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Examples of Boundary Layers Associated with the Incompressible Navier-Stokes Equations**

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The author surveys a few examples of boundary layers for which the Prandtl boundary layer theory can be rigorously validated. All of them are associated with the incompressible Navier-Stokes equations for Newtonian fluids equipped with various Dirichlet boundary conditions (specified velocity). These examples include a family of (nonlinear 3D) plane parallel flows, a family of (nonlinear) parallel pipe flows, as well as flows with uniform injection and suction at the boundary. We also identify a key ingredient in establishing the validity of the Prandtl type theory, i.e., a spectral constraint on the approximate solution to the Navier-Stokes system constructed by combining the inviscid solution and the solution to the Prandtl type system. This is an additional difficulty besides the well-known issue related to the well-posedness of the Prandtl type system. It seems that the main obstruction to the verification of the spectral constraint condition is the possible separation of boundary layers. A common theme of these examples is the inhibition of separation of boundary layers either via suppressing the velocity normal to the boundary or by injection and suction at the boundary so that the spectral constraint can be verified. A meta theorem is then presented which covers all the cases considered here.

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1 Introduction

One of the most useful models in fluid dynamics is the following incompressible Navier-Stokes equations for Newtonain fluids:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{u}$ is the Eulerian fluid velocity, p is the kinematic pressure, ν is the kinematic viscosity and $\mathbf{f}$ is a (given) applied external body force. The system is customary equipped with initial

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condition

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \tag{1.3}$$

and boundary conditions

$$\mathbf{u}\big|_{\partial\Omega} = \text{given}, \tag{1.4}$$

where we have considered the physically most important case with fluid velocity specified at the boundary (the classical no-slip boundary condition is the case with the given velocity on the boundary to be identically zero).

For many practical fluids like air and water, the kinematic viscosity is small. If we formally set the viscosity to zero, we arrive at the classical Euler equations for incompressible inviscid fluids

$$\frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 + \nabla p^0 = \mathbf{f}, \tag{1.5}$$

$$\nabla \cdot \mathbf{u}^0 = 0, \tag{1.6}$$

and equipped with the same initial condition (1.3). However, the boundary conditions cannot be the same since we have dropped the viscous (diffusion) term which has the highest spatial derivative. The classical boundary condition associated is to specify the normal velocity at the boundary

$$\mathbf{u}^0 \cdot \mathbf{n}\big|_{\partial\Omega} = \text{given}, \tag{1.7}$$

but we may need to augment the system with additional boundary conditions in the case when there is injection at the boundary (see Example 4.3 below).

A natural question then is if such a heuristic inviscid limit procedure can be rigorously justified, i.e., if the solution of the Navier-Stokes equations converge to that of the Euler equations at vanishing viscosity: $\mathbf{u} \to \mathbf{u}^0$, as $\nu \to 0$? This is the well-known problem of vanishing viscosity and remains a major conundrum in applied mathematics and theoretical fluid dynamics.

The difficulty associated can be partially understood via the following simple example of linear plane parallel channel flow.

Example 1.1 (Linear Plane Parallel Channel Flow) Here we consider a two dimensional channel, i.e., assuming the fluids occupying the domain (with the y variable suppressed) $\Omega = \mathbf{R}^1 \times (0, H)$, and we consider the following ansatz (with characteristic boundary condition):

$$\mathbf{u} = (u_1(z,t), 0), \quad p \equiv 0, \quad u_1\big|_{z=0,H} = \text{given}.$$

It is easy to check that this form is preserved under the Navier-Stokes dynamics and the Euler dynamics provided that the initial data and the external forcing satisfy the same ansatz.

