda > 0$ , there exists a constant  $C_{14} = C_{14}(\lambda) > 0$  such that*

$$\tau \int_{-T}^T \|ue^{\tau\phi}\|_{H^{1-s}(\Omega)}^2 dt \leq C_{14} \int_{-T}^T \|e^{\tau\phi} Pu\|_{H^{-s}(\mathbb{R}^n)}^2 dt$$

*for all  $u \in C_0^2(Q)$ , provided that  $\tau > 0$  is large enough.*

**Proof** By Proposition 2.1, we know that for large enough  $\lambda$ , there exists a constant  $C_{15} = C_{15}(\lambda) > 0$  such that if  $\tau$  is large enough, then

$$\sum_{|\alpha|=0}^1 \tau^{3-2|\alpha|} \int_{Q^{2\delta_1}} |\partial_x^{\alpha} v_0|^2 dx dt \leq C_{15} \int_{Q^{2\delta_1}} |P_{\phi} v_0|^2 dx dt \quad \text{for all } v_0 \in C_0^2(Q^{2\delta_1}).$$

Let  $\chi \in C_0^\infty(Q^{2\delta_1})$ ,  $\chi = 1$  in  $Q^{\delta_1}$ . Applying the Carleman estimate to  $\chi \wedge_\tau^{-s} v$ , we have

$$\begin{aligned} & \int_{Q^{2\delta_1}} \left[ \tau^3 \chi^2 |\wedge_\tau^{-s} v|^2 + \tau \sum_{j=1}^n |\chi \partial_j (\wedge_\tau^{-s} v) + (\partial_j \chi) \wedge_\tau^{-s} v|^2 \right] dx dt \\ & \leq C_{16} \int_{Q^{2\delta_1}} |P_\phi(\chi \wedge_\tau^{-s} v)|^2 dx dt \\ & \leq C_{17} \int_{Q^{2\delta_1}} [|P_\phi(\wedge_\tau^{-s} v)|^2 + \tau^2 |\wedge_\tau^{-s} v|^2 + |\nabla(\wedge_\tau^{-s} v)|^2] dx dt. \end{aligned} \quad (2.7)$$

Furthermore, we can estimate

$$\int_{Q^{2\delta_1}} |\wedge_\tau^{-s} v|^2 dx dt \leq \frac{1}{\tau^{2s}} \int_{Q^{2\delta_1}} |v|^2 dx dt. \quad (2.8)$$

From (2.7) and (2.8), we have

$$\begin{aligned} & \int_{Q^{\delta_1}} [\tau^3 |\wedge_\tau^{-s} v|^2 + \tau |\nabla(\wedge_\tau^{-s} v)|^2] dx dt \\ & \leq C_{18} \int_{Q^{2\delta_1}} [|P_\phi(\wedge_\tau^{-s} v)|^2 + \tau^{2-2s} |v|^2 + |\nabla(\wedge_\tau^{-s} v)|^2] dx dt. \end{aligned} \quad (2.9)$$

We estimate

$$\begin{aligned} \tau \int_{-T}^T \int_{\Omega_{\delta_1}} |v|^2 dx dt & \leq \tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_\tau^s \wedge_\tau^{-s} v|^2 dx dt \\ & \leq C_{19} \int_{-T}^T \int_{\mathbb{R}^n} [\tau^3 |\wedge_\tau^{-s} v|^2 + \tau |\nabla(\wedge_\tau^{-s} v)|^2] dx dt. \end{aligned} \quad (2.10)$$

We also have

$$\begin{aligned} & \tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 dx dt \\ & \leq \int_{-T}^T \int_{\mathbb{R}^n} \tau |\wedge_0^{1-s} \wedge_\tau^s \wedge_\tau^{-s} v|^2 dx dt \\ & \leq \tau \int_{-T}^T \int_{\mathbb{R}^n} |F^{-1}[(\langle \xi \rangle + \langle \xi \rangle^{1-s} \tau^s) F(\wedge_\tau^{-s} v)]|^2 dx dt \\ & \leq 2\tau \int_{-T}^T \int_{\mathbb{R}^n} |\langle \xi \rangle F(\wedge_\tau^{-s} v)|^2 d\xi dt + 2\tau \int_{-T}^T \int_{\mathbb{R}^n} |\langle \xi \rangle^{1-s} \tau^s F(\wedge_\tau^{-s} v)|^2 d\xi dt. \end{aligned}$$

For the last inequality, we have used Parseval's identity. By the Young's inequality, we have

$$\begin{aligned} \tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 dx dt & \leq 2\tau \int_{-T}^T \int_{\mathbb{R}^n} |\langle \xi \rangle F(\wedge_\tau^{-s} v)|^2 d\xi dt \\ & \quad + 2\tau \int_{-T}^T \int_{\mathbb{R}^n} \left| \left( \frac{\langle \xi \rangle^{4(1-s)}}{4} + \frac{3}{4} \tau^{\frac{4}{3}s} \right) F(\wedge_\tau^{-s} v) \right|^2 d\xi dt. \end{aligned}$$

By noting  $0 \leq 4(1-s) \leq 1$  and  $\frac{4}{3}s \leq \frac{4}{3}$ , we can obtain

$$\tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 dx dt \leq C_{20} \int_{-T}^T \int_{\mathbb{R}^n} [\tau^3 |\wedge_\tau^{-s} v|^2 + \tau |\nabla(\wedge_\tau^{-s} v)|^2] dx dt. \quad (2.11)$$

By (2.9)–(2.11) and Lemma 2.1, we have

$$\begin{aligned}
& \int_{-T}^T \int_{\mathbb{R}^n} [\tau^3 |\wedge_{\tau}^{-s} v|^2 + \tau |v|^2 + \tau |\wedge_0^{1-s} v|^2 + \tau |\nabla(\wedge_{\tau}^{-s} v)|^2] dx dt \\
& \leq C_{21} \int_{-T}^T \int_{\mathbb{R}^n} [\tau^3 |\wedge_{\tau}^{-s} v|^2 + \tau |\nabla(\wedge_{\tau}^{-s} v)|^2] dx dt \\
& = C_{21} \int_{-T}^T \int_{\mathbb{R}^n \setminus \Omega_{\delta_1}} [\tau^3 |\wedge_{\tau}^{-s} v|^2 + \tau |\nabla(\wedge_{\tau}^{-s} v)|^2] dx dt \\
& \quad + C_{21} \int_{-T}^T \int_{\Omega_{\delta_1}} [\tau^3 |\wedge_{\tau}^{-s} v|^2 + \tau |\nabla(\wedge_{\tau}^{-s} v)|^2] dx dt \\
& \leq C_{22} \int_{-T}^T \int_{\Omega_{2\delta_1}} |\wedge_{\tau}^{-s} P_{\phi} v|^2 dx dt + C_{22} \tau^{4-4s} \int_Q |v|^2 dx dt + C_{22} \tau^{2-2s} \int_{-T}^T \int_{\mathbb{R}^n} |\nabla(\wedge_{\tau}^{-s} v)|^2 dx dt \\
& \quad + C_{22} \int_{-T}^T \int_{\mathbb{R}^n \setminus \Omega_{\delta_1}} [\tau^3 |\wedge_{\tau}^{-s} v|^2 + \tau |\nabla(\wedge_{\tau}^{-s} v)|^2] dx dt. \tag{2.12}
\end{aligned}$$

Now we eliminate the last integral. From Lemma 2.2, we have

$$\begin{aligned}
|\partial_x^{\alpha} \wedge_{\tau}^{-s} v(x, t)| & \leq (2\pi)^{-n} \int_{\Omega} |v(y, t)| |\partial_x^{\alpha} K(x, y; \tau)| dy \\
& \leq C_{23}(\delta_1) \tau^{-2s} \int_{\Omega} |x - y|^{-2n-3} |v(y, t)| dy, \quad |\alpha| \leq 1. \tag{2.13}
\end{aligned}$$

