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On a Strongly Damped Wave Equation for the Flame Front*****

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(*Dedicated to Roger Temam on his* 70*th Birthday, with Respect and Admiration*)

Abstract In two-dimensional free-interface problems, the front dynamics can be modeled by single parabolic equations such as the Kuramoto-Sivashinsky equation (K-S). However, away from the stability threshold, the structure of the front equation may be more involved. In this paper, a generalized K-S equation, a nonlinear wave equation with a strong damping operator, is considered. As a consequence, the associated semigroup turns out to be analytic. Asymptotic convergence to K-S is shown, while numerical results illustrate the dynamics.

Keywords Front dynamics, Wave equation, Kuramoto-Sivashinsky equation, Stability, Analytic semigroups, Spectral method
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1 Introduction and Physical Background

The reduction of a free-interface problem to an explicit equation for the interface dynamics is a very challenging question (see [3–5]). A paradigm two-dimensional problem in combustion theory is the model for the Near Equidiffusive Flames (see [16] and for a brief exposure [5, Appendix A]), a system for the (rescaled) temperature θ, the enthalpy S and the moving flame front $\xi = \xi(t, y)$ which reads:

$$\frac{\partial \theta}{\partial t} = \Delta\theta, \qquad x < \xi(t, y), \tag{1.1}$$

$$\theta = 1, \qquad x \geq \xi(t, y), \tag{1.2}$$

$$\frac{\partial S}{\partial t} = \Delta S - \alpha\Delta\theta, \qquad x \neq \xi(t, y). \tag{1.3}$$

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At the front, θ and S are continuous and the following jump conditions occur for the normal derivatives:

$$\Big[\frac{\partial \theta}{\partial n}\Big] = -\exp(S), \quad \Big[\frac{\partial S}{\partial n}\Big] = \alpha\Big[\frac{\partial \theta}{\partial n}\Big], \tag{1.4}$$

where α is the reduced Lewis number. We recall that in combustion theory the Lewis number (Le) is the ratio of thermal and molecular diffusivities. Equidiffusivity corresponds to $Le = 1$ and, in System (1.1)–(1.4), to $\alpha = 0$. We consider only the case where α is positive, i.e., the case of high mobility of the deficient reactant.

System (1.1)–(1.4) can be formulated (i) either in the whole space $\mathbb{R}$ (see [6, 11]) or (ii) in a strip $\mathbb{R} \times [-\frac{\ell}{2}, \frac{\ell}{2}]$ with periodic or Neumann boundary conditions (see [12–14]). It is easy to see that in both cases System (1.1)–(1.4) admits a planar Traveling Wave (TW) solution $(\overline{\theta}, \overline{S})$, with velocity -1. The threshold of (orbital) stability of this TW occurs at a critical value α_c. In the former case (i), one has $\alpha_c = 1$, in the latter case (ii), one has $\alpha_c < 1$ and $\alpha_c \to 1$ as $\ell \to +\infty$.

The main issue is the dynamics of the perturbation of the planar front $\varphi(t, y) = \xi(t, y) + t$ when $\alpha > \alpha_c$. It was already observed by Turing (see [21] and [17, Chapter 14]), sixty years ago, that spatially inhomogeneous patterns can evolve by diffusion driven instability when equidiffusion does not hold. One of the authors in [19] introduced the small positive parameter $\varepsilon = \alpha - 1$ and considered the rescaled independent and dependent variables

$$\tau = \varepsilon^2 t, \quad \eta = \sqrt{\varepsilon} y, \quad \varphi = \varepsilon \psi. \tag{1.5}$$

As $\varepsilon \to 0$, he derived asymptotically the Kuramoto-Sivashinsky equation (K-S) for the formal limit Φ of ψ:

$$\Phi_\tau + 4\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0. \tag{1.6}$$

In accordance with Turing's prediction, K-S generates a cellular structure and chaotic behavior in an appropriate range of parameters (see [10]). We refer to [20] and its extensive bibliography.

However, when $\alpha - 1$ is positive but not necessarily small, namely away from the stability threshold, the structure of the front equation may be far more involved. In a previous paper, assuming quasi-steadiness of the temperature and enthalpy, and neglecting some higher order terms for the sake of simplicity, we came to the following fully nonlinear equation for the front perturbation φ, assuming periodic boundary conditions:

$$\frac{\partial}{\partial t}\mathscr{B}(\varphi) = \mathscr{S}(\varphi) + \mathscr{F}((\varphi_y)^2), \tag{1.7}$$

where $\mathscr{S}$ is the fourth-order differential operator

$$\mathscr{S}(\varphi) = -4\varphi_{yyyy} - (\alpha - 1)\varphi_{yy}$$

and $\mathscr{B}$ and $\mathscr{F}$ are pseudo-differential operators.

Our goal is to get rid of the quasi-steadiness hypothesis. The main feature is that, as already observed in [1] for the κ-θ model, Equation (1.7) turns out to be a wave equation. We will give a full derivation of the equation in Fourier-Laplace variables in Appendix. However, for simplicity we will focus in this paper on a paradigm model that we derive heuristically for the convenience of the reader.

Let us consider in fixed coordinates (see Appendix) the perturbation of temperature u and enthalpy v in (1.1)–(1.4):

$$\theta = \overline{\theta} + u, \quad S = \overline{S} + v.$$

The conventional linear stability analysis of the travelling wave solution $(\overline{\theta}, \overline{S})$ yields the following set of relations between the perturbation of the interface $\varphi(t, y)$ and its temperature $\chi(t, y) = u(t, 0, y)$, valid in the long-wavelength limit (see [8]):

$$\varphi_t = \varphi_{yy} + \chi, \tag{1.8}$$

$$\chi_t = \chi_{yy} - \frac{1}{4}\alpha\varphi_{yy} - \frac{1}{4}\chi. \tag{1.9}$$

Upon elimination of χ, Equations (1.8) and (1.9) yield

$$\varphi_t + (\alpha - 1)\varphi_{yy} + 4\varphi_{yyyy} + 4\varphi_{tt} - 8\varphi_{tyy} = 0. \tag{1.10}$$

One can easily check that Equation (1.10) implies exponential amplification of long wavelength disturbances at $\alpha > 1$. In reality, this amplification is checked by effects represented by certain nonlinear terms not present in (1.10). The structure of these terms may be estimated via the following semiheuristic arguments. Consider a curved front in the model (1.1)–(1.4). If the characteristic radius of curvature of the flame is significantly greater than its scaled thermal thickness (= 1), then the scaled propagation speed of the flame relative to the gas may be considered a constant (= 1). In a coordinate system at rest with respect to the undisturbed planar flame, the front $x = \varphi(t, y)$ of such a curved flame is described by the equation (see [18])

$$\varphi_t = 1 - \sqrt{1 + (\varphi_y)^2}.$$

Near the stability threshold α_c, one expects that $(\varphi_y)^2 \ll 1$. Hence,

$$\varphi_t + \frac{1}{2}(\varphi_y)^2 = 0.$$

Comparing this weakly nonlinear equation, which disregards effects due to distortion of the flame structure, with Equation (1.10), in which these effects are included, one reaches the reasonable conclusion that $\frac{1}{2}(\varphi_y)^2$ is precisely the nonlinear term missing from (1.10). One thus ends up with the following equation for the nonlinear evolution of the disturbed flame front:

$$\varphi_t + \frac{1}{2}(\varphi_y)^2 + (\alpha - 1)\varphi_{yy} + 4\varphi_{yyyy} + 4\varphi_{tt} - 8\varphi_{tyy} = 0.$$

There are of course alternative possibilities for reduction of the free-interface problem (1.1)–(1.4) to an explicit equation of the flame front. In [8], for example, the reduced model is obtained through a geometrically-invariant extrapolation of System (1.8)–(1.9), resulting in a coupled strongly nonlinear system of second-order equations for the flame front and its temperature.