Utilizing the proposed ansatz, the Navier-Stokes equations reduce to the following simple heat equation with small diffusive coefficient (and possibly inhomogeneous boundary condition):

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial z^2} = f_1, \quad u_1\big|_{z=0,H} = \text{given}.$$

The Euler system reduces to the following ODE (with no boundary condition): $\frac{\partial u_1^0}{\partial t} = f_1$. It is then easy to see [34] that there exists two boundary layers (one at $z = 0$ and another at $z = H$) in the sense that there exists two smooth functions $\theta_0,\ \theta_H$ such that $u_1 - u_1^0 \approx \theta_0(\frac{z}{\sqrt{\nu t}}) + \theta_H(\frac{H-z}{\sqrt{\nu t}})$. Consequently, we observe that vorticities are generated in the NSE near the boundary and vortex sheets form at the boundary in the vanishing viscosity limit. This implies that the problem of vanishing viscosity is a genuine singular problem since singular structure (vortex sheet) form at the boundary. This also indicates that no uniform (in viscosity) estimates in the space of H^1 or L^∞ is possible. This gives us a hint on the possible difficulty associated with the nonlinear problem since a common technique in passing the limit in the nonlinear term is to obtain uniform estimates in space with derivative (see [15]).

Boundary layer is not only a mathematical challenge, it is also of great importance in application. Indeed, in the absence of body force, it is the vorticities generated in the boundary layer and later advected into the main stream that drive the flow. The interested reader is referred to [27] for much more on the physics of the boundary layer theory.

Due to the existence of boundary layers, it is natural to study effective equations for the boundary layer directly. We briefly touch on the Prandtl boundary layer theory in Section 2. Suppose that the well-posedness of the Prandtl equations is established together with appropriate decay estimates at infinity. We naturally combine the solution of the Prandtl equation and that of the Euler equation to form an approximate solution to the Navier-Stokes system. The general belief is that these kind of approximate solution would converge to that of the Navier-Stokes system (1.1) at vanishing viscosity. However, no general results like this is known, even in the case of very well-behaved Euler system and Prandtl system. Indeed, we illustrate that a spectral constraint is involved to ensure the convergence. The obstruction to the verification of the spectral constraint lies in the advection normal to the boundary near the boundary. This mathematical difficulty is associated with the physical mechanism of boundary layer separation. We illustrate the approximation procedure and the difficulty involved in Section 3. We also provide a meta theorem which states that we do have convergence under the assumptions that the Prandtl type system is well behaved and that a suitable spectral constraint is satisfied. The relationship between the spectral constraint and inhibition to boundary layer separation mechanism will be briefly discussed as well. In Section 4, we provide three nonlinear examples where the conditions of the meta theorem are satisfied. These examples include the nonlinear 3D plane parallel channel flow, nonlinear parallel pipe flow, and the case of channel flow with uniform injection and suction at the boundary. We provide summary in the last section.

One of the main contributions of this manuscript is the identification of a spectral constraint on approximate solutions constructed via Prandtl theory in order to ensure the rigorous validity of the Prandtl theory.

We choose to focus on boundary layers associated with the incompressible Navier-Stokes system equipped with Dirichlet boundary condition in this manuscript. Other types of boundary layer may emerge if different types of boundary conditions are imposed or different type of physical mechanisms are incorporated in the model (see [17] among others).

2 The Prandtl Theory

Since the example above suggests that the solution to the Navier-Stokes system (1.1) and that of the Euler system (1.5) are close to each other except in the boundary layers, it is natural to derive an effective equation for the NSE (1.1) within the boundary layers. We assume a flat boundary at $z = 0$ for simplicity. Noticing that the example also suggests that the solution to the NSE in the boundary layer follows the stretched coordinate $Z = \frac{z}{\sqrt{\nu}}$, Prandtl [24, 27] proposed the following seminal Prandtl's equation which can be formally derived by utilizing the stretched coordinate and omitting lower order terms in the Navier-Stokes system (for simplicity we present the two dimensional version with the y coordinate suppressed):

$$\partial_t u_1 + u_1 \partial_x u_1 + u_3 \partial_z u_1 - \nu \partial_{zz}^2 u_1 + \partial_x p^0\big|_{z=0} = f_1\big|_{z=0}, \tag{2.1}$$

$$\partial_x u_1 + \partial_z u_3 = 0, \tag{2.2}$$

together with the following initial and boundary conditions:

$$u_1\big|_{z=0} = u_3\big|_{z=0} = 0, \quad u_1\big|_{z=\infty} = u_1^0(x, 0, t), \quad u_1\big|_{t=0} = u_{01}. \tag{2.3}$$

The external forcing $\mathbf{f}$ is usually set to zero in most theoretical studies.