Here we take  $\mu = \frac{1}{2} \text{dist}(\partial\Omega_{\delta_1}, \Omega) = \frac{1}{2} \delta_1 > 0$  when we apply Lemma 2.1. Since  $\Omega$  is bounded, we can choose a constant  $C_{24} > 1$  such that  $|y| < \frac{\delta_1 C_{24}}{2}$  for  $y \in \Omega$ . Since

$$|x - y| = \frac{1}{2}|x - y| + \frac{1}{2}|x - y| \geq \frac{\delta_1}{2} + \frac{1}{2C_{24}}|x| - \frac{1}{2C_{24}}|y| \geq \frac{\delta_1}{4} + \frac{1}{2C_{24}}|x| \geq \frac{1 + |x|}{C_{25}},$$

for  $x \in \mathbb{R}^n \setminus \Omega_{\delta_1}$ ,  $y \in \Omega$ , we have

$$|\partial_x^{\alpha} \wedge_{\tau}^{-s} v(x, t)| \leq C_{26} \tau^{-2s} (1 + |x|)^{-2n-3} \left( \int_{\Omega} |v|^2 dx \right)^{\frac{1}{2}}, \quad |\alpha| \leq 1. \tag{2.14}$$

Consequently,

$$\begin{aligned}
\int_{-T}^T \int_{\mathbb{R}^n \setminus \Omega_{\delta_1}} |\partial_x^{\alpha} \wedge_{\tau}^{-s} v(x, t)|^2 dx dt & \leq C_{27} \tau^{-4s} \int_{\mathbb{R}^n \setminus \Omega_{\delta_1}} (1 + |x|)^{-4n-6} dx \int_{-T}^T \int_{\Omega} |v|^2 dx dt \\
& \leq C_{28} \tau^{-4s} \int_Q |v|^2 dx dt, \quad |\alpha| \leq 1. \tag{2.15}
\end{aligned}$$

From (2.12), (2.14) and (2.15), we have

$$\begin{aligned}
& \tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 dx dt + \tau \int_Q |v|^2 dx dt \\
& \leq C_{29} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} P_{\phi} v|^2 dx dt + C_{29} \tau^{4-4s} \int_Q |v|^2 dx dt. \tag{2.16}
\end{aligned}$$

Since  $4 - 4s < 1$ , the last term on the right-hand side of (2.16) can be absorbed by the left-hand side if  $\tau > 0$  large enough. Then we have

$$\tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 dx dt + \tau \int_Q |v|^2 dx dt \leq C_{29} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_{\tau}^{-s} P_{\phi} v|^2 dx dt \tag{2.17}$$

by choosing  $\tau > 0$  large enough. By the definitions of  $\wedge_0^{1-s}$  and  $\wedge_\tau^{-s}$ , we see that

$$\begin{cases} \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 dx dt = \int_{-T}^T \|v\|_{H^{1-s}(\mathbb{R}^n)}^2 dt, \\ \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_\tau^{-s} P_\phi v|^2 dx dt \leq C_{30} \int_{-T}^T \|P_\phi v\|_{H^{-s}(\mathbb{R}^n)}^2 dt. \end{cases} \quad (2.18)$$

Consequently, from (2.17) and (2.18), we have

$$\tau \int_Q |v|^2 dx dt + \tau \int_{-T}^T \|v\|_{H^{1-s}(\mathbb{R}^n)}^2 dt \leq C_{31} \int_{-T}^T \|P_\phi v\|_{H^{-s}(\mathbb{R}^n)}^2 dt. \quad (2.19)$$

By taking  $v = ue^{\tau\phi}$ , we complete the proof of Lemma 2.3.

**Proof of Theorem 1.1** We divide the proof of Theorem 1.1 into two steps.

**Step 1** We first prove that (1.5) holds for all  $u \in C^2(Q)$  with compact support in  $\Omega^0 \times (-T_1, T_1)$ . Thanks to Lemma 2.3, it suffices to prove that there exists a constant  $C_{32} > 0$  such that

$$\|v_1\|_{H^{-s}(\mathbb{R}^n)} \leq C_{32} \|v_1\|_{H^{-s}(\Omega)} \quad (2.20)$$

for all  $v_1 \in C(\Omega)$  with compact support in  $\Omega^0$ .

Let  $\kappa = \text{dist}(\partial\Omega, \partial\Omega^0)$ . We choose  $\chi_1 \in C_0^\infty(\Omega)$  such that

$$\chi_1 = \begin{cases} 1, & x \in \left\{ x \in \Omega; \text{dist}(x, \partial\Omega) \geq \frac{2}{3}\kappa \right\}, \\ 0, & x \in \left\{ x \in \Omega; \text{dist}(x, \partial\Omega) < \frac{1}{3}\kappa \right\}, \end{cases}$$

and  $\|\nabla\chi_1\|_{L^\infty(\Omega)} \leq C_{33}\kappa^{-1}$ . For every  $\varrho \in H^s(\mathbb{R}^n)$  satisfying  $\|\varrho\|_{H^s(\mathbb{R}^n)} = 1$ , we set  $\varrho_1 = \frac{\chi_1 \varrho}{\|\chi_1 \varrho\|_{H_0^s(\Omega)}}$ . Then  $\varrho_1 \in H_0^s(\Omega)$  and  $\|\varrho_1\|_{H_0^s(\Omega)} = 1$ . Hence we have

$$\begin{aligned} H^{-s}(\mathbb{R}^n) \langle v_1, \varrho \rangle_{H^s(\mathbb{R}^n)} &=_{H^{-s}(\mathbb{R}^n)} \langle v_1, \chi_1 \varrho \rangle_{H^s(\mathbb{R}^n)} \\ &= \|\chi_1 \varrho\|_{H_0^s(\Omega)} \langle v_1, \varrho_1 \rangle_{H_0^s(\Omega)} \leq C_{34}(1 + \kappa^{-1}) \langle v_1, \varrho_1 \rangle_{H_0^s(\Omega)} \end{aligned}$$

for all  $v_1 \in C(\Omega)$  with compact support in  $\Omega^0$ . Therefore

$$\begin{aligned} \|v_1\|_{H^{-s}(\mathbb{R}^n)} &= \sup_{\|\varrho\|_{H^s(\mathbb{R}^n)}=1} |H^{-s}(\mathbb{R}^n) \langle v_1, \varrho \rangle_{H^s(\mathbb{R}^n)}| \\ &\leq C_{35}(1 + \kappa^{-1}) \sup_{\|\varrho_1\|_{H_0^s(\Omega)}=1} |H^{-s}(\Omega) \langle v_1, \varrho_1 \rangle_{H^s(\Omega)}| \\ &= C_{35}(1 + \kappa^{-1}) \|v_1\|_{H^{-s}(\Omega)} \end{aligned}$$

for all  $v_1 \in C(\Omega)$  with compact support in  $\Omega^0$ . Thus (2.20) is proved.

**Step 2** We finish the proof of Theorem 1.1 by an argument based on approximation. Let  $\varsigma \in C_0^\infty(\mathbb{R}^{n+1})$  be a mollifier such that  $\text{supp } \varsigma \subset \{(x, t); |x|^2 + t^2 < 1\}$ ,  $\varsigma \geq 0$  and  $\int_{\mathbb{R}^{n+1}} \varsigma(x, t) dx dt = 1$ . We set  $\varsigma_\epsilon(x, t) = \epsilon^{-n-1} \varsigma(\frac{x}{\epsilon}, \frac{t}{\epsilon})$ ,  $\epsilon > 0$ . Let  $(\varsigma_\epsilon * u)(x, t)$  be the convolution of  $\varsigma_\epsilon$  and  $u$  (see e.g., [9, Chapter I]). Then we have  $\varsigma_\epsilon * u \in C^\infty(\mathbb{R}^{n+1})$  and  $\text{supp}(\varsigma_\epsilon * u) \subset \text{supp } \varsigma_\epsilon + \text{supp } u$ . By choosing

$\epsilon > 0$  sufficiently small, we see that  $\varsigma_\epsilon * u \in C^\infty(Q)$  with support in  $\Omega_\epsilon^0 \times (-T_1 - \epsilon, T_1 + \epsilon) \subset Q$ . Here  $\Omega_\epsilon^0 = \{x \in \mathbb{R}^n; \text{dist}(x, \Omega^0) < \epsilon\}$ . By the result of Step 1, we have

$$\tau \int_{-T}^T \|(\varsigma_\epsilon * u)e^{\tau\phi}\|_{H^{1-s}(\Omega)}^2 dt \leq C_{35} \int_{-T}^T \|e^{\tau\phi} P(\varsigma_\epsilon * u)\|_{H^{-s}(\Omega)}^2 dt. \quad (2.21)$$