This paper is devoted to the paradigm model

$$4(\varphi_{tt} + \varphi_{yyyy}) + (I - 8D_{yy})\varphi_t + (\alpha - 1)\varphi_{yy} + \frac{1}{2}(\varphi_y)^2 = 0, \tag{1.11}$$

a nonlinear wave equation with a strongly damping operator $I - 8D_{yy}$ acting on φ_t, which will play a crucial role hereafter. Equation (1.11) is set on a strip $\mathbb{R} \times [-\frac{\ell}{2}, \frac{\ell}{2}]$ with periodic boundary conditions and initial conditions for φ and φ_t respectively:

$$\varphi(0, \cdot) = \varphi_0, \quad \varphi_t(0, \cdot) = \varphi_1.$$

The paper is organized as follows. In Section 2, we introduce the notation and the functional spaces. Section 3 is devoted to the stability issue. We write (1.11) as a first-order system and study the semigroup $(T(t))_{t\geq 0}$ associated with the linear operator. In wave problems, the semigroup is a priori only strongly continuous. However, in the case of a damped wave equation and if the damping is strong enough, the semigroup may be analytic (see [7] and the references therein). This is what we prove for (1.11). Using arguments from the semigroup theory, we can establish the following first main result of the paper as follows:

Theorem 1.1 *Let*

$$\alpha_c = 1 + \frac{16\pi^2}{\ell^2}. \tag{1.12}$$

The following properties are satisfied:

(a) *If $\alpha < \alpha_c$, then the null solution to Equation* (1.11) *is* (*orbitally*) *stable, with asymptotic phase, with respect to sufficiently smooth and small perturbations.*

(b) *If $\alpha > \alpha_c$, then the null solution to Equation* (1.11) *is unstable.*

In Section 4, we set $\alpha = 1 + \varepsilon$ and perform the change of independent and dependent variables (1.5) and after division by ε^3 we get to

$$4\varepsilon^2\psi_{\tau\tau} + (I - 8\varepsilon D_{\eta\eta})\psi_\tau + 4\psi_{\eta\eta\eta\eta} + \psi_{\eta\eta} + \frac{1}{2}(\psi_\eta)^2 = 0. \tag{1.13}$$

Then, we anticipate, in the limit $\varepsilon \to 0$, that $\psi \sim \Phi$, where Φ solves (1.6). The idea is to link the small positive parameter ε and the width of the strip, which will blow up as $\varepsilon \to 0$. For $\ell_0 > 0$, we take ℓ of the form $\ell_\varepsilon = \frac{\ell_0}{\sqrt{\varepsilon}}$, hence $\alpha_c = 1 + \frac{16\pi^2}{\ell_0^2}\varepsilon$. Thus, ℓ_0 becomes the new bifurcation parameter, which may be renormalized (see [20, Chapter III] and [10]) as

$$\widetilde{\ell}_0 = \frac{\ell_0}{4\pi}.$$

Therefore we assume $\widetilde{\ell}_0 > 1$ in order to have $\alpha_c \in (1, 1+\varepsilon)$, i.e., $\alpha > \alpha_c$, otherwise the trivial solution is stable and the dynamics is trivial.

The second main result of the paper is the following theorem.

Theorem 1.2 *Let $\Phi_0 \in H^m$ be a periodic function of period ℓ_0. Further, let Φ be the periodic solution of* (1.6) (*with period ℓ_0*) *on a fixed time interval $[0, T]$, satisfying the initial condition $\Phi(0, \cdot) = \Phi_0$. Then, if m is large enough, there exists $\varepsilon_0 = \varepsilon_0(T) \in (0, 1)$ such that, for $0 < \varepsilon \leq \varepsilon_0$, Problem* (1.11) *admits a unique solution φ on $[0, \frac{T}{\varepsilon^2}]$, which is periodic with period $\frac{\ell_0}{\sqrt{\varepsilon}}$ with respect to y, and satisfies, for $|y| \leq \frac{\ell_0}{2\sqrt{\varepsilon}}$,*

$$\varphi(0, y) = \varepsilon\Phi_0(y\sqrt{\varepsilon}),$$
$$\varphi_t(0, y) = -\varepsilon^3\Big\{4\Phi_0^{(4)}(y\sqrt{\varepsilon}) + \Phi_0''(y\sqrt{\varepsilon}) + \frac{1}{2}(\Phi_0'(y\sqrt{\varepsilon}))^2\Big\}.$$

Moreover, there exists a positive constant C, independent of $\varepsilon \in (0, \varepsilon_0]$, such that

$$|\varphi(t,y) - \varepsilon\Phi(t\varepsilon^2, y\sqrt{\varepsilon})| \le C\varepsilon^2, \quad 0 \le t \le \frac{T}{\varepsilon^2},\ |y| \le \frac{\ell_0}{2\sqrt{\varepsilon}} \tag{1.14}$$

for any $\varepsilon \in (0, \varepsilon_0]$.

In other words, starting from the same configuration, the solution of (1.11) remains on a fixed time interval close to the solution of K-S up to some renormalization, uniformly in ε sufficiently small. Note that the initial conditions for φ are of special type, compatible with Φ_0 and (1.6) at $\tau = 0$. Initial conditions of this type have been already considered in [1, 2, 4, 5].

Numerical computations on Equation (1.13) are presented in Section 5.

Eventually, in Appendix, we follow the framework of [5, Section 3] and derive a self-consistent equation for φ in the Fourier-Laplace variables. The latter has the same linear part as Equation (1.11) and will be studied in a forthcoming paper.

2 Some Mathematical Setting

In this section, we briefly introduce some notation and the functional spaces we will use below. We use the discrete Fourier transform with respect to the variable y. For this purpose, given a function $f : (-\frac{\ell}{2}, \frac{\ell}{2}) \to \mathbb{C}$, we denote by $\widehat{f}(k)$ its kth Fourier coefficient, that is, we write

$$f(y) = \sum_{k=0}^{+\infty} \widehat{f}(k) w_k(y), \quad y \in \Big(-\frac{\ell}{2}, \frac{\ell}{2}\Big),$$

where $\{w_k\}$ is a complete set of (complex valued) eigenfunctions of the operator

$$D_{yy} : H^2\Big(-\frac{\ell}{2}, \frac{\ell}{2}\Big) \to L^2\Big(-\frac{\ell}{2}, \frac{\ell}{2}\Big),$$

with ℓ-periodic boundary conditions, corresponding to the non-positive eigenvalues

$$0, -\frac{4\pi^2}{\ell^2}, -\frac{4\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{36\pi^2}{\ell^2}, \cdots.$$

We shall find it convenient to label this sequence as

$$0 = -\lambda_0(\ell) > -\lambda_1(\ell) = -\lambda_2(\ell) > -\lambda_3(\ell) = -\lambda_4(\ell) > \cdots.$$

In Section 3, we simply write λ_k instead of $\lambda_k(\ell)$.

For any integer s, we denote by $H^s_\sharp$ the usual Sobolev space of order s consisting of ℓ-periodic (generalized) functions, which we will conveniently represent as

$$H^s_\sharp = \Big\{u = \sum_{k=0}^{+\infty} \widehat{u}(k) w_k :\ \sum_{k=0}^{+\infty} \lambda_k^s \widehat{u}(k)^2 < +\infty\Big\}.$$

For $s = 0$, we simply write L^2 instead of $H^0_\sharp$ and we denote by $|\cdot|_2$ the usual L^2-norm.

By Πu, we denote the mean value of the function $u \in L^2$, i.e.,

$$\Pi u = \frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} u(y) \mathrm{d}y.$$

When $s>0$, we endow the space $(I-\Pi)H^s_\sharp$ with the norm

$$\|u\|_s^2=\sum_{k=0}^{+\infty}\lambda_k^s\widehat{u}(k)^2,\quad u\in(I-\Pi)H^s_\sharp.$$

3 Proof of Theorem 1.1

We begin this section by rewriting the initial value problem $\varphi(0,\cdot)=\varphi_0$, $\varphi_t(0,\cdot)=\varphi_1$ for Equation (1.11) in the following abstract form:

$$\begin{cases}\varphi_{tt}=B\varphi_t+A\varphi-\dfrac{1}{8}(\varphi_y)^2,\\ \varphi(0,\cdot)=\varphi_0,\\ \varphi_t(0,\cdot)=\varphi_1,\end{cases}\tag{3.1}$$

where $A=-D_{yyyy}-\frac{\alpha-1}{4}D_{yy}$, and $B=2D_{yy}-\frac{1}{4}I$ is the damping operator.