A natural question then is the validity of the Prandtl theory. In particular, the following two important issues need to be addressed:

(1) Well-posedness of the Prandtl system (2.1);

(2) Matching of the solutions of the Prandtl system (2.1) and the Euler system (1.5) to that of the Navier-Stokes system (1.1).

The well-posedness of the Prandtl system is a very challenging problem due to its degeneracy and nonlinearity. There is a very nice treatise on the Prandtl system by Oleinik [22] together with the recent progress (see [37]) as well as the very interesting finite time blow-up result (see [8]). There are also interesting results suggesting various instability within the Prandtl system (see [11, 12]). More results can be found in the references therein. Even with the well-posedness of the Prandtl system, the matching problem seems to be a challenge and is essentially open in the general case. There are interesting partial results, especially those for self-similar solution in a wedge (see [9, 28]), half space in the analytic setting (see [3, 25, 26]), and the case of channel flow with uniform injection and suction at the boundary (see [31, 32] and Section 4 below). It seems that the difficulty of the matching lies in the physical mechanism of separation of variables which can be interpreted as the verification of a spectral constraint on the approximate solution as we illustrate in the next section.

There is a slightly different approach to the boundary layer problem advocated by Lyusternik and Vishik [33] as well as Lions [16]. In this alternative approach, the effective equation is in terms of the difference of the solutions to the Navier-Stokes system and that of the Euler system, i.e., approximating $\mathbf{u} - \mathbf{u}^0$ in the boundary layer. Following the same argument as in the derivation of the original Prandtl theory, we deduce the following Prandtl type system for $\boldsymbol{\theta} \approx \mathbf{u} - \mathbf{u}^0$ (we consider the 2D case again suppressing the y dependence):

$$\partial_t \theta_1 + (u_1^0\big|_{z=0} + \theta_1)\partial_x \theta_1 + \theta_1 \partial_x u_1^0\big|_{z=0} + \theta_3 \partial_z \theta_1 + u_3^0 \partial_z \theta_1 - \nu \partial_{zz}^2 \theta_1 = 0, \tag{2.4}$$

$$\partial_x \theta_1 + \partial_z \theta_3 = 0, \tag{2.5}$$

together with the following boundary and initial conditions:

$$\theta_1\big|_{z=0} = -u_1^0\big|_{z=0}, \quad \theta_1\big|_{z=\infty} = 0, \quad \theta_3\big|_{z=0} = 0, \quad \theta\big|_{t=0} = 0. \tag{2.6}$$

The term u_3^0 can be replaced by $\partial_z u_3^0\big|_{z=0} z$ if needed (so that the system can be written in terms of the stretched coordinate Z). The quantity $\boldsymbol{\theta}$ is usually called **corrector** in the literature (see [33, 16]). The issues associated with this alternative approach is exactly the same as Prandtl's original approach, i.e., we need to deal with the well-posedness of the Prandtl type system (2.4) and the matching issue. An obvious advantage of this alternative approach is that the matching problem is conceptually simple in the sense that $\mathbf{u}^a = \mathbf{u}^0 + \boldsymbol{\theta}$ should be a natural candidate for approximate solution to the Navier-Stokes system, and our goal is to demonstrate that $\mathbf{u} - \mathbf{u}^a \to 0,$ as $\nu \to 0$. A disadvantage of this approach is the complexity of the Prandtl type system (2.4) versus the original Prandtl system (2.1) since it has now variable coefficients among others. This is on top of the degeneracy and nonlinearity.

3 Matching, Convergence and Spectral Constraint

We follow the alternative approach of Vishik, Lyusternik and Lions by working with the Prandtl type system (2.4) instead of the original Prandtl system (2.1). As we have discussed earlier, the matching issue here is conceptually simple in the sense that, provided that the Prandtl type system (2.4) is well-poseded and the solutions decay fast enough at infinity, there is a natural candidate for approximate solution to the Navier-Stokes equations:

$$\mathbf{u}^a = \mathbf{u}^0 + \boldsymbol{\theta}. \tag{3.1}$$

Our goal is to show that the approximation is valid in the sense that the approximation error defined as

$$\mathbf{u}^e = \mathbf{u} - \mathbf{u}^a \tag{3.2}$$

is small in some appropriate sense.