By an argument similar to the proof of Friedrichs' lemma (P.9) in [10], we obtain

$$\int_{-T}^T \|e^{\tau\phi} P(\varsigma_\epsilon * u) - e^{\tau\phi} [\varsigma_\epsilon * (Pu)]\|_{H^{-s}(\Omega)}^2 dt \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Moreover, since it is easily verified that

$$\int_{-T}^T \|e^{\tau\phi} [\varsigma_\epsilon * (Pu)] - e^{\tau\phi} Pu\|_{H^{-s}(\Omega)}^2 dt \rightarrow 0, \quad \epsilon \rightarrow 0,$$

we have

$$\int_{-T}^T \|e^{\tau\phi} P(\varsigma_\epsilon * u) - e^{\tau\phi} Pu\|_{H^{-s}(\Omega)}^2 dt \rightarrow 0, \quad \epsilon \rightarrow 0.$$

On the other hand, we have

$$\int_{-T}^T \|(\varsigma_\epsilon * u)e^{\tau\phi} - ue^{\tau\phi}\|_{H^{1-s}(\Omega)}^2 dt \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Thus, letting  $\epsilon \rightarrow 0$  in (2.21), we complete the proof of Theorem 1.1.

**Proof of Corollary 1.2** By virtue of Theorem 1.1, we see that if  $\zeta_1 ue^{\tau\phi} \in L^2(-T, T; H^{-s}(\Omega))$ , then

$$\begin{aligned} & \tau \|ue^{\tau\phi}\|_{L^2(-T, T; H^{1-s}(\Omega))}^2 \\ & \leq C_{36} \|e^{\tau\phi}(P + \zeta_1 - \zeta_1)u\|_{L^2(-T, T; H^{-s}(\Omega))}^2 \\ & \leq 2C_{36} (\|e^{\tau\phi}(P + \zeta_1)u\|_{L^2(-T, T; H^{-s}(\Omega))}^2 + \|\zeta_1 ue^{\tau\phi}\|_{L^2(-T, T; H^{-s}(\Omega))}^2) \end{aligned} \quad (2.22)$$

for all  $u \in L^2(Q)$  with compact support in  $\Omega^0 \times (-T_1, T_1)$  such that  $Pu \in L^2(-T, T; H^{-s}(\Omega))$ .

Next we estimate the last term on the right-hand side. We see that

$$\begin{aligned} \int_{-T}^T {}_{H^{-s}(\Omega)} \langle \zeta_1 ue^{s\phi}, \mu \rangle_{H^s(\Omega)} dt &= \int_{-T}^T {}_{W_{p'}^{-\gamma}(\Omega)} \langle \zeta_1, ue^{s\phi} \mu \rangle_{W_p^\gamma(\Omega)} dt \\ &\leq \int_{-T}^T \|\zeta_1\|_{W_{p'}^{-\gamma}(\Omega)} \|ue^{s\phi} \mu\|_{W_p^\gamma(\Omega)} dt, \end{aligned} \quad (2.23)$$

if  $ue^{s\phi} \mu \in W_p^\gamma(\Omega)$ . From the corollary in [39, p. 189], we know that if  $n \geq 2$ ,  $0 < \gamma < 1 - s$  and  $1 - \gamma > n(1 - \frac{1}{p})$ , then

$$H^{1-s}(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) \subset W_p^\gamma(\mathbb{R}^n)$$

for  $\frac{3}{4} < s < 1$ . By using an argument of taking the 0-extension of  $\mu$  and  $ue^{s\phi}$  outside  $\Omega \times (-T, T)$ , it is easy to prove

$$\|ue^{s\phi} \mu\|_{W_p^\gamma(\Omega)} \leq \|ue^{s\phi}\|_{H^{1-s}(\Omega)} \|\mu\|_{H_0^s(\Omega)}. \quad (2.24)$$

From (2.23) and (2.24), we obtain

$$\begin{aligned}
& \int_{-T}^T H^{-s}(\Omega) \langle \zeta_1 u e^{s\phi}, \mu \rangle_{H_0^s(\Omega)} dt \\
& \leq \int_{-T}^T \|\zeta_1\|_{W_{p'}^{-\gamma}(\Omega)} \|u e^{s\phi}\|_{H^{1-s}(\Omega)} \|\mu\|_{H_0^s(\Omega)} dt \\
& \leq \|\zeta_1\|_{L^\infty(-T, T; W_{p'}^{-\gamma}(\Omega))} \|u e^{s\phi}\|_{L^2(-T, T; H^{1-s}(\Omega))} \|\mu\|_{L^2(-T, T; H_0^s(\Omega))}. \tag{2.25}
\end{aligned}$$

Inequality (2.25) means  $\zeta_1 u e^{\tau\phi} \in L^2(-T, T; H^{-s}(\Omega))$  and

$$\|\zeta_1 u e^{\tau\phi}\|_{L^2(-T, T; H^{-s}(\Omega))} \leq \|\zeta_1\|_{L^\infty(-T, T; W_{p'}^{-\gamma}(\Omega))} \|u e^{s\phi}\|_{L^2(-T, T; H^{1-s}(\Omega))}. \tag{2.26}$$

From (2.22) and (2.26), we obtain (1.6) by taking  $\tau$  large enough. Thus the proof of Corollary 1.2 is completed.

### 3 Proof of Theorem 1.3

**Proof of Theorem 1.3** Let  $q(x) = q_1(x) - q_2(x)$ ,  $y_1(x, t) = y(q_1, b, y_0, y_\Sigma)(x, t)$ ,  $y_2(x, t) = y(q_2, b, y_0, y_\Sigma)(x, t)$  and  $y(x, t) = y_2(x, t) - y_1(x, t)$ . Then we have

$$\begin{cases} i\partial_t y(x, t) + \operatorname{div}(b(x)\nabla y(x, t)) - q_2(x)y(x, t) = qy_1, & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(\cdot, 0) = 0, & \text{in } \Omega. \end{cases} \tag{3.1}$$

Since  $|y_0(x)| > c > 0$ ,  $x \in \overline{\Omega(\delta_0)}$  and  $y_1 \in C(\overline{\Omega(\delta_0)} \times (-T, T))$ , there exist constants  $\epsilon_1 > 0$  and  $c_1 > 0$  such that

$$y_1(x, t) > c_1 > 0, \quad (x, t) \in \overline{\Omega(\delta_0)} \times [-\epsilon_1, \epsilon_1].$$

It is easy to see that

$$\frac{i\partial_t y + \operatorname{div}(b(x)\nabla y(x, t)) - q_2(x)y}{y_1} = q(x), \quad \text{in } \Omega(\delta_0) \times (-\epsilon_1, \epsilon_1). \tag{3.2}$$

Because  $q(x)$  is independent of  $t$ , the differentiation of both sides of (3.2) with respect to  $t$  eliminates  $q(x)$ . Hence

$$\begin{aligned}
& \partial_t [i\partial_t y + \operatorname{div}(b(x)\nabla y(x, t)) - q_2(x)y(x, t)] \\
& = \frac{\partial_t y_1}{y_1} [i\partial_t y + \operatorname{div}(b(x)\nabla y(x, t)) - q_2(x)y(x, t)], \quad \text{in } (\mathcal{D}(\Omega(\delta_0) \times (-\epsilon_1, \epsilon_1)))'. \tag{3.3}
\end{aligned}$$

We set

$$h_1(x, t) = \frac{\partial_t y_1(x, t)}{y_1(x, t)}, \quad h(x, t) = \partial_t y(x, t) - h_1(x, t)y(x, t).$$

Then we have

$$y(x, t) = \int_0^t \frac{y_1(x, \theta)}{y_1(x, \theta)} \cdot h(x, \theta) d\theta. \tag{3.4}$$