We write $\varphi(t,y)=\Pi(\varphi(t,\cdot))+((I-\Pi)\varphi(t,\cdot))(y):=r(t)+u(t,y)$ and split Problem (3.1) into the two problems

$$\begin{cases}r_{tt}(t)=-\dfrac{1}{4}r_t(t)-\dfrac{1}{8\ell}\Pi((u_y(t,\cdot))^2)=-\dfrac{1}{4}r_t(t)-\dfrac{1}{8\ell}\displaystyle\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}(u_y(t,y))^2\mathrm{d}y,\quad t>0,\\ r(0)=\Pi\varphi_0,\\ r_t(0)=\Pi\varphi_1\end{cases}\tag{3.2}$$

and

$$\begin{cases}u_{tt}(t,\cdot)=Bu_t(t,\cdot)+Au(t,\cdot)-\dfrac{1}{8}(I-\Pi)((u_y(t,\cdot))^2),\quad t>0,\\ u(0,\cdot)=(I-\Pi)\varphi_0,\\ u_t(0,\cdot)=(I-\Pi)\varphi_1.\end{cases}\tag{3.3}$$

Problem (3.2) can be immediately solved once the solution to Problem (3.3) is known. Indeed,

$$\begin{aligned}r(t)=&\Big(-4\Pi\varphi_1+\frac{1}{2}\int_0^t\mathrm{e}^{\frac{s}{4}}\Pi((u_y(s,\cdot))^2)\mathrm{d}s\Big)\mathrm{e}^{-\frac{1}{4}t}\\ &+\Pi\varphi_0+4\Pi\varphi_1-\frac{1}{2}\int_0^t\Pi((u_y(s,\cdot))^2)\mathrm{d}s,\quad t>0.\end{aligned}\tag{3.4}$$

Hence, the core of our analysis is Problem (3.3) which can be written as a first order system for the unknown $\mathbf{U}:=(u,u_t)$ as follows:

$$\begin{cases}\mathbf{U}_t(t,\cdot)=\mathscr{A}\mathbf{U}(t,\cdot)+\mathscr{F}(\mathbf{U}(t,\cdot)),\quad t>0,\\ \mathbf{U}(0,\cdot)=(I-\Pi)\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix},\end{cases}\tag{3.5}$$

where

$$\mathscr{A}\mathbf{u}:=\begin{pmatrix}0 & I\\ A & B\end{pmatrix}\mathbf{u},\quad \mathscr{F}(\mathbf{u})=\begin{pmatrix}0\\ -\frac{1}{8}(I-\Pi)((u_y)^2)\end{pmatrix},$$

on smooth vector-valued functions $\mathbf{u}=(u,v)$.

3.1 Study of the operator $\mathscr{A}$: generation of an analytic semigroup

In the next proposition, we will study the main properties of the operator $\mathscr{A}$ showing that it generates an analytic strongly continuous semigroup $(T(t))_{t\geq 0}$ in the space $\mathscr{X} = (I-\Pi)H^2_\sharp \times (I-\Pi)L^2$. We also characterize the spectrum of $\mathscr{A}$.

Proposition 3.1 *The operator $\mathscr{A}$ with domain $D(\mathscr{A}) = (I-\Pi)H^4_\sharp \times (I-\Pi)H^2_\sharp$ is the generator of an analytic strongly continuous semigroup in $\mathscr{X}$. Its spectrum $\sigma(\mathscr{A})$ consists of real eigenvalues only; it contains positive eigenvalues if and only if $\alpha > \alpha_c$ (see* (1.12)*).*

Proof To begin with we prove that the operator $\mathscr{A}$ is sectorial. For this purpose, we split $\mathscr{A}$ into the sum $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$, where

$$\mathscr{A}_0 = \begin{pmatrix} 0 & I \\ D_{yyyy} & B \end{pmatrix}, \quad \mathscr{A}_1 = \frac{1-\alpha}{4}\begin{pmatrix} 0 & 0 \\ D_{yy} & 0 \end{pmatrix}.$$

Since $\mathscr{A}_1$ is a bounded operator in $\mathscr{X}$, to prove that $\mathscr{A}$ generates a strongly continuous analytic semigroup it suffices to show that $\mathscr{A}_0$ generates an analytic semigroup in $\mathscr{X}$ with domain $D(\mathscr{A}_0) = D(\mathscr{A})$ (see e.g., [15, Proposition 2.4.1(i)]). For this purpose, we fix $\mathbf{f} = (f,g) \in \mathscr{X}$ and consider the resolvent equation

$$\begin{cases} \mu u = v + f, \\ \mu v = -u_{yyyy} + 2v_{yy} - \dfrac{1}{4}v + g \end{cases}$$

with periodic boundary conditions at $\pm\frac{\ell}{2}$. Plugging the first equation into the second one, we get the following self-consistent equation for u:

$$u_{yyyy} - 2\mu u_{yy} + \Big(\mu^2 + \frac{\mu}{4}\Big)u = \mu f - 2f_{yy} + \frac{1}{4}f + g. \tag{3.6}$$

We rewrite Equation (3.6) in Fourier variables. It gives the infinitely many equations

$$\Big(\lambda_n^2 + 2\mu\lambda_n + \mu^2 + \frac{\mu}{4}\Big)\widehat{u}(n) = \Big(\mu + 2\lambda_n + \frac{1}{4}\Big)\widehat{f}(n) + \widehat{g}(n), \quad n = 1,2,\cdots,$$

where we observe that $2\mu\lambda_n$ and $\frac{\mu}{4}$ are the contribution of the damping operator B. These terms play a crucial role in the estimates below.

If $\lambda_n^2 + 2\mu\lambda_n + \mu^2 + \frac{\mu}{4} \neq 0$, we get

$$\widehat{u}(n) = \frac{4\mu + 8\lambda_n + 1}{4\lambda_n^2 + 8\mu\lambda_n + 4\mu^2 + \mu}\widehat{f}(n) + \frac{4}{4\lambda_n^2 + 8\mu\lambda_n + 4\mu^2 + \mu}\widehat{g}(n) := a_n\widehat{f}(n) + b_n\widehat{g}(n)$$

for any $n = 1,2,\cdots$. Note that

$$\begin{aligned} &\Big|\lambda_n^2 + 2\mu\lambda_n + \mu^2 + \frac{\mu}{4}\Big|^2 \\ =\ &\frac{1}{16}\{16x^4 + 8x^3(8\lambda_n+1) + x^2(32y^2 + 96\lambda_n^2 + 16\lambda_n + 1) \\ &+ 8x(y^2+\lambda_n^2)(8\lambda_n+1) + 16y^4 + y^2(32\lambda_n^2 + 16\lambda_n + 1) + 16\lambda_n^4\}, \end{aligned} \tag{3.7}$$

where we set $\mu = x + \mathrm{i}y$. Since $\lambda_n > 0$ for any $n = 1,2,\cdots$, it follows immediately that $\lambda_n^2 + 2\mu\lambda_n + \mu^2 + \frac{\mu}{4}$ never vanishes in the right halfplane. Therefore, $\widehat{u}(n)$ is well-defined for any $n = 1, 2, \cdots$ and any $\mu \in \mathbb{C}$ with nonnegative real part. For any such fixed μ, it holds that $a_n \sim 2\lambda_n^{-1}$ and $b_n \sim \lambda_n^{-2}$ as $n \to +\infty$. Thus, $(a_n \widehat{f}(n))$ and $(b_n \widehat{g}(n))$ are the Fourier coefficients of a function in $(I - \Pi)H^4_\sharp$. Since $v = \mu u - f$, it follows that $v \in (I - \Pi)H^2_\sharp$.