It is a straightforward calculation to check that the approximate solution $\mathbf{u}^a$ satisfies the same Navier-Stokes system with an extra body force

$$\frac{\partial \mathbf{u}^a}{\partial t} + (\mathbf{u}^a \cdot \nabla)\mathbf{u}^a - \nu\Delta\mathbf{u}^a + \nabla p^a = \mathbf{f} + \mathbf{f}^e, \tag{3.3}$$

$$\nabla \cdot \mathbf{u}^a = 0, \tag{3.4}$$

where $\mathbf{f}^e$ is an additional external forcing due to approximation error and it satisfies the following type of estimates: $\mathbf{f}^e \approx \sqrt{\nu}\psi(\frac{z}{\sqrt{\nu}})$.

We may then deduce that the approximation error $\mathbf{u}^e$ satisfies the following error equations (A subtle issue that we have omitted here is that the boundary condition for the approximation error is not necessarily zero. In fact, $\mathbf{u}^e$ constructed above will not satisfy the homogeneous Dirichlet boundary condition although the corrector $\boldsymbol{\theta}$ decay at infinity. This complication can be resolved by utilizing truncation at the stream function level. The truncation introduces additional error that is of the same order as the error represented in the extra body force $\mathbf{f}^e$. See [31, 32] for details related to Example 4.3, and [18] for details related to Example 4.1):

$$\frac{\partial \mathbf{u}^e}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}^e + (\mathbf{u}^e \cdot \nabla)\mathbf{u}^a - \nu\Delta\mathbf{u}^e + \nabla q = \mathbf{f}^e, \tag{3.5}$$

$$\nabla \cdot \mathbf{u}^e = 0, \tag{3.6}$$

$$\mathbf{u}^e\big|_{\partial\Omega} = 0, \tag{3.7}$$

$$\mathbf{u}^e\big|_{t=0} = 0. \tag{3.8}$$

Our overall **GOAL** is to demonstrate that $\mathbf{u}^e$ is small in some appropriate sense (in various spaces).

For this purpose, we multiply the error equation (3.5) by $\mathbf{u}^e$ and integrate over the domain, and we have

$$\frac{1}{2}\frac{\mathrm{d}\|\mathbf{u}^e\|_{L^2}^2}{\mathrm{d}t} + \nu\|\nabla\mathbf{u}^e\|_{L^2}^2 + \int_\Omega(\mathbf{u}^e\cdot\nabla)\mathbf{u}^a\cdot\mathbf{u}^e = \int_\Omega \mathbf{f}^e\cdot\mathbf{u}^e. \tag{3.9}$$

It is then easy to see that a sufficient condition for $\mathbf{u}^e \to 0$ in L^2 is the following spectral constraint on the approximate solution $\mathbf{u}^a$:

$$\inf_{\mathbf{v}\in\mathcal{A}} \frac{\int_\Omega(\nu|\nabla\mathbf{v}|^2 + (\mathbf{v}\cdot\nabla)\mathbf{u}^a\cdot\mathbf{v})}{\|\mathbf{v}\|_{L^2}^2} \ge \Lambda > -\infty, \tag{3.10}$$

where Λ should be a constant independent of the viscosity ν, and the admissible set $\mathcal{A}$ is defined as

$$\mathcal{A} = \{\mathbf{v}\in(H_0^1(\Omega))^d,\ \nabla\cdot\mathbf{v}=0,\ \mathbf{v}\neq 0\}. \tag{3.11}$$

This admissible set can be replaced by smaller sets if the flow satisfies certain symmetry as those illustrated in Examples 4.1 and 4.2 below.

Such kind of spectral constraints are ubiquitous in classical fluid stability analysis (see [6]) and in terms of estimating bulk dissipative quantities associated with incompressible flows (see [5]). However, we believe that this is the first time that such a spectral constraint is identified in terms of boundary layers associated with incompressible flows.