Consequently, we have

$$i\partial_t h + \operatorname{div}(b(x)\nabla h) - q_2 h = \int_0^t \left( \sum_{j=1}^n K_{1j}(x, t, \theta) h(x, \theta) + \sum_{j=1}^n K_{2j}(x, t, \theta) \partial_j h(x, \theta) \right) d\theta, \tag{3.5}$$

in  $(\mathcal{D}(\Omega(\delta_0) \times (-\epsilon_1, \epsilon_1)))'$ , where

$$\begin{aligned} K_{1j}(x, t, \theta) &= 2b(x)(\partial_j h_1) \partial_j \left( \frac{y_1(x, t)}{y_1(x, \theta)} \right) + (i\partial_t h_1 + \operatorname{div}(b(x)\nabla h_1)) \frac{y_1(x, t)}{y_1(x, \theta)}, \\ K_{2j}(x, t, \theta) &= 2b(x)(\partial_j h_1) \frac{y_1(x, t)}{y_1(x, \theta)}. \end{aligned}$$

Let  $\delta > 0$  be a constant. Define

$$Q(\delta) = \{(x, t) \in Q; |x - x_0|^2 - \beta t^2 > \delta^2\}.$$

Here we choose a constant  $\beta > 0$  such that  $Q(\delta_0) \subset \Omega \times (-\epsilon_1, \epsilon_1)$ . We introduce a cut-off function  $\chi_2 \in C^\infty(Q)$  such that

$$\chi_2(x, t) = \begin{cases} 1, & (x, t) \in Q(\delta_0 + 3\delta_1), \\ 0, & (x, t) \in Q \setminus \overline{Q(\delta_0 + 2\delta_1)}, \end{cases} \quad (3.6)$$

where  $\delta_1 > 0$  is a small constant.

Let  $z = \chi_2 h$ . By (3.6) and (1.13), we know that  $z \in L^2(\Omega(\delta_0 + \delta_1) \times (-T, T))$  with  $\operatorname{supp} z \subset (\Omega(\delta_0 + 2\delta_1) \setminus \omega) \times (-\epsilon_1, \epsilon_1)$ .

From (3.5), we have

$$\begin{aligned} & i\partial_t z + \operatorname{div}(b(x)\nabla z) - \int_0^t \sum_{j=1}^n K_{2j}(x, t, \theta) \partial_j z(x, \theta) d\theta \\ &= q_2 z + \chi_2 \int_0^t \sum_{j=1}^n K_{1j}(x, t, \theta) h(x, \theta) d\theta - \int_0^t \sum_{j=1}^n K_{2j}(x, t, \theta) h(x, \theta) \partial_j \chi_2(x, \theta) d\theta \\ & \quad + i(\partial_t \chi_2) h + b(\Delta \chi_2) h + 2b\nabla \chi_2 \cdot \nabla h + h\nabla b \cdot \nabla \chi_2, \quad \text{in } (\mathcal{D}(\Omega(\delta_0) \times (-\epsilon_1, \epsilon_1)))'. \end{aligned} \quad (3.7)$$

By noting also (1.12), we apply Corollary 1.1 to  $z$ . Then we obtain

$$\begin{aligned} & \tau \int_{-T}^T \int_{\Omega(\delta_0 + \delta_1)} |z|^2 e^{2\tau\phi_1} dx dt \\ & \leq \int_{-T}^T \left\| q_2 z e^{\tau\phi_1} + \int_0^t \sum_{j=1}^n K_{1j}(\cdot, t, \theta) \chi_2(\cdot, \theta) h(\cdot, \theta) e^{\tau\phi_1} d\theta \right\|_{H^{-1}(\Omega(\delta_0 + \delta_1))}^2 dt \\ & \quad + \int_{-T}^T \left\| \int_0^t \sum_{j=1}^n K_{2j}(\cdot, t, \theta) h(\cdot, \theta) (\partial_j \chi_2)(\cdot, \theta) e^{2\tau\phi_1} d\theta \right\|_{H^{-1}(\Omega(\delta_0 + \delta_1))}^2 dt \\ & \quad + \int_{-T}^T \|i(\partial_t \chi_2) h e^{2\tau\phi_1} + b(\Delta \chi_2) h e^{2\tau\phi_1} + 2b(\nabla \chi_2 \cdot \nabla h) e^{2\tau\phi_1} \\ & \quad + h(\nabla b \cdot \nabla \chi_2) e^{2\tau\phi_1}\|_{H^{-1}(\Omega(\delta_0 + \delta_1))}^2 dt \end{aligned} \quad (3.8)$$

for large  $\lambda$ ,  $\tau > 0$ . By (1.12), (3.6) and  $q_2 \in L^p(\Omega)$  with  $p > \max\{n, 2\}$ , we have

$$\begin{aligned} & \tau \int_{-T}^T \int_{\Omega(\delta_0 + \delta_1)} |z|^2 e^{2\tau\phi_1} dx dt \\ & \leq \int_{-T}^T \left\| \int_0^t \sum_{j=1}^n K_{1j}(x, t, \theta) \chi_2(x, \theta) h(x, \theta) e^{\tau\phi_1} d\theta \right\|_{L^2(\Omega(\delta_0 + \delta_1))}^2 dt \end{aligned}$$

$$\begin{aligned}
& + \int_{-T}^T \|q_2\|_{L^p(\Omega)}^2 \|ze^{\tau\phi_1}\|_{L^2(\Omega(\delta_0+\delta_1))}^2 dt + C_2\tau e^{2\tau(\delta_0+3\delta_1)} \\
& \leq C_3 \int_{-T}^T \int_{\Omega(\delta_0+\delta_1)} |z|^2 e^{2\tau\phi_1} dx dt + C_4\tau e^{2\tau(\delta_0+3\delta_1)}. \tag{3.9}
\end{aligned}$$

By taking  $\tau$  large enough, the first term on the right-hand side of (3.9) can be absorbed by the term on the left-hand side. Then we have

$$\tau \int_{-T}^T \int_{\Omega(\delta_0+\delta_1)} |h\chi_2|^2 e^{2\tau\phi_1} dx dt \leq C_5\tau e^{2\tau(\delta_0+3\delta_1)}$$

for large  $\tau > 0$ . By (3.6), we obtain

$$\int_{Q(\delta_0+4\delta_1)} |h|^2 dx dt \leq C_6 e^{-2\tau\delta_1}$$

for large  $\tau > 0$ . As  $\tau \rightarrow +\infty$ , we have  $\int_{Q(\delta_0+4\delta_1)} |h|^2 dx dt = 0$ . This implies that  $h(x, t) = 0$ ,  $(x, t) \in Q(\delta_0 + 4\delta_1)$ . By (3.4), we have  $y(x, t) = 0$  in  $Q(\delta_0 + 4\delta_1)$ . Consequently, we have  $q(x) = 0$  in  $Q(\delta_0 + 4\delta_1)$  by noting (3.2). Since  $\delta_1$  is arbitrary,  $q_2(x) = q_1(x)$ ,  $x \in \Omega(\delta_0)$ . Thus the proof of Theorem 1.3 is completed.