Let us now prove that, for any $M > 0$, there exists a positive constant C, such that

$$\|R(\mu, \mathscr{A}_0)\mathbf{f}\|_{\mathscr{X}} \le \frac{C}{|\mu|}\|\mathbf{f}\|_{\mathscr{X}}, \quad \mathrm{Re}\,\mu \ge M. \tag{3.8}$$

Proposition 2.1.11 of [15] then will imply that $\mathscr{A}_0$ is sectorial. For this purpose, we observe that Formula (3.7) implies that

$$\left|\lambda_n^2 + 2\mu\lambda_n + \mu^2 + \frac{\mu}{4}\right| \ge \sqrt{|\mu|^4 + 2\lambda_n^2|\mu|^2} \ge \frac{1}{\sqrt{2}}|\mu|^2 + \lambda_n|\mu|$$

for any $\mu \in \mathbb{C}$ with positive real part and any $n = 1, 2, \cdots$. Hence,

$$|a_n| \le \left(\sqrt{2} + 1 + \frac{\sqrt{2}}{4M}\right)\frac{1}{|\mu|} \quad \text{and} \quad |b_n| \le \frac{1}{\lambda_n|\mu|} \tag{3.9}$$

for any $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu \ge M$ and any $n = 1, 2, \cdots$. Since

$$\|u\|_2^2 = \sum_{n=1}^{+\infty} \lambda_n^2|\widehat{u}(n)|^2 \le 2\sum_{n=1}^{+\infty} \lambda_n^2|a_n|^2|\widehat{f}(n)|^2 + 2\sum_{n=1}^{+\infty} \lambda_n^2|b_n|^2|\widehat{g}(n)|^2,$$

from (3.9) we deduce that

$$\|u\|_2 \le \left(2 + \sqrt{2} + \frac{1}{2M}\right)\frac{1}{|\mu|}(\|f\|_2 + |g|_2), \quad \mathrm{Re}\,\mu \ge M. \tag{3.10}$$

Let us now consider the function $v = \mu u - f$. As it is immediately seen

$$\widehat{v}(n) = -\frac{4\lambda_n^2}{4\mu^2 + 8\lambda_n\mu + \mu + 4\lambda_n^2}\widehat{f}(n) + \frac{4\mu}{4\mu^2 + 8\lambda_n\mu + \mu + 4\lambda_n^2}\widehat{g}(n) := c_n\widehat{f}(n) + d_n\widehat{g}(n)$$

for any $n = 1, 2, \cdots$. Since $|c_n| \le \lambda_n|\mu|^{-1}$ and $|d_n| \le \sqrt{2}|\mu|^{-1}$ for any $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu \ge M$ and any $n = 1, 2, \cdots$, we get

$$|v|_2^2 = \sum_{n=1}^{+\infty} |\widehat{v}(n)|^2 \le 2\sum_{n=1}^{+\infty} |c_n|^2|\widehat{f}(n)|^2 + 2\sum_{n=1}^{+\infty} |d_n|^2|\widehat{g}(n)|^2 \le \frac{2}{|\mu|^2}(\|f\|_2^2 + 2|g|_2^2),$$

i.e.,

$$|v|_2 \le \frac{2}{|\mu|}(\|f\|_2 + |g|_2). \tag{3.11}$$

From (3.10) and (3.11), estimate (3.8) follows with $C = 4 + \sqrt{2} + \frac{1}{2M}$.

Let us now characterize the spectrum of the operator $\mathscr{A}$. Since $D(\mathscr{A}) = (I - \Pi)H^4_\sharp \times (I - \Pi)H^2_\sharp$ compactly embeds in $\mathscr{X}$, $\sigma(\mathscr{A})$ consists of eigenvalues only. Let $\mu \in \mathbb{C}$ be any such eigenvalues. Setting $\mathbf{u} = (u, v)$ we are led to the system

$$\begin{cases} \mu u = v, \\ \mu v = -u_{yyyy} - \dfrac{\alpha - 1}{4}u_{yy} + 2v_{yy} - \dfrac{1}{4}v \end{cases}$$

with periodic boundary conditions at $\pm\frac{\ell}{2}$. Arguing as in the first part of the proof, we consider the infinitely many equations

$$\Big\{\lambda_n^2 - \Big(\frac{\alpha-1}{4} - 2\mu\Big)\lambda_n + \Big(\mu^2 + \frac{\mu}{4}\Big)\Big\}u_n = 0, \quad n = 1, 2, \cdots .$$

Clearly, μ is an eigenvalue of $\mathscr{A}$ if and only if there exists $n \in \mathbb{N}$ such that

$$\mu^2 + \Big(\frac{1}{4} + 2\lambda_n\Big)\mu + \lambda_n^2 - \frac{\alpha-1}{4}\lambda_n = 0,$$

i.e., if and only if $\mu = \mu_n^{\pm}$, where

$$\mu_n^{\pm} = \frac{-(1+8\lambda_n) \pm \sqrt{16\alpha\lambda_n + 1}}{8}, \quad n = 1, 2, \cdots .$$

Let us determine the values of α such that $\mathscr{A}$ admits eigenvalues with positive real parts. Clearly, we have only to consider the eigenvalues μ_n^+. Observe that $\mu_n^+ > 0$ if and only if $\sqrt{16\alpha\lambda_n + 1} \geq 1 + 8\lambda_n$, i.e., if and only if $\alpha \geq 4\lambda_n + 1$. It follows that the spectrum of $\mathscr{A}$ admits positive eigenvalues if and only if $\alpha > \alpha_c$.

3.2 Proof of Theorem 1.1

(a) Fix $\mathbf{u_0} = (\varphi_0, \varphi_1) \in \mathscr{X}$. From the remarks at the very beginning of this section, it is clear that the main point of the proof consists in proving that Problem (3.3) admits a unique solution, defined for all positive times, which decreases to zero as $t \to +\infty$. This property follows immediately from applying [9, Theorem 5.1.1] due to the results in Proposition 3.1. More precisely, for any $\omega \in (0, -\omega_0)$, where ω_0 denotes the maximum of the eigenvalues of the operator $\mathscr{A}$, there exists a positive constant C, such that

$$\|\mathbf{u}(t,\cdot)\|_{\mathscr{X}} \leq C\mathrm{e}^{-\omega t}\|\mathbf{u}_0\|_{\mathscr{X}}, \quad t > 0,$$

provided that the $\mathscr{X}$-norm of $\mathbf{u_0}$ is sufficiently small.

Let us now go back to Problem (3.2) whose solution is given by (3.4). A straightforward computation reveals that $r(t)$ converges to r_∞ as $t \to +\infty$, where

$$r_\infty = \Pi\varphi_0 + 4\Pi\varphi_1 - \frac{1}{2\ell}\int_0^{+\infty} \Pi((u_y(s,\cdot))^2)\mathrm{d}s.$$

Indeed,

$$\Big|\mathrm{e}^{-\frac{1}{4}t}\int_0^t \mathrm{e}^{\frac{s}{4}}\Pi((u_y(s,\cdot))^2)\mathrm{d}s\Big| = \Big|\int_0^t \mathrm{e}^{-\frac{s}{4}}\Pi((u_y(t-s,\cdot))^2)\mathrm{d}s\Big| \leq K\mathrm{e}^{-2\omega t}\int_0^t \mathrm{e}^{-\frac{s}{4}}\mathrm{e}^{2\omega s}\mathrm{d}s$$

for some $K > 0$. If we fix $\omega < \frac{1}{8}$, then the last side of the previous chain of inequalities vanishes as $t \to +\infty$. Since

$$r'(t) = \Big(\Pi\varphi_1 - \frac{1}{8\ell}\int_0^t \mathrm{e}^{\frac{s}{4}}\Pi((u_y(s,\cdot))^2)\mathrm{d}s\Big)\mathrm{e}^{-\frac{1}{4}t}$$

for any $t > 0$, the above arguments show that $r'(t)$ tends to 0 as $t \to +\infty$.

We have so proved that, if the data φ_0 and φ_1 have sufficiently small $H^2_\sharp$- and L^2-norms, respectively, then Problem (3.1) admits a solution φ defined in $[0, +\infty)$, such that

$$\lim_{t\to+\infty}\varphi(t) = \Pi\varphi_0 + 4\Pi\varphi_1 - \frac{1}{2\ell}\int_0^{+\infty} \Pi((u_y(s,\cdot))^2)\mathrm{d}s,$$

whereas $\varphi_t(t,\cdot)$ tends to zero as $t\to+\infty$ with exponential rate.

Actually, the previous one is the only solution to Problem (3.1). This can be proved, i.e., adapting the arguments in [11].

(b) To prove the second part of Theorem 1.1, it suffices to apply [9, Theorem 5.1.3], which gives the instability of the null solution to the differential equation $\mathbf{U}_t=\mathscr{A}\mathbf{U}+\mathscr{F}(\mathbf{U})$. Note that, since the eigenvalues of $\mathscr{A}$ define two sequences (μ_n^+) and (μ_n^-) which tend to $-\infty$ as $n\to+\infty$, there is a gap between the imaginary axis and the part of $\sigma(\mathscr{A})$ in the open rightplane which is a finite set. Hence, in particular, it is a spectral set.