Heuristically, the tangential derivative part of the advection term does not pose any serious difficulty in terms of the spectral constraint, since we anticipate that the behavior of the tangential derivative of the approximate solution $\mathbf{u}^a$ is very much the same as that of $\mathbf{u}^a$ itself. Indeed, we formally have

$$\Big|\int_\Omega v_1\partial_x\mathbf{u}^a\cdot\mathbf{v}\Big| \le \|\partial_x\mathbf{u}^a\|_{L^\infty}\|\mathbf{v}\|_{L^2}^2 \le C\|\mathbf{v}\|_{L^2}^2.$$

The difficulty associated with the verification of this spectral constraint (3.10) lies in the advection normal to the boundary term. Indeed, heuristic calculations lead to

$$\begin{aligned}\Big|\int_\Omega v_3\partial_z\mathbf{u}^a\cdot\mathbf{v}\Big| &\sim \frac{1}{\sqrt{\nu}}\|\mathbf{v}\|_{L^2}^2,\\ \Big|\int_\Omega v_3\partial_z\mathbf{v}\cdot\mathbf{u}^a\Big| &\sim \|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2},\\ \Big|\int_\Omega v_3\partial_z\mathbf{v}\cdot\boldsymbol{\theta}\Big| &\sim \sqrt{\nu}\|\nabla\mathbf{v}\|_{L^2},\end{aligned}$$

and these estimates are not sufficient for the spectral constraint. This advection normal to the boundary is the physical mechanism that is responsible for the possible separation of boundary layers. If the separation of the boundary is inhibited, we would anticipate the verification of the spectral constraint which further leads to the validity of the Prandtl type theory. Therefore, we have the following meta theorem.

Theorem 3.1 (Meta Theorem on the Validity of Prandtl Type Theory) *Suppose that the following two conditions are satisfied:*

(1) *The well-posedness of the Prandtl type system* (2.4) *together with appropriate decay at infinity*;

(2) *The verification of the spectral constraint* (3.10) *on the approximate solution* (3.1). *Then the Prandtl type approximation is valid in the sense that*

$$\|\mathbf{u}^e\|_{L^\infty(L^2)} \leq C\nu^{\frac{3}{4}}, \tag{3.12}$$

$$\|\mathbf{u}^e\|_{L^2(H^1)} \leq C\nu^{\frac{1}{4}}. \tag{3.13}$$

The spectral constraint (3.10) *can be verified in the case of no normal flow near the boundary.*

The verification of the spectral constraint (3.10) under the assumption that there is no normal to the boundary flow near the boundary is obvious.

4 Nonlinear Examples of Boundary Layers

The purpose of this section is to provide examples where the assumptions in the meta theorem presented in the previous section are satisfied so that the Prandtl theory is validated. Besides, the trivial plane parallel flows presented in Example 1.1, we show that certain type of nonlinear flows verify the assumptions as well. These examples include a class of 3D nonlinear plane parallel flows, nonlinear parallel pipe flows, and the case of channel flow with injection and suction at the boundary. The first two special type of nonlinear flows were proposed in [35] where the vanishing viscosity problem was resolved following a strategy due to Kato [14]. The case of channel flow with injection and suction at the boundary was investigated in [1] for vanishing viscosity and in [31, 32] for the boundary layer issue.

The common feature for the nonlinear 3D plane parallel channel flow and the nonlinear parallel pipe flow is the suppression of normal velocity. Indeed, the normal velocity in these two models are identically zero which leads to the verification of the spectral constraint. For the channel flow with uniform injection and suction at the boundary, the suction makes the boundary layer narrower which leads to stabilization (verification of the spectral constraint).

The next example is a slight generalization of the nonlinear parallel channel flow proposed in [35].

Example 4.1 (Nonlinear Plane Parallel 3D Channel Flow) In this case, the domain is assumed to be a channel of the form $\Omega = \mathbf{R}^1 \times \mathbf{R}^1 \times (0, 1)$ with periodicity assumed in the x, y direction.