#### 4 Proof of Theorem 1.4

**Proof of Theorem 1.4** Let  $d_0 = \inf_{x \in \Omega} \exp\{|x - x_0|^2\}$ . We choose  $\beta > 0$  such that  $\sup_{x \in \Omega} |x - x_0|^2 < T^2\beta$ . For  $\lambda > 1$ , we set  $\phi_1(x, t) = \exp\{\lambda(|x - x_0|^2 - \beta t^2)\}$ . Then we have

$$\phi_1(x, 0) \geq d_0, \quad \phi_1(x, -T) = \phi_1(x, T) < d_0, \quad x \in \overline{\Omega}.$$

Thus, for given  $\epsilon > 0$ , we choose a sufficiently small  $\delta_1 = \delta_1(\epsilon) > 0$  such that

$$\phi_1(x, t) \geq d_0 - \epsilon, \quad (x, t) \in \overline{\Omega} \times [-\delta_1, \delta_1], \tag{4.1}$$

$$\phi_1(x, t) \leq d_0 - 2\epsilon, \quad (x, t) \in \overline{\Omega} \times ([-T, -T + 2\delta_1] \cup [T - 2\delta_1, T]). \tag{4.2}$$

We introduce a cut-off function  $\chi_3$  satisfying  $0 \leq \chi_3 \leq 1$ ,  $\chi_3 \in C^\infty[0, T]$  and

$$\chi_3(t) = \begin{cases} 0, & t \in [-T, -T + \delta_1] \cup [T - \delta_1, T], \\ 1, & t \in [-T + 2\delta_1, T - 2\delta_1]. \end{cases} \tag{4.3}$$

Let  $\Omega^0 \subset \omega$  satisfy  $\partial\Omega^0 \supset \partial\Omega$ . We introduce another cut-off function  $\chi_4$  satisfying  $0 \leq \chi_4 \leq 1$ ,  $\chi_4 \in C_0^\infty(\Omega)$  and

$$\chi_4(x) = \begin{cases} 0, & x \in \Omega^0, \\ 1, & x \in \Omega \setminus \omega. \end{cases} \tag{4.4}$$

Multiply the first equation in (1.14) by  $\overline{\eta}$ , take the imaginary parts and integrate by parts. Then we have

$$\|\eta(\cdot, t)\|_{L^2(\Omega)} = \|\eta(\cdot, 0)\|_{L^2(\Omega)}, \quad t \in [0, T]. \tag{4.5}$$



Consequently, we have

$$\int_{-T}^T \int_{\omega} |\eta|^2 dx dt \leq C_1 \|\eta_0\|_{L^2(\Omega)}^2.$$

By a density argument, we can see that in order to prove Theorem 1.4 it suffices to prove that (1.15) holds for  $\eta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ . Then we have

$$\eta \in C([-T, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([-T, T]; L^2(\Omega)). \quad (4.6)$$

Let  $z = \chi_3 \chi_4 \eta$ . Then by (4.3)–(4.6), we have

$$z \in H^1(Q), \quad \text{supp } z \subset (\Omega \setminus \omega) \times (-T + 2\delta_1, T - 2\delta_1). \quad (4.7)$$

Since  $i\partial_t z + \text{div}(b(x)\nabla z) \in L^2(Q)$ , we apply Corollary 1.1 to  $z$  for the case when  $s = 1$ . Then we have

$$\begin{aligned} \tau \int_Q |ze^{\tau\phi_1}|^2 dx dt &\leq C_2 \int_{-T}^T \|e^{\tau\phi_1}[i\partial_t z + \text{div}(b(x)\nabla z)]\|_{H^{-1}(\Omega)}^2 dt \\ &= C_2 \int_{-T}^T \|e^{\tau\phi_1}\{\chi_3 \chi_4 [i\partial_t \eta + \text{div}(b\nabla \eta)] + i\chi_4 \eta \partial_t \chi_3 \\ &\quad + 2\chi_3 \text{div}(b\eta \nabla \chi_4) - \chi_3 \eta \nabla b \cdot \nabla \chi_4 - \chi_3 b \eta \Delta \chi_4\}\|_{H^{-1}(\Omega)}^2 dt \\ &= C_2 \int_{-T}^T \|e^{\tau\phi_1}\{\chi_3 \chi_4 q \eta + i\chi_4 \eta \partial_t \chi_3 + 2\chi_3 \text{div}(b\eta \nabla \chi_4) \\ &\quad - \chi_3 \eta \nabla b \cdot \nabla \chi_4 - \chi_3 b \eta \Delta \chi_4\}\|_{H^{-1}(\Omega)}^2 dt, \end{aligned} \quad (4.8)$$

provided that  $\lambda > 1$ ,  $\tau > 1$  are large enough. By noting that  $\partial_t \chi_3 \neq 0$  only in the case where  $\phi_1(t, x) \leq d_0 - 2\epsilon$ , and  $\partial_j \chi_4$ ,  $1 \leq j \leq n$ , are supported in  $\omega$ , we have

$$\begin{aligned} \tau \int_Q |ze^{\tau\phi_1}|^2 dx dt &\leq C_3 \int_{-T}^T \|q\|_{L^p(\Omega)}^2 \|ze^{\tau\phi_1}\|_{L^2(\Omega)}^2 dt + C_3 e^{2\tau(d_0-2\epsilon)} \int_Q |\chi_4 \eta|^2 dx dt \\ &\quad + C_3 e^{2\tau\Phi} \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt + C_3 \int_{-T}^T \|e^{\tau\phi_1} 2b\chi_3 \text{div}(\eta \nabla \chi_4)\|_{H^{-1}(\Omega)}^2 dt, \end{aligned} \quad (4.9)$$

where  $\Phi = \sup_{(x,t) \in Q} \phi_1(x, t)$ . The first term on the right-hand side of (4.9) can be absorbed by the left-hand term when we choose  $\tau > 1$  large enough. Thus we have

$$\begin{aligned} \tau \int_Q |ze^{\tau\phi_1}|^2 dx dt &\leq C_4 e^{2\tau(d_0-2\epsilon)} \int_Q |\chi_4 \eta|^2 dx dt + C_4 e^{2\tau\Phi} \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt \\ &\quad + C_4 \int_{-T}^T \|e^{\tau\phi_1} 2\chi_3 \text{div}(b\eta \nabla \chi_4)\|_{H^{-1}(\Omega)}^2 dt. \end{aligned} \quad (4.10)$$

By noting that  $e^{\tau\phi_1} 2\chi_3 \text{div}(b\eta \nabla \chi_4) = 2 \sum_{j=1}^n \partial_j (e^{\tau\phi_1} b \chi_3 \eta \partial_j \chi_4) - 2e^{\tau\phi_1} \tau b \chi_3 \eta (\nabla \phi_1 \cdot \nabla \chi_4)$ , we can obtain

$$\tau \int_Q |ze^{\tau\phi_1}|^2 dx dt \leq C_5 e^{2\tau(d_0-2\epsilon)} \int_Q |\chi_4 \eta|^2 dx dt + C_5 (1 + \tau) e^{2\tau\Phi} \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt. \quad (4.11)$$

Noting  $z = \chi_3 \chi_4 \eta$ , (4.1) and (4.3), we can obtain

$$\begin{aligned} & \tau e^{2\tau(d_0-\epsilon)} \int_{-\delta_1}^{\delta_1} \int_{\Omega} |\chi_4 \eta|^2 dx dt \\ & \leq C_6 e^{2\tau(d_0-2\epsilon)} \int_Q |\eta|^2 dx dt + C_6 (1+\tau) e^{2\tau\Phi} \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt. \end{aligned} \quad (4.12)$$

By (4.4), we have

$$\begin{aligned} & \tau e^{2\tau(d_0-\epsilon)} \int_{-\delta_1}^{\delta_1} \int_{\Omega \setminus \omega} |\eta|^2 dx dt \\ & \leq C_7 e^{2\tau(d_0-2\epsilon)} \int_Q |\eta|^2 dx dt + C_7 (1+\tau) e^{2\tau\Phi} \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt. \end{aligned} \quad (4.13)$$

Consequently, we have

$$\begin{aligned} & \tau e^{2\tau(d_0-\epsilon)} \int_{-\delta_1}^{\delta_1} \int_{\Omega} |\eta|^2 dx dt \\ & \leq C_8 e^{2\tau(d_0-2\epsilon)} \int_Q |\eta|^2 dx dt + C_8 [(1+\tau) e^{2\tau\Phi} + \tau e^{2\tau(d_0-\epsilon)}] \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt \end{aligned} \quad (4.14)$$

for large  $\tau > 1$ . By noting (4.5) we obtain

$$\begin{aligned} & 2\tau \delta_1 e^{2\tau(d_0-\epsilon)} \left(1 - \frac{C_9 T \delta_1^{-1}}{\tau e^{2\tau\epsilon}}\right) \|\eta_0\|_{L^2(\Omega)}^2 dx dt \\ & \leq C_9 [(1+\tau) e^{2\tau\Phi} + \tau e^{2\tau(d_0-\epsilon)}] \int_{-T}^T \|\eta\|_{L^2(\omega)}^2 dt. \end{aligned} \quad (4.15)$$

Taking  $\tau > \max\{1, \frac{\log 2C_9 T \delta_1^{-1}}{2\epsilon}\}$ , we have

$$1 - \frac{C_9 T \delta_1^{-1}}{\tau e^{2\tau\epsilon}} > \frac{1}{2}.$$

Thus, by (4.15), we complete the proof of Theorem 1.4.