This shows that either the solution u to Problem (3.3) does not exist for any positive t or at least one between u and u_t does not become small when the datum (φ_0,φ_1) vanishes in $\mathscr{X}$. Hence, the null solution to Problem (3.1) is unstable.

4 Convergence to K-S

Let φ be a solution to (1.11). We set $\alpha=1+\varepsilon$ and define the rescaled dependent and independent variables:

$$t=\frac{\tau}{\varepsilon^2},\quad y=\frac{\eta}{\sqrt{\varepsilon}},\quad \varphi=\varepsilon\psi. \tag{4.1}$$

The spatial period is now $\ell_\varepsilon=\frac{\ell_0}{\sqrt{\varepsilon}}$, for some $\ell_0>4\pi$ fixed (see Section 1). Obviously the function ψ satisfies Equation (1.13) that we recall

$$4\varepsilon^2\psi_{\tau\tau}+(I-8\varepsilon D_{\eta\eta})\psi_\tau+4\psi_{\eta\eta\eta\eta}+\psi_{\eta\eta}+\frac{1}{2}(\psi_\eta)^2=0.$$

We split ψ as follows:

$$\psi=\Phi+\varepsilon\rho,\quad \rho=\rho(\varepsilon), \tag{4.2}$$

where Φ solves the K-S equation

$$\Phi_\tau+4\Phi_{\eta\eta\eta\eta}+\Phi_{\eta\eta}+\frac{1}{2}(\Phi_\eta)^2=0$$

on the interval $[-\frac{\ell_0}{2},\frac{\ell_0}{2}]$ with periodic boundary conditions. For the convenience of the reader, we recall the regularity theorem (see [5, Appendix B]) as follows.

Theorem 4.1 *Let $\Phi_0\in H^m_\sharp$ for some $m\ge 4$ and fix $T>0$. Then, the Cauchy problem*

$$\begin{cases}\Phi_\tau(\tau,\eta)=-4\Phi_{\eta\eta\eta\eta}(\tau,\eta)-\Phi_{\eta\eta}(\tau,\eta)-\dfrac{1}{2}(\Phi_\eta(\tau,\eta))^2, & \tau\ge 0,\ |\eta|\le\dfrac{\ell_0}{2},\\ D^k_\eta\Phi\Big(\tau,-\dfrac{\ell_0}{2}\Big)=D^k_\eta\Phi\Big(\tau,\dfrac{\ell_0}{2}\Big), & \tau\ge 0,\ k=0,1,2,3,\\ \Phi(0,\eta)=\Phi_0(\eta), & |\eta|\le\frac{\ell_0}{2}\end{cases}$$

admits a unique solution $\Phi\in C([0,T];H^m_\sharp)$ such that $\Phi_\tau\in C([0,T];H^{m-4}_\sharp)$.

We assume m sufficiently large to justify all our estimates below. By assumptions (see Theorem 1.2), the initial conditions for ρ are

$$\rho(0,\cdot)=\rho_\tau(0,\cdot)=0.$$

Replacing $\psi = \Phi + \varepsilon\rho$ in (1.13) we get, after simplifying by ε,

$$4\varepsilon^2\rho_{\tau\tau} + (I - 8\varepsilon D_{\eta\eta})\rho_\tau + 4\rho_{\eta\eta\eta\eta} + \rho_{\eta\eta} + \frac{1}{2}\varepsilon(\rho_\eta)^2 + \Phi_\eta\rho_\eta = -4\varepsilon\Phi_{\tau\tau} + 8D_{\eta\eta}\Phi_\tau. \tag{4.3}$$

Since all the operators appearing in (4.3) commute with D_η, the *differentiated* problem for $\zeta := \rho_\eta \in (I - \Pi)L^2$ reads as follows:

$$4\varepsilon^2\zeta_{\tau\tau} + (I - 8\varepsilon D_{\eta\eta})\zeta_\tau + 4\zeta_{\eta\eta\eta\eta} + \zeta_{\eta\eta} + \varepsilon\zeta\zeta_\eta + \Psi_\eta\zeta + \Psi\zeta_\eta = -4\varepsilon\Psi_{\tau\tau} + 8D_{\eta\eta}\Psi_\tau, \tag{4.4}$$

where we have set $\Psi = \Phi_\eta$. Obviously, Equation (4.4) is to be solved with zero initial conditions for ζ and ζ_τ at time $\tau = 0$.

4.1 Formal a priori estimates

In this section, we determine a priori L^2-estimates for the solution to Equation (4.4) and some of its derivatives.

(i) We formally multiply both sides of Equation (4.4) by ζ_τ and integrate by parts over $(-\frac{\ell_0}{2}, \frac{\ell_0}{2})$. We thus get

$$2\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_\tau)^2\mathrm{d}\eta\Big) + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_\tau)^2\mathrm{d}\eta + 8\varepsilon\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta})^2\mathrm{d}\eta + 2\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\eta\eta})^2\mathrm{d}\eta\Big) = F,$$

where

$$F = -\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\eta\eta} + \varepsilon\zeta\zeta_\eta + \Psi_\eta\zeta + \Psi\zeta_\eta)\zeta_\tau\mathrm{d}\eta + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(-4\varepsilon\Psi_{\tau\tau} + 8D_{\eta\eta}\Psi_\tau)\zeta_\tau\mathrm{d}\eta.$$

We estimate the integrals in the right-hand side F, by the Cauchy-Schwarz inequality. Below the c_i's are positive constants independent of ε. We have

$$|F| \le \frac{1}{2}|\zeta_\tau|_2^2 + c_0(|\zeta_{\eta\eta}|_2^2 + \varepsilon^2|\zeta\zeta_\eta|_2^2 + \|\Psi_\eta\|_\infty^2|\zeta|_2^2 + \|\Psi\|_\infty^2|\zeta_\eta|_2^2) + c_0|-4\varepsilon\Psi_{\tau\tau} + 8D_{\eta\eta}\Psi_\tau|_2^2.$$

Using several times the Poincaré-Wirtinger inequality, it is not difficult to see that

$$|F| \le \frac{1}{2}|\zeta_\tau|_2^2 + c_1\varepsilon|\zeta_{\eta\eta}|_2^4 + c_2|\zeta_{\eta\eta}|_2^2 + c_3.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(2\varepsilon^2|\zeta_\tau|_2^2 + 2|\zeta_{\eta\eta}|_2^2) + \frac{1}{2}|\zeta_\tau|_2^2 \le c_1\varepsilon|\zeta_{\eta\eta}|_2^4 + c_2|\zeta_{\eta\eta}|_2^2 + c_3. \tag{4.5}$$

(ii) Next, we multiply both sides of Equation (4.4) by $-D_{\eta\eta}\zeta_\tau$. Integrating by parts over $(-\frac{\ell_0}{2}, \frac{\ell_0}{2})$, it comes

$$2\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta})^2\mathrm{d}\eta\Big) + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta})^2\mathrm{d}\eta + 8\varepsilon\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta\eta})^2\mathrm{d}\eta + 2\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\eta\eta\eta})^2\mathrm{d}\eta\Big) = G,$$

where

$$G = -\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}D_\eta(\zeta_{\eta\eta} + \varepsilon\zeta\zeta_\eta + \Psi_\eta\zeta + \Psi\zeta_\eta)\zeta_{\tau\eta}\mathrm{d}\eta + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(-4\varepsilon\Psi_{\tau\tau\eta} + 8D_{\eta\eta\eta}\Psi_\tau)\zeta_{\tau\eta}\mathrm{d}\eta.$$

As above,

$$|G| \leq \frac{1}{2}|\zeta_{\tau\eta}|_2^2 + c_4\varepsilon|\zeta_{\eta\eta\eta}|_2^4 + c_5|\zeta_{\eta\eta\eta}|_2^2 + c_6$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(2\varepsilon^2|\zeta_{\tau\eta}|_2^2 + 2|\zeta_{\eta\eta\eta}|_2^2) + \frac{1}{2}|\zeta_{\tau\eta}|_2^2 \leq c_4\varepsilon|\zeta_{\eta\eta\eta}|_2^4 + c_5|\zeta_{\eta\eta\eta}|_2^2 + c_6. \tag{4.6}$$