We consider the following family of plane parallel flow:

$$\mathbf{u} = (u_1(t, z), u_2(t, x, z), 0), \quad \mathbf{u}\big|_{z=j} = (\beta_1^j(t), \beta_2^j(t, x), 0), \quad j = 0, 1$$

provided that the initial data and external forcing also satisfy the same ansatz, i.e.,

$$\mathbf{u}_0 = (u_{1,0}(z), u_{2,0}(x, z), 0), \quad \mathbf{f} = (f_1(t, z), f_2(t, x, z), 0).$$

It is easy to check that this family of solution is invariant under the Navier-Stokes flow. Indeed, the NSE (1.1) reduces to the following nonlinear system:

$$\partial_t u_1 - \nu\partial_{zz} u_1 = f_1, \quad \partial_t u_2 + u_1\partial_x u_2 - \nu\partial_{xx} u_2 - \nu\partial_{zz} u_2 = f_2,$$

and the Euler system (1.5) reduces to

$$\partial_t u_1^0 = f_1, \quad \partial_t u_2^0 + u_1^0\partial_x u_2^0 = f_2,$$

and the associated Prandtl type system for the corrector $\boldsymbol{\theta}^0$ (for the boundary layer at $z=0$) is given by

$$\begin{aligned} \partial_t\theta_1^0 - \nu\partial_{zz}\theta_1^0 &= 0, \\ \partial_t\theta_2^0 - \nu\partial_{zz}\theta_2^0 + \theta_1^0\partial_x\theta_2^0 + u_1^0(t,0)\partial_x\theta_2^0 + \theta_1^0\partial_x u_2^0(t,x,0) &= 0, \\ (\beta_1^0(t) - u_1^0(t,0), \beta_2^0(t,x) - u_2^0(t,x,0)) &= (\theta_1^0, \theta_2^0)\big|_{z=0}. \end{aligned}$$

The Prandtl type equation for the corrector $\boldsymbol{\theta}^1$ (for boundary layer at $z=1$) is derived in a similar fashion.

It is then straightforward to verify that the assumptions in the meta theorem are satisfied, i.e., the Prandtl type system is well-posed with solution decay at infinity and the spectral constraint is verified due to the lack of normal velocity (see [18]). Indeed, one can show that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}^0 - \boldsymbol{\theta}^1\|_{L^\infty(L^2)} &\le C\nu^{\frac{3}{4}}, \\ \|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}^0 - \boldsymbol{\theta}^1\|_{L^2(H^1)} &\le C\nu^{\frac{1}{4}}, \\ \|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}^0 - \boldsymbol{\theta}^1\|_{L^\infty(L^\infty)} &\le C\nu^{\frac{1}{4}}. \end{aligned}$$

The interested reader is referred to [18] for more details (the uniform in space and time estimate requires additional tools involving anisotropic Sobolev spaces and related embedding). Related $L^\infty(L^2)$ result was derived earlier in [35] using Kato type technique and $L^\infty(L^\infty)$ type result has been obtained in [20] using parametrix technique and with different kind of correctors (not related to the Prandtl approach).

The next example is a slight generalization of the nonlinear parallel pipe flow introduced in [35].

Example 4.2 (Nonlinear Parallel Pipe Flow) In this case, the domain Ω is set to be a cylinder with radius 1 parallel to the x axis. Periodicity in x is assumed. The ansatz for flow is

$$\mathbf{u} = u_\phi(t,r)\mathbf{e}_\phi + u_x(t,r,\phi)\mathbf{e}_x, \quad \mathbf{u}\big|_{r=1} = (0, \beta_\phi(t), \beta_x(t,\phi,x))$$

with the initial data and external forcing satisfying the same ansatz, and ϕ denotes the azimuthal variable and r denotes the radial variable.