## Appendix

**Proof of Proposition 2.1** We first prove (2.1). Let  $a(x, t) = \frac{1}{g(x, t)}$ ,  $P^0 v = ia\partial_t v + \Delta v$ ,  $w = ve^{\tau\phi}$  and  $Lw = e^{\tau\phi} P^0(e^{-\tau\phi} w)$ . Then we have

$$\begin{aligned} Lw + i\tau(\partial_t \phi)aw &= L_1 w + L_2 w, \\ L_1 w &= ia\partial_t w + \Delta w + \tau^2 |\nabla \phi|^2 w, \\ L_2 w &= -2\tau \nabla \phi \cdot \nabla w - \tau w \Delta \phi. \end{aligned} \quad (A.1)$$

Then

$$\int_Q |Lw + i\tau(\partial_t \phi)aw|^2 dx dt = \int_Q |L_1 w|^2 dx dt + \int_Q |L_2 w|^2 dx dt + 2\operatorname{Re} \int_Q L_1 w \overline{L_2 w} dx dt. \quad (A.2)$$

We calculate the last term:

$$\begin{aligned}
2\operatorname{Re} \int_Q L_1 w \overline{L_2 w} dx dt &= -4\tau \operatorname{Re} \int_Q i(\partial_t w) a \nabla \phi \cdot \nabla \overline{w} dx dt - 4\tau \operatorname{Re} \int_Q (\Delta w) \nabla \phi \cdot \nabla \overline{w} dx dt \\
&\quad - 2\tau \operatorname{Re} \int_Q i a \overline{w} (\partial_t w) \Delta \phi dx dt - 2\tau \operatorname{Re} \int_Q \overline{w} (\Delta w) \Delta \phi dx dt \\
&\quad - 4\tau^3 \operatorname{Re} \int_Q |\nabla \phi|^2 w \nabla \phi \cdot \nabla \overline{w} dx dt - 2\tau^3 \int_Q |\nabla \phi|^2 (\Delta \phi) |w|^2 dx dt \\
&= \sum_{j=1}^6 I_j. \tag{A.3}
\end{aligned}$$

Integrating by parts and using  $w|_{\partial\Omega} = 0$  and (A.1), we have

$$\begin{aligned}
I_1 &= -2\tau \operatorname{Im} \int_Q \{(-2\lambda^2 \beta t) a \phi (\nabla d \cdot \nabla \overline{w}) w - a \partial_t \overline{w} (\lambda \phi \Delta d + \lambda^2 \phi |\nabla d|^2) w\} dx dt \\
&\quad + 2\tau \operatorname{Re} \int_Q \lambda \phi (\nabla \log a \cdot \nabla d) w L_1 \overline{w} dx dt - 2\tau \operatorname{Im} \int_Q (\partial_t a) w \nabla \overline{w} \cdot \nabla \phi dx dt \\
&\quad + 2\tau \operatorname{Re} \int_Q [\lambda \phi w \nabla (\nabla \log a \cdot \nabla d) \cdot \nabla \overline{w} + \lambda^2 \phi (\nabla \log a \cdot \nabla d) (\nabla d \cdot \nabla \overline{w}) w] dx dt \\
&\quad + 2\tau \int_Q \lambda \phi (\nabla \log a \cdot \nabla d) |\nabla w|^2 dx dt - 2\tau^3 \int_Q \lambda^3 \phi^3 (\nabla \log a \cdot \nabla d) |\nabla d|^2 |w|^2 dx dt, \\
I_2 &= 2\tau \int_Q -(\lambda \phi \Delta d + \lambda^2 \phi |\nabla d|^2) |\nabla w|^2 dx dt \\
&\quad + 4\tau \operatorname{Re} \int_Q \sum_{j,k=1}^n [\lambda \phi \partial_j \partial_k d + \lambda^2 \phi (\partial_j d) \partial_k d] (\partial_j w) \partial_k \overline{w} \\
&\quad + 2\tau \int_{-T}^T \int_{\partial\Omega} \lambda \phi \frac{\partial d}{\partial \nu} |\nabla w|^2 d\Sigma - 4\tau \operatorname{Re} \int_{-T}^T \int_{\partial\Omega} \lambda \phi (\nabla d \cdot \nabla \overline{w}) \frac{\partial w}{\partial \nu} dx dt, \\
I_3 &= -2\tau \operatorname{Im} \int_Q a w (\partial_t \overline{w}) (\lambda \phi \Delta d + \lambda^2 \phi |\nabla d|^2) dx dt, \\
I_4 &= 2\tau \int_Q (\lambda \phi \Delta d + \lambda^2 \phi |\nabla d|^2) |\nabla w|^2 dx dt - \tau \lambda \int_Q \phi (\Delta^2 d) |w|^2 dx dt \\
&\quad - \tau \lambda^2 \int_Q \phi (|\Delta d|^2 + 2\nabla d \cdot \nabla (\Delta d) + \Delta (|\nabla d|^2)) |w|^2 dx dt \\
&\quad - 2\tau \lambda^3 \int_Q \phi (|\nabla d|^2 \Delta d + \nabla d \cdot \nabla (|\nabla d|^2)) |w|^2 dx dt - \tau \lambda^4 \int_Q \phi |\nabla d|^4 |w|^2 dx dt, \\
I_5 &= 2\tau^3 \lambda^3 \int_Q \phi^3 (|\nabla d|^2 \Delta d + \nabla d \cdot \nabla (|\nabla d|^2)) |w|^2 dx dt + 6\tau^3 \lambda^4 \int_Q \phi^3 |\nabla d|^4 |w|^2 dx dt, \\
I_6 &= -2\tau^3 \lambda^3 \int_Q \phi^3 |\nabla d|^2 (\Delta d) |w|^2 dx dt - 2\tau^3 \lambda^4 \int_Q \phi^3 |\nabla d|^4 |w|^2 dx dt.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&2\operatorname{Re} \int_Q L_1 w \overline{L_2 w} dx dt \\
&= 2\tau \lambda \int_Q \phi \left[ |\nabla w|^2 \nabla \log a \cdot \nabla d + 2 \sum_{j,k=1}^n (\partial_j \partial_k d) (\partial_j w) \partial_k \overline{w} \right] dx dt
\end{aligned}$$

$$\begin{aligned}
& + 4\tau\lambda^2 \int_Q \phi |\nabla d \cdot \nabla w|^2 dxdt + 4\tau^3\lambda^4 \int_Q \phi^3 |\nabla d|^4 |w|^2 dxdt \\
& + 2\tau \int_{-T}^T \int_{\partial\Omega} \lambda\phi \frac{\partial d}{\partial \nu} |\nabla w|^2 d\Sigma - 4\tau \operatorname{Re} \int_{-T}^T \int_{\partial\Omega} \lambda\phi (\nabla d \cdot \nabla \bar{w}) \frac{\partial w}{\partial \nu} d\Sigma + X_1 + X_2, \quad (\text{A.4})
\end{aligned}$$

where

$$\left\{ \begin{aligned}
X_1 &= -2\tau^3 \int_Q \lambda^3 \phi^3 (\nabla \log a \cdot \nabla d) |\nabla d|^2 |w|^2 dxdt + 2\tau^3 \lambda^3 \int_Q \phi^3 \nabla d \cdot \nabla (|\nabla d|^2) |w|^2 dxdt, \\
X_2 &= -2\tau \operatorname{Im} \int_Q (-2\lambda^2 \beta t) a \phi (\nabla d \cdot \nabla \bar{w}) w dxdt + 2\tau \operatorname{Re} \int_Q \lambda \phi (\nabla \log a \cdot \nabla d) w L_1 \bar{w} dxdt \\
& + 2\tau \operatorname{Re} \int_Q [\lambda \phi w \nabla (\nabla \log a \cdot \nabla d) \cdot \nabla \bar{w} + \lambda^2 \phi (\nabla \log a \cdot \nabla d) (\nabla d \cdot \nabla \bar{w}) w] dxdt \\
& - 2\tau \operatorname{Im} \int_Q (\partial_t a) w \nabla \bar{w} \cdot \nabla \phi dxdt - \tau \lambda \int_Q \phi (\Delta^2 d) |w|^2 dxdt \\
& - \tau \lambda^2 \int_Q \phi (|\Delta d|^2 + 2\nabla d \cdot \nabla (\Delta d) + \Delta (|\nabla d|^2)) |w|^2 dxdt \\
& - 2\tau \lambda^3 \int_Q \phi (|\nabla d|^2 \Delta d + \nabla d \cdot \nabla (|\nabla d|^2)) |w|^2 - \tau \lambda^4 \int_Q \phi |\nabla d|^4 |w|^2 dxdt.
\end{aligned} \right. \quad (\text{A.5})$$