(iii) Finally, we multiply both sides of Equation (4.4) by $D_{\eta\eta\eta\eta}\zeta_\tau$ and get

$$2\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta\eta})^2\mathrm{d}\eta\Big) + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta\eta})^2\mathrm{d}\eta + 8\varepsilon\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\tau\eta\eta\eta})^2\mathrm{d}\eta + 2\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}(\zeta_{\eta\eta\eta\eta})^2\mathrm{d}\eta\Big) = H,$$

where

$$H = -\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} D_{\eta\eta}(\zeta_{\eta\eta} + \varepsilon\zeta\zeta_\eta + \Psi_\eta\zeta + \Psi\zeta_\eta)\zeta_{\tau\eta\eta}\mathrm{d}\eta + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} D_{\eta\eta}(-4\varepsilon\Psi_{\tau\tau} + 8D_{\eta\eta}\Psi_\tau)\zeta_{\tau\eta\eta}\mathrm{d}\eta.$$

Again,

$$|H| \leq \frac{1}{2}|\zeta_{\tau\eta\eta}|_2^2 + c_7\varepsilon|\zeta_{\eta\eta\eta\eta}|_2^4 + c_8|\zeta_{\eta\eta\eta\eta}|_2^2 + c_9,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(2\varepsilon^2|\zeta_{\tau\eta\eta}|_2^2 + 2|\zeta_{\eta\eta\eta\eta}|_2^2) + \frac{1}{2}|\zeta_{\tau\eta\eta}|_2^2 \leq c_7\varepsilon|\zeta_{\eta\eta\eta\eta}|_2^4 + c_8|\zeta_{\eta\eta\eta\eta}|_2^2 + c_9. \tag{4.7}$$

We recall the following technical lemma (see [1]).

Lemma 4.1 *Assume that a family of nonnegative functions $A_\varepsilon \in C^1([0, T_0])$, $\varepsilon \in (0, 1]$, satisfies*

$$A'_\varepsilon \leq C_0 + C_1 A_\varepsilon + C_2\varepsilon A_\varepsilon^2 + C_3\varepsilon^2 A_\varepsilon^3, \quad A_\varepsilon(0) \leq A_0$$

with positive constants A_0, C_i, independent of ε. Then there exist $\varepsilon_0 > 0$, $K_0 > 0$ such that $A_\varepsilon(\tau) \leq K_0$ for all $\tau \in [0, T_0]$ whenever $0 < \varepsilon \leq \varepsilon_0$.

Adding Formulae (4.5)–(4.7), we can take $A_0 = 0$ and

$$A_\varepsilon = \varepsilon^2(|\zeta_\tau|_2^2 + |\zeta_{\tau\eta}|_2^2 + |\zeta_{\tau\eta\eta}|_2^2) + (|\zeta_{\eta\eta}|_2^2 + |\zeta_{\eta\eta\eta}|_2^2 + |\zeta_{\eta\eta\eta\eta}|_2^2).$$

Therefore there exists a constant $K_1 > 0$ such that

$$\begin{aligned}&\sup_{\tau\in[0,T_0]} \varepsilon^2(|\zeta_\tau(\tau,\cdot)|_2^2 + |\zeta_{\tau\eta}(\tau,\cdot)|_2^2 + |\zeta_{\tau\eta\eta}(\tau,\cdot)|_2^2)\\ &+ \sup_{[0,T_0]}(|\zeta_{\eta\eta}(\tau,\cdot)|_2^2 + |\zeta_{\eta\eta\eta}(\tau,\cdot)|_2^2 + |\zeta_{\eta\eta\eta\eta}(\tau,\cdot)|_2^2)\\ &+ \int_0^{T_0}(|\zeta_\tau(\tau,\cdot)|_2^2 + |\zeta_{\tau\eta}(\tau,\cdot)|_2^2 + |\zeta_{\tau\eta\eta}(\tau,\cdot)|_2^2)\mathrm{d}\tau \leq K_1.\end{aligned} \tag{4.8}$$

As a byproduct, it follows from Equation (4.4) that the map $\tau \mapsto \varepsilon^3|\zeta_{\tau\tau}(\tau,\cdot)|_2$ is also bounded in the sup-norm.

4.2 Existence and uniqueness

We have the following result about ζ.

Proposition 4.1 *Assume that m is large enough in Theorem 4.1. Then, for any $T > 0$, there exists $\varepsilon_0(T) > 0$ such that, for any $0 < \varepsilon \leq \varepsilon_0(T)$, Equation (4.4) subject to the initial conditions $\zeta(0,\cdot) = \zeta_\tau(0,\cdot) = 0$ has a unique solution $\zeta \in C([0,T]; H^4_\sharp)$ such that $\zeta_\tau \in C([0,T]; H^2_\sharp)$ and $\zeta_{\tau\tau} \in C([0,T]; L^2)$.*

The proof of Proposition 4.1, by mean of a Faedo-Galerkin method, will not be elaborated here (see [20], especially Section IV(4), for details). The Fourier variational framework uses the eigenfunctions $\{w_k\}$ defined in Section 2, with $\ell = \ell_0$.

We now return to ρ and Equation (4.3). The mean value of ρ verifies

$$\begin{cases} 4\varepsilon^2(\Pi\rho)_{\tau\tau} + (\Pi\rho)_\tau = -\dfrac{1}{2\ell_0}\varepsilon \displaystyle\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \zeta^2 \mathrm{d}\eta - \dfrac{1}{\ell_0}\displaystyle\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \Phi_\eta \zeta \mathrm{d}\eta - \dfrac{4}{\ell_0}\varepsilon \displaystyle\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \Phi_{\tau\tau} \mathrm{d}\eta, \\ (\Pi\rho)(0) = (\Pi\rho)_\tau(0) = 0. \end{cases} \tag{4.9}$$

Denote by χ the right-hand side of (4.9). From Theorem 4.1, it follows that χ is a continuous function in $[0,T]$. A straightforward computation now reveals that

$$(\Pi\rho)(\tau) = 4\varepsilon \int_0^\tau \chi(s)\mathrm{d}s - 4\varepsilon \int_0^\tau \mathrm{e}^{-\frac{s}{4\varepsilon}} \chi(\tau - s)\mathrm{d}s, \quad \tau \in [0,T]. \tag{4.10}$$

Let $\widetilde{\rho}$ be the primitive of ζ with 0 mean value on $[-\frac{\ell_0}{2}, \frac{\ell_0}{2}]$. We uniquely define ρ as $\rho(\tau) = \widetilde{\rho}(\tau) + (\Pi\rho)(\tau)$ for any $\tau \in [0,T]$.

4.3 Proof of Theorem 1.2

To complete the proof of Theorem 1.2, let us check that there exists $M > 0$ such that

$$\sup_{\substack{\tau\in[0,T] \\ \eta\in[-\frac{\ell_0}{2},\frac{\ell_0}{2}]}} |\rho(\tau,\eta)| \leq M, \tag{4.11}$$

uniformly in $0 < \varepsilon \leq \varepsilon_0(T)$.

Estimate (4.8) provides us with a uniform estimate of $\rho_\eta = \zeta$ on $[0,T] \times [-\frac{\ell_0}{2}, \frac{\ell_0}{2}]$ thanks to the Poincaré-Wirtinger inequality, since ζ has zero mean value. Again, the Poincaré-Wirtinger inequality gives us a uniform estimate on $[0,T] \times [-\frac{\ell_0}{2}, \frac{\ell_0}{2}]$ of $\rho - \Pi\rho$. Finally, from (4.10) we immediately deduce that $\Pi\rho$ is bounded in $[0,T]$. Estimate (4.11) follows at once.

Using (4.1) and (4.2), it is now immediate to check that the function φ satisfies (1.14).