It is easy to verify that this family of flow is also invariant under the Navier-Stokes dynamics. Indeed, the NSE (1.1) reduces to the following form:

$$\begin{aligned} -(u_\phi)^2 + r\partial_r p &= 0, \\ \partial_t u_\phi &= \frac{\nu}{r}\partial_r(r\partial_r u_\phi) - \frac{\nu}{r^2}u_\phi + f_\phi, \\ \partial_t u_x + \frac{u_\phi}{r}\partial_\phi u_x &= \frac{\nu}{r}\partial_r(r\partial_r u_x) + \frac{\nu}{r^2}\partial_{\phi\phi}u_x + f_x, \end{aligned}$$

while the Euler system reduces to

$$-(u_\phi^0)^2 + r\partial_r p^0 = 0, \quad \partial_t u_\phi^0 = f_\phi, \quad \partial_t u_x^0 + \frac{u_\phi^0}{r}\partial_\phi u_x^0 = f_x,$$

and the associated Prandtl type system takes the form

$$\begin{aligned} \partial_t\theta_\phi - \nu\partial_{rr}^2\theta_\phi &= 0, \\ \partial_t\theta_x + \theta_\phi\partial_\phi u_x^0(t,1) + \theta_\phi\partial_\phi\theta_x + u_\phi^0(t,1)\partial_\phi\theta_x &= \nu\partial_{rr}^2\theta_x. \end{aligned}$$

It is then straightforward to verify that the assumptions in the meta theorem are satisfied, i.e., the Prandtl type system is well-posed with solution decay at infinity and the spectral constraint is verified due to the lack of normal velocity (see [19]). Indeed, one can show that

$$\|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^\infty(L^2)} \le C\nu^{\frac{3}{4}}, \quad \|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^2(H^1)} \le C\nu^{\frac{1}{4}}.$$

Uniform in space and time estimates can be also derived via appropriate application and generalization of anisotropic embedding developed earlier (see [34, 29]). The interested reader is referred to [19] for more details. Related $L^\infty(L^2)$ result was derived earlier in [35] using Kato type technique and H^1 type result can be found in [21] using parametrix technique and an approach not related to the Prandtl theory.

The previous examples all require special symmetry of the flows and suppressed the normal velocity completely. The next example is for general Navier-Stokes flows but with uniform injection and suction at the boundary (non-characteristic boundary).

Example 4.3 (Channel Flow with Uniform Injection and Suction at the Boundary) In this case, we set the domain to be a 2D channel (3D can be discussed analogously), i.e.,

$$\Omega = \mathbf{R}^1 \times (0, H),$$

with periodicity assumed in the x direction.

We assume that there exist uniform injection at the top of the channel and uniform suction at the bottom of the channel, i.e.,

$$\mathbf{u}\big|_{z=0,H} = (0, -U), \quad U > 0,$$

where U is a positive constant.

It is worthwhile to point out that an additional (upwind) boundary condition is needed for the well-posedness of the Euler equation in this case:

$$u_1^0 = 0.$$

This is a manifestation of the hyperbolic mechanism of the Euler system under this non-characteristic boundary condition. The local in time well-posedness of the Euler system with this non-conventional additional upwind boundary condition can be found in [2] for 2D and [23] for 3D.

It is a pleasant surprise that the Prandtl type system takes the following linear elliptic form in this case with uniform injection and suction at the boundary despite the nonlinearity of the Navier-Stokes and Euler systems:

$$-U\frac{\partial \theta_1}{\partial z} - \nu \frac{\partial^2 \theta_1}{\partial z^2} = 0, \quad \theta_1\big|_{z=0} = -v_1^0\big|_{z=0}, \quad \theta_1\big|_{z=\infty} = 0, \quad \frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_3}{\partial z} = 0.$$

We notice that the θ_1 equation coincides with Friedrichs' classical example of boundary layer (see [10]).

In this case, the well-posedness of the Prandtl type equation is trivial. One also notices that there is only one boundary layer at the downwind location ($z = 0$) with a thickness proportional to $\frac{\nu}{U}$. And therefore the spectral constraint (3.10) may be verified for short time at least since we have

$$\Big| \int_\Omega v_3 \partial_z \mathbf{v} \cdot \boldsymbol{\theta} \Big| \le |z\boldsymbol{\theta}|_{L^\infty} \|\nabla \mathbf{v}\|_{L^2} \Big\| \frac{v_3}{z} \Big\|_{L^2} \le C|z\boldsymbol{\theta}|_{L^\infty} \|\nabla \mathbf{v}\|_{L^2}^2 \le C \times C_{\boldsymbol{\theta}}(t)\nu \|\nabla \mathbf{v}\|_{L^2}^2,$$

thanks to Hardy's inequality and the explicit form of the boundary layer solution with thickness proportional to $\frac{\nu}{U}$ and initial thickness 0. One can then show that

$$\|\mathbf{u} - \mathbf{u}^0 - \boldsymbol{\theta}\|_{L^\infty(L^2), L^\infty(L^\infty), L^2(H^1)} \to 0.$$

The interested reader is referred to [31, 32] for more details. Further refinements and generalizations of this example can be found in [13, 36] among others.