By (1.2), we obtain

$$\begin{aligned}
& \int_Q \phi \left[ |\nabla w|^2 \nabla \log a \cdot \nabla d + 2 \sum_{j,k=1}^n (\partial_j \partial_k d) (\partial_j w) \partial_k \bar{w} \right] dxdt + \int_Q \phi |\nabla d \cdot \nabla w|^2 dxdt \\
& \geq C_1 \int_Q \phi |\nabla w|^2 dxdt.
\end{aligned}$$

By taking  $\lambda > \frac{1}{2}$ , we have

$$\begin{aligned}
& \int_Q \phi \left[ |\nabla w|^2 \nabla \log a \cdot \nabla d + 2 \sum_{j,k=1}^n (\partial_j \partial_k d) (\partial_j w) \partial_k \bar{w} \right] dxdt + 2\lambda \int_Q \phi |\nabla d \cdot \nabla w|^2 dxdt \\
& \geq C_1 \int_Q \phi |\nabla w|^2 dxdt. \quad (\text{A.6})
\end{aligned}$$

From (A.4) and (A.6), we have

$$\begin{aligned}
\int_Q |Lw + i\tau a (\partial_t \phi) w|^2 dxdt & \geq \int_Q |L_1 w|^2 dxdt + \int_Q |L_2 w|^2 dxdt \\
& + 2\tau \lambda C_1 \int_Q \phi |\nabla w|^2 dxdt + 4\tau^3 \lambda^4 \int_Q \phi^3 |\nabla d|^4 |w|^2 dxdt \\
& + 2\tau \int_{-T}^T \int_{\partial\Omega} \lambda \phi \frac{\partial d}{\partial \nu} |\nabla w|^2 d\Sigma \\
& - 4\tau \lambda \operatorname{Re} \int_{-T}^T \int_{\partial\Omega} \phi (\nabla d \cdot \nabla \bar{w}) \frac{\partial w}{\partial \nu} d\Sigma + X_1 + X_2.
\end{aligned}$$

Since  $|\nabla d| > 0$  on  $\overline{\Omega}$ , there exists a constant  $C_2 > 0$  such that  $|\nabla d|^4 > C_2$  on  $\overline{\Omega}$ . Then we have

$$\begin{aligned} \int_Q |Lw + i\tau a(\partial_t \phi)w|^2 dxdt &\geq \int_Q |L_1 w|^2 dxdt + \int_Q |L_2 w|^2 dxdt \\ &\quad + 2\tau\lambda C_1 \int_Q \phi |\nabla w|^2 dxdt + 4C_2\tau^3\lambda^4 \int_Q \phi^3 |w|^2 dxdt \\ &\quad + 2\tau \int_{-T}^T \int_{\partial\Omega} \lambda \phi \frac{\partial d}{\partial \nu} |\nabla w|^2 d\Sigma \\ &\quad - 4\tau\lambda \operatorname{Re} \int_{-T}^T \int_{\partial\Omega} \phi (\nabla d \cdot \nabla \overline{w}) \frac{\partial w}{\partial \nu} d\Sigma + X_1 + X_2. \end{aligned} \quad (\text{A.7})$$

By (A.5) and (A.7), we see that there exists a constant  $\lambda_1 > \frac{1}{2}$  such that for arbitrary  $\lambda > \lambda_1$ , the terms of  $X_1$  can be absorbed by  $4C_2\tau^3\lambda^4 \int_Q \phi^3 |w|^2 dxdt$ , and we have

$$\begin{aligned} &\int_Q |Lw + i\tau a(\partial_t \phi)w|^2 dxdt \\ &\geq \int_Q |L_1 w|^2 dxdt + \int_Q |L_2 w|^2 dxdt + 2\tau\lambda C_1 \int_Q \phi |\nabla w|^2 dxdt + C_3\tau^3\lambda^4 \int_Q \phi^3 |w|^2 dxdt \\ &\quad + 2\tau \int_{-T}^T \int_{\partial\Omega} \lambda \phi \frac{\partial d}{\partial \nu} |\nabla w|^2 d\Sigma - 4\tau\lambda \operatorname{Re} \int_{-T}^T \int_{\partial\Omega} \phi (\nabla d \cdot \nabla \overline{w}) \frac{\partial w}{\partial \nu} d\Sigma + X_2. \end{aligned}$$

Since  $\phi > 0$  on  $\overline{Q}$  for  $\lambda > \lambda_1$ , there exist constants  $C_4 = C_4(\lambda)$  and  $\tau_1 = \tau_1(\lambda)$  such that for all  $\tau > \tau_1$ ,

$$\begin{aligned} &\int_Q |Lw + i\tau a(\partial_t \phi)w|^2 dxdt \\ &\geq \int_Q |L_1 w|^2 dxdt + \int_Q |L_2 w|^2 dxdt + C_4(\lambda)\tau \int_Q |\nabla w|^2 dxdt + C_4(\lambda)\tau^3 \int_Q |w|^2 dxdt \\ &\quad + 2\tau \int_{-T}^T \int_{\partial\Omega} \lambda \phi \frac{\partial d}{\partial \nu} |\nabla w|^2 d\Sigma - 4\tau\lambda \operatorname{Re} \int_{-T}^T \int_{\partial\Omega} \phi (\nabla d \cdot \nabla \overline{w}) \frac{\partial w}{\partial \nu} d\Sigma + X_2. \end{aligned}$$

Then we choose  $\tau_2 = \tau_2(\lambda) > 0$  such that all the terms of  $X_2$  can be absorbed into  $\|L_1 w\|_{L^2(Q)}^2$ ,  $\|L_2 w\|_{L^2(Q)}^2$ ,  $C_4\|\nabla w\|_{L^2(Q)}^2$  and  $C_4\tau^3\|w\|_{L^2(Q)}^2$ , if we take  $\tau > \tau_2$ . Hence since  $\nabla w = \frac{\partial w}{\partial \nu}\nu$  by  $w = 0$  on  $\partial\Omega$  and  $\int_Q |Lw + i\tau a(\partial_t \phi)w|^2 dxdt \leq 2 \int_Q |Lw|^2 dxdt + C_5\tau^2 \int_Q |w|^2 dxdt$ , taking  $\tau > 0$  sufficiently large, we obtain

$$\begin{aligned} C_6 \int_Q |Lw|^2 dxdt &\geq \int_Q |L_1 w|^2 dxdt + \int_Q |L_2 w|^2 dxdt + \tau \int_Q |\nabla w|^2 dxdt + \tau^3 \int_Q |w|^2 dxdt \\ &\quad - C_7\tau \int_{-T}^T \int_{\partial\Omega} \phi (\nabla d \cdot \nu) \left| \frac{\partial w}{\partial \nu} \right|^2 d\Sigma. \end{aligned}$$

Now we rewrite our inequality with  $v$  instead of  $w$ . By  $w = ve^{\tau\phi}$ ,  $|v|^2 e^{2\tau\phi} = |w|^2$ ,  $Lw = e^{\tau\phi} P^0 v$ , and  $|\nabla v|^2 e^{2\tau\phi} \leq 2|\nabla w|^2 + 2\tau^2\lambda^2\phi^2|\nabla d|^2|w|^2$ ,  $|\frac{\partial v}{\partial \nu}|^2 e^{2\tau\phi} = |\frac{\partial w}{\partial \nu}|^2$  on  $\partial\Omega \times (-T, T)$ , we see that there exist positive constants  $C_8(\lambda)$  and  $\tau_0 > \tau_2(\lambda)$  such that for all  $\tau > \tau_0$ ,

$$\begin{aligned} &\tau \int_Q |\nabla v|^2 e^{2\tau\phi} dxdt + \tau^3 \int_Q |v|^2 e^{\tau\phi} dxdt \\ &\leq C_8 \int_Q |P^0 v|^2 e^{2\tau\phi} dxdt + C_8\tau \int_{-T}^T \int_{\partial\Omega} \phi \left| \frac{\partial v}{\partial \nu} \right|^2 e^{2\tau\phi} (\nabla d \cdot \nu) d\Sigma. \end{aligned}$$

By noting  $P^0 u = \frac{1}{g(x,t)} P_0 u$  and  $g \in C^1(\overline{Q})$ ,  $g > 0$  on  $\overline{Q}$ , we complete the proof of (2.1). Thus the proof of Proposition 2.1 is completed.