5 Numerical Experiments

We follow the framework of [10]. Equation (1.13) is now formulated on the fixed interval $[0, 2\pi]$ with periodic boundary conditions, setting $x = 2\pi\frac{y}{\ell_0} = \frac{y}{2\widetilde{\ell}_0}$. As in [10], we introduce the bifurcation parameter $\beta = 4(\widetilde{\ell}_0)^2$. After multiplication by β^2, it comes

$$4\varepsilon^2\beta^2\psi_{\tau\tau} + (\beta^2 I - 8\varepsilon\beta D_{xx})\psi_\tau + 4\psi_{xxxx} + \beta\Big(\psi_{xx} + \frac{1}{2}(\psi_x)^2\Big) = 0.$$

Changing the time $t = \frac{\tau}{\beta^2}$,

$$4\frac{\varepsilon^2}{\beta^2}\psi_{tt} + \Big(I - 8\frac{\varepsilon}{\beta}D_{xx}\Big)\psi_t + 4\psi_{xxxx} + \beta\Big(\psi_{xx} + \frac{1}{2}(\psi_x)^2\Big) = 0,$$

and setting $\varepsilon' = \frac{\varepsilon}{\beta}$, the prime being omitted, we eventually get

$$4\varepsilon^2\psi_{tt} + (I - 8\varepsilon D_{xx})\psi_t + 4\psi_{xxxx} + \beta\Big(\psi_{xx} + \frac{1}{2}(\psi_x)^2\Big) = 0.$$

The mean value $\Pi\psi = \frac{1}{2\pi}\int_0^{2\pi}\psi(\cdot, x)\mathrm{d}x$ satisfies the drift equation

$$4\varepsilon^2(\Pi\psi)'' + (\Pi\psi)' + \frac{\beta}{4\pi}\int_0^{2\pi}(\psi_x(\cdot, x))^2\mathrm{d}x = 0.$$

To normalize this drift to zero, we numerically solve the equation for $v(t, x) = \psi(t, x) - (\Pi\psi)(t)$:

$$4\varepsilon^2 v_{tt} + (I - 8\varepsilon D_{xx})v_t + 4v_{xxxx} + \beta\Big(v_{xx} + \frac{1}{2}(v_x)^2 - \frac{1}{4\pi}\int_0^{2\pi}(v_x(t, x))^2\mathrm{d}x\Big) = 0. \tag{5.1}$$

A complete numerical algorithm requires a discretization strategy in both time and space. Since the Fourier method is one of the most suitable approximation for periodic problems, it will be employed to handle the spatial discretization (see [22]). The time discretization combines a Newmark schema for the second-order derivative in time, backward-Euler schema for the first-order derivative in time, implicit treatment for all linear terms and explicit treatment for all nonlinear terms. Precisely, the time schema reads

$$\begin{aligned}&\frac{4\varepsilon^2}{\Delta t^2}(u^{n+1} - 2u^n + u^{n-1}) - \frac{8\varepsilon}{\Delta t}(u^{n+1}_{xx} - u^n_{xx}) + \frac{1}{\Delta t}(u^{n+1} - u^n) + 4u^{n+1}_{xxxx} + \beta u^{n+1}_{xx}\\ &= -\frac{\beta}{2}(u^n_x)^2 + \frac{\beta}{4\pi}\int_0^{2\pi}(u^n_x)^2\mathrm{d}x\end{aligned}$$

for $n \geq 1$, and

$$\begin{aligned}&\frac{4\varepsilon^2}{\Delta t^2}(u^1 - u^0 - \Delta t\sigma^0) - \frac{8\varepsilon}{\Delta t}(u^1_{xx} - u^0_{xx}) + \frac{1}{\Delta t}(u^1 - u^0) + 4u^1_{xxxx} + \beta u^1_{xx}\\ &= -\frac{\beta}{2}(u^0_x)^2 + \frac{\beta}{4\pi}\int_0^{2\pi}(u^0_x)^2\mathrm{d}x\end{aligned}$$

for the first step calculation. The initial conditions are given by

$$u^0(x) = u(0, x) = \varepsilon\phi_0(x), \quad \sigma_0(x) = u_t(0, x) = -\varepsilon\Big(4\phi_0^{(4)}(x) + \phi_0''(x) + \frac{1}{2}(\phi_0')^2(x)\Big).$$

This method is of first order accuracy with respect to the time step. The use of such a schema is motivated by the following considerations: the implicit treatment of the fourth- and second-order terms allows to reduce the associated stability constraint, while the explicit treatment of the nonlinear terms avoids the expensive process of solving nonlinear equations at each time step.

The Fourier method in space consists in finding an approximate solution $u^n_K(x)$ in form of a truncated Fourier expansion (for convenience here the notation slightly differs from the one in Section 2):

$$u^n_K(x) = \sum_{k=-K}^{K}\widehat{u}^n_k\exp(-\mathrm{i}kx),$$

where K is a positive integer. By applying Fourier transformation to the semi-discretized equations, we obtain a set of equations for each mode k in the Fourier space

$$\frac{4\varepsilon^2}{\Delta t^2}(\widehat{u}_k^{n+1}-2\widehat{u}_k^n+\widehat{u}_k^{n-1})+\frac{8\varepsilon k^2}{\Delta t}(\widehat{u}_k^{n+1}-\widehat{u}_k^n)+\frac{1}{\Delta t}(\widehat{u}_k^{n+1}-\widehat{u}_k^n)+4k^4\widehat{u}_k^{n+1}-\beta k^2\widehat{u}_k^{n+1}$$
$$=\Big\{-\frac{\beta}{2}(u_x^n)^2+\frac{\beta}{4\pi}\int_0^{2\pi}(u_x^n)^2\mathrm{d}x\Big\}_k, \tag{5.2}$$

where, as $\widehat{f}_k$, $\{f\}_k$ also represents the kth Fourier coefficient of the function f. The Fourier coefficients of the nonlinear terms are calculated by performing the discrete fast Fourier transform (FFT). In practical calculations, we work in the spectral space. An additional FFT is needed to recover the physical nodal values u_k $(-K\le k\le K)$ from $\widehat{u}$. By using (5.2), the kth Fourier coefficient $\widehat{u}_k^{n+1}$ can be obtained by a simple inversion

$$\widehat{u}_k^{n+1}=\Big[\Big(\frac{4\varepsilon^2}{\Delta t^2}+\frac{1}{\Delta t}\Big)+\Big(\frac{8\varepsilon}{\Delta t}-\beta\Big)k^2+4k^4\Big]^{-1}\Big[\Big(\frac{8\varepsilon^2}{\Delta t^2}+\frac{1}{\Delta t}\Big)\widehat{u}_k^n+\frac{8\varepsilon k^2}{\Delta t}\widehat{u}_k^n-\frac{4\varepsilon^2}{\Delta t^2}\widehat{u}_k^{n-1}$$
$$+\Big\{-\frac{\beta}{2}(u_x^n)^2+\frac{\beta}{4\pi}\int_0^{2\pi}(u_x^n(t,x))^2\mathrm{d}x\Big\}_k\Big].$$

The purpose of the numerical tests is to check the behavior of the solutions of Equation (5.1) as compared to the Kuramoto-Sivashinsky equation when ε tends to zero. To this end, we first fix $\beta=10$, and let ε vary. In Figures 1–3, we plot consecutive front positions computed by (5.2) with $\beta=10$ for $\varepsilon=0.04$, $\varepsilon=0.001$ and $\varepsilon=0$ respectively. Note that the case $\varepsilon=0$ corresponds to the K-S equation. It is observed that the solutions of (5.1) converge to the solution of the K-S equation as ε tends to zero.