The boundary layer in this case with injection and suction can be easily understood at the linear level. Indeed, we can decompose the velocity field in the following form:

$$\mathbf{u} = (v_1, -U + v_3), \quad v_j\big|_{z=0,H} = 0, \quad j = 1, 3.$$

Inserting this into the Navier-Stokes system (1.1) and dropping the nonlinear terms, we arrive at the following advection-diffusion equation:

$$\frac{\partial v_1}{\partial t} - U\frac{\partial v_1}{\partial z} - \nu\frac{\partial^2 v_1}{\partial z^2} = f_1, \quad v_1\big|_{z=0,H} = 0,$$

and the Euler system reduces to the following linear transport equation:

$$\frac{\partial v_1^0}{\partial t} - U\frac{\partial v_1^0}{\partial z} = f_1, \quad v_1^0\big|_{z=H} = 0.$$

It is then clear that the additional (upwind) boundary condition for the tangential velocity v_1^0 is exactly the upwind boundary condition required for transport equation.

The steady state case reduces to Friedrichs' example and has the following exact solution (assuming f_1 is a constant):

$$v_1(z) = -\frac{f_1}{U}z + \frac{f_1 H}{U(1 - \exp(-\frac{UH}{\nu}))}\Big(1 - \exp\Big(-\frac{Uz}{\nu}\Big)\Big).$$

The existence of a boundary layer at the downwind location ($z = 0$) with thickness proportional to $\frac{\nu}{U}$ is then transparent

$$u_1 - u_1^0 = v_1 - v_1^0 = \text{boundary layer}\Big(\frac{z}{\nu}\Big).$$

A complete analysis of the boundary layer for the time-dependent linear case can be found in [4].

5 Summary

We have surveyed a few linear and nonlinear examples where the validity of the Prandtl type theory can be rigorously established. Besides the well-known issue of well-posedness of the Prandtl type system, another important ingredient in validating the Prandtl theory is the spectral constraint (3.10) on the approximate solution. The main obstruction to the verification of the spectral constraint is the advection normal to the boundary which corresponds to the mechanism that leads to the separation of boundary layers. For the nonlinear plane parallel flow and nonlinear parallel pipe flow, the normal velocities are suppressed in both cases (no separation of boundary layer), and hence we can rigorously demonstrate the validity of the Prandtl type theory. Another physical mechanism that can be used to inhibit the separation of boundary layer is suction at the boundary. The case of channel flow with uniform injection and suction is presented in Example 4.3, where the spectral constraint is verified and the validity of the Prandtl theory is established for short time.

We believe that a useful and important discovery here is the spectral constraint on the approximate solution for the validity of the Prandtl type theory even under the assumption of the well-posedness of the Prandtl type system. Although, the verification of such a spectral constraint was implicit in previous works on validation of Prandtl type theory, it seems appropriate and beneficial for us to identify this constraint explicitly here.

The study of boundary layers associated with the incompressible Navier-Stokes flow remains a great challenge. Heuristically, we expect the following type of results:

(1) short time well-posedness of the Prandtl type system,

(2) short time validity of the Prandtl type theory.

However, we are far away from reaching our goal except for a few special cases that we mentioned above and in the introduction.

Beyond the short time validity of the Prandtl type theory, it is of great importance to develop theory after the boundary layer separates (and hence the Prandtl theory cease to be viable). We still expect the validity of the vanishing viscosity limit. But different methods must be used after the separation of boundary layer. Kato type approach is one of these methods that do not rely on the Prandtl theory but its success is very much limited so far (see [14, 35, 30]).

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