**Proof of Theorem 1.2** Let  $0 < T' < T$ , and  $\Omega'$  be a subset of  $\Omega$  satisfying  $(x^1, t^1) \in \Omega' \times (-T', T')$  and  $\overline{\Omega'} \times \overline{(-T', T')} \subset Q$ . Let  $\mathcal{U}'$  be a neighbourhood of  $(x^1, t^1)$  satisfying  $\overline{\mathcal{U}'} \subset \Omega' \times (-T', T')$ . We introduce a cut-off function  $\chi_5 \in C_0^\infty(Q)$  such that  $\chi_5(x, t) = 1$ ,  $(x, t) \in \mathcal{U}'$  and  $\chi_5(x, t) = 0$ ,  $(x, t) \in Q \setminus \Omega' \times (-T', T')$ . We set  $z(x, t) = \chi_5(x, t)u(x, t)$ . Then  $z \in L^2(-T, T; H^{1-s}(\Omega))$  and  $\text{supp } z \subset \Omega' \times (-T', T')$ . Next we check that  $P_1 z \in L^2(-T, T; H^{-s}(\Omega))$ . In fact, by (1.7), we obtain

$$\begin{aligned} \|P_1 z\|_{L^2(-T, T; H^{-s}(\Omega))} &= \|iu\partial_t \chi_5 + \text{div}(\rho u \nabla \chi_5) + \rho \nabla u \cdot \nabla \chi_5\|_{L^2(-T, T; H^{-s}(\Omega))}^2 \\ &\leq C_1 \|u\|_{L^2(-T, T; H^{1-s}(\Omega))}^2 < \infty. \end{aligned}$$

Here we use  $\|\nabla v\|_{H^{-s}(\Omega)} \leq C_1' \|v\|_{H^{1-s}(\Omega)}$ , which can be proved for example by an interpolation inequality (see e.g., [34, Section 5, Chapter 1, Vol. 1]), because  $\nabla$  is a bounded operator from  $H_0^1(\Omega)$  to  $L^2(\Omega)$  and from  $L^2(\Omega)$  to  $H^{-1}(\Omega)$ .

Let  $\delta_1 > 0$  be a constant. We take  $d(x) = |x - x_0|^2 - \delta_1 |x - x^1|^2$  in Corollary 1.2. From (1.8), by choosing  $\delta_1 > 0$  small enough, we apply Corollary 1.2 to  $P_1$  and  $z$ . Let

$$\phi_2(x, t) = e^{\lambda(|x-x_0|^2 - \beta t^2 - \delta_1 |x-x^1|^2)}.$$

Then we have

$$\tau \int_{-T}^T \|ze^{\tau \phi_2}\|_{H^{1-s}(\Omega)}^2 dt \leq C_2 \int_{-T}^T \|e^{\tau \phi_2} P_1 z\|_{H^{-s}(\Omega)}^2 dt$$

for large  $\tau$ ,  $\lambda > 1$ . By noting that  $P_1 u = 0$  in  $Q$ , we obtain

$$\begin{aligned} \tau \int_{-T}^T \|\chi_5 u e^{\tau \phi_2}\|_{H^{1-s}(\Omega)}^2 dt &\leq C_3 \int_{-T}^T \|\{i(\partial_t \chi_5)u + (\Delta \chi_5)u\rho \\ &\quad + 2\rho \nabla \chi_5 \cdot \nabla u + u \nabla \chi_5 \cdot \nabla \rho\} e^{\tau \phi_2}\|_{H^{-s}(\Omega)}^2 dt \end{aligned}$$

for large  $\tau$ ,  $\lambda > 1$ . Consequently,

$$\begin{aligned} \tau \int_{-T}^T \|\chi_5 u e^{\tau \phi_2}\|_{H^{1-s}(\Omega)}^2 dt &\leq C_4 \int_{-T}^T \{\|i(\partial_t \chi_5)u e^{\tau \phi_2}\|_{H^{-s}(\Omega)}^2 + \|(\Delta \chi_5)u \rho e^{\tau \phi_2}\|_{H^{-s}(\Omega)}^2 \\ &\quad + \|e^{\tau \phi_2} u \nabla \chi_5 \cdot \nabla \rho\|_{H^{-s}(\Omega)}^2 + \|2(\nabla \chi_5 \cdot \nabla u) \rho e^{\tau \phi_2}\|_{H^{-s}(\Omega)}^2\} dt \\ &\leq C_5 \int_{-T}^T \{\|i(\partial_t \chi_5)u e^{\tau \phi_2}\|_{L^2(\Omega)}^2 + \|(\Delta \chi_5)u e^{\tau \phi_2}\|_{L^2(\Omega)}^2 \\ &\quad + \|3u e^{\tau \phi_2} \nabla \chi_5 \cdot \nabla \rho\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \|2(\partial_i \chi_5)u \rho e^{\tau \phi_2}\|_{H^{1-s}(\Omega)}^2 \\ &\quad + \|2(\Delta \chi_5)u \rho e^{\tau \phi_2}\|_{L^2(\Omega)}^2 + \|2\tau(\nabla \chi_5 \cdot \nabla \phi_2)u \rho e^{\tau \phi_2}\|_{L^2(\Omega)}^2\} dt. \quad (\text{A.8}) \end{aligned}$$

Since  $u = 0$  in  $\{(x, t) \in Q; e^{\lambda(|x-x_0|^2 - \beta t^2)} > e^{\lambda(|x^1-x_0|^2 - \beta |t^1|^2)}\}$ , we see that there exists a constant  $\epsilon > 0$  such that either  $u = 0$  or all of the derivatives of  $\chi_5$  are equal to zero in  $\{(x, t) \in Q; \phi_2(x, t) > \phi_2(x^1, t^1) - \epsilon\}$ . Hence all the terms on the right-hand side of (A.8) are equal to zero in  $\{(x, t) \in Q; \phi_2(x, t) > \phi_2(x^1, t^1) - \epsilon\}$ . Then we obtain

$$\tau \int_{-T}^T \int_{\Omega} |\chi_5 u|^2 e^{2\tau \phi_2} dx dt \leq C_6 \tau e^{2\tau \phi_2(x^1, t^1) - \epsilon} \int_{-T}^T \|u\|_{H^{1-s}(\Omega)}^2 dt.$$

Consequently,

$$e^{2\tau e^{\phi_2(x^1, t^1) - \frac{\epsilon}{2}}} \int_{\mathcal{U}' \cap \{(x, t) \in Q; \phi_2(x, t) > \phi_2(x^1, t^1) - \frac{\epsilon}{2}\}} |u|^2 dx dt \leq C_7 e^{2\tau e^{\phi_2(x^1, t^1) - \epsilon}} \int_{-T}^T \|u\|_{H^{1-s}(\Omega)}^2 dt.$$

By taking  $\tau \rightarrow +\infty$ , we obtain  $u = 0$  in  $\mathcal{U}' \cap \{(x, t) \in Q; \phi_2(x, t) > \phi_2(x^1, t^1) - \frac{\epsilon}{2}\}$ . Hence we can choose a neighbourhood  $\mathcal{U}$  of  $(x^1, t^1)$  such that  $u = 0$  in  $\mathcal{U}$ . Thus, the proof of Theorem 1.2 is completed.

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