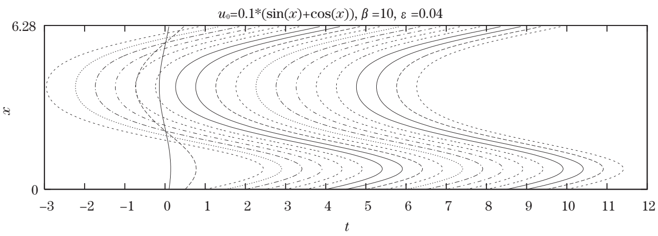

Figure 1 Front evolution with $\beta=10,\ \varepsilon=0.04$

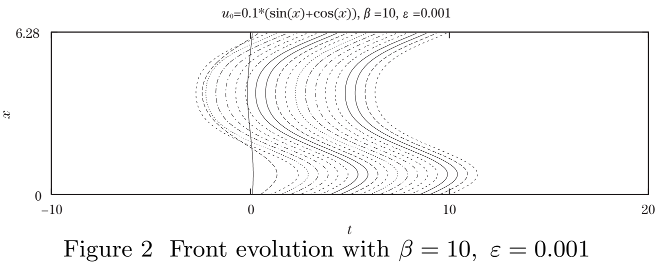

Figure 2 Front evolution with $\beta=10,\ \varepsilon=0.001$

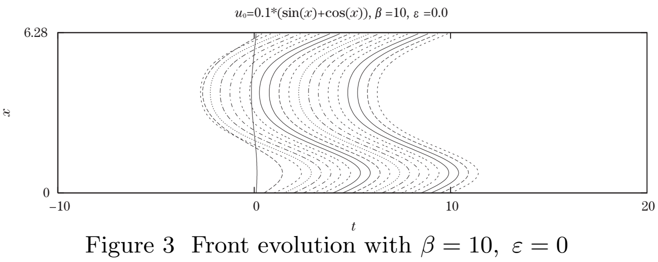

Figure 3 Front evolution with $\beta=10,\ \varepsilon=0$

In order to investigate the dynamics of Equation (5.1) with respect to the parameter β, we now fix $\varepsilon = 0.01$.

We have confirmed that, similar to the K-S equation, for $1 \leq \beta \leq 4$, 0 is a global attractor for the solution to Equation (5.1). A non-trivial attractor is expected for larger β. In Figures 4–5, we can see the front evolutions generated by (5.2) with $\beta = 18$ for two different initial conditions. The same calculation is repeated with $\beta = 30$, and the result is given in Figures 6–7. In all these figures, the periodic orbit is clearly observed.

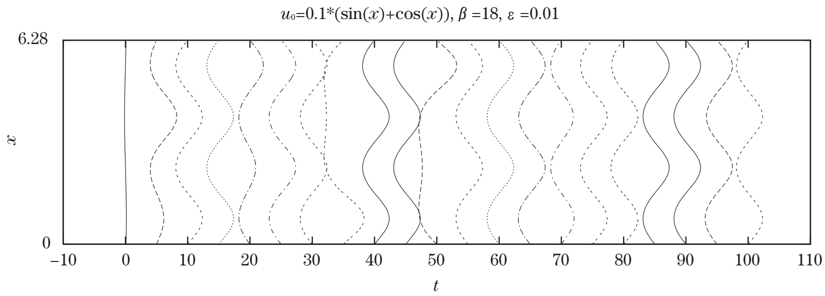

Figure 4 Front evolution with $\beta = 18$, $\varepsilon = 0.01$, and $u_0 = 0.1(\sin(x) + \cos(x))$

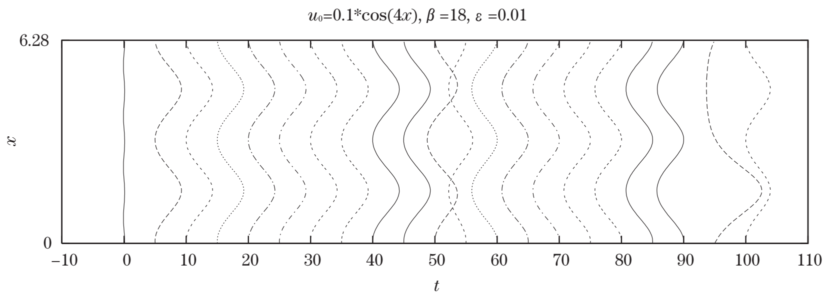

Figure 5 Front evolution with $\beta = 18$, $\varepsilon = 0.01$, and $u_0 = 0.1\cos(4x)$

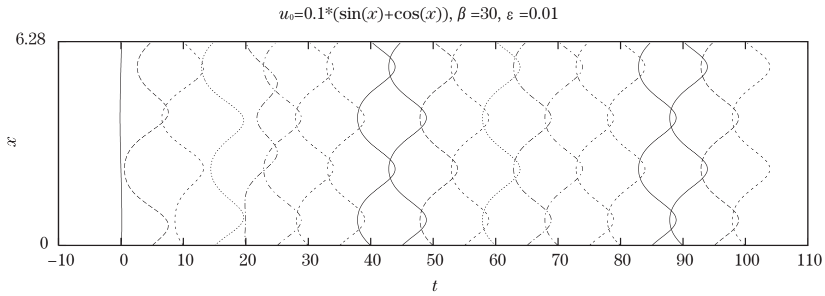

Figure 6 Front evolution with $\beta = 30$, $\varepsilon = 0.01$, and $u_0 = 0.1(\sin(x) + \cos(x))$

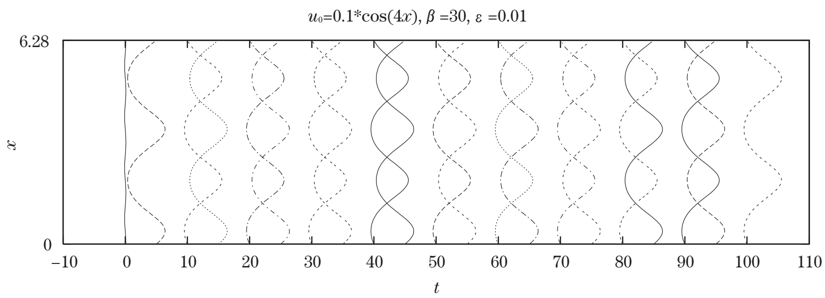

Figure 7 Front evolution with $\beta = 30$, $\varepsilon = 0.01$, and $u_0 = 0.1\cos(4x)$

These numerical results indicate that the solution of Equation (5.1) preserves the same structure as K-S equation. Richer dynamics can be generated by using even larger β. Finally, we plot in Figure 8 the front propagation captured from calculation with $\beta = 105$. As expected from the paper [10], the front evolves toward an essentially quadrimodal global attractor.

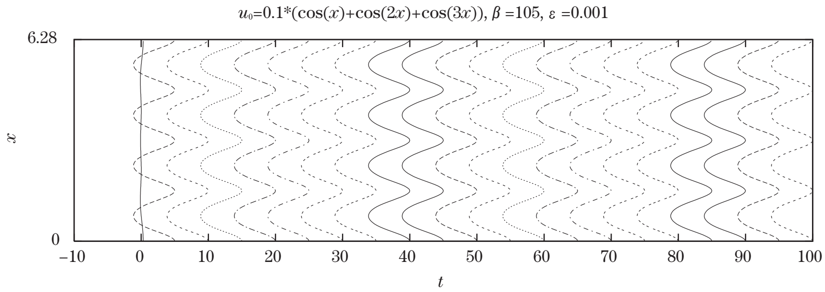

Figure 8 Front propagation with $\beta = 105$, $\varepsilon = 0.001$

Appendix The Derivation of a Self-consistent Equation for the Front

System (1.1)–(1.4), set in $\mathbb{R} \times [-\frac{\ell}{2}, \frac{\ell}{2}]$, admits a planar TW solution, with velocity -1:

$$\overline{\theta}(x) = \begin{cases} \exp(x), & x \le 0, \\ 1, & x > 0, \end{cases} \qquad \overline{S}(x) = \begin{cases} \alpha x \exp(x), & x \le 0, \\ 0, & x > 0. \end{cases}$$

As usual one fixes the free boundary. We set $\xi(t, y) = -t + \varphi(t, y)$, $x' = x - \xi(t, y)$. In this new framework, the system reads

$$\begin{aligned} \theta_t + (1 - \varphi_t)\theta_{x'} &= \Delta_\varphi \theta, && x' < 0, \\ \theta(x') &= 1, && x' > 0, \\ S_t + (1 - \varphi_t)S_{x'} &= \Delta_\varphi S - \alpha \Delta_\varphi \theta, && x' \neq 0, \end{aligned}$$

where

$$\Delta_\varphi = (1 + (\varphi_y)^2) D_{x'x'} + D_{yy} - \varphi_{yy} D_{x'} - 2\varphi_y D_{x'y}.$$

The front is now fixed at $x' = 0$. The first condition in (1.4) reads

$$\sqrt{1 + (\varphi_y)^2} \left[\frac{\partial \theta}{\partial x'}\right] = -\exp(S);$$

the second one becomes

$$\left[\frac{\partial S}{\partial x'}\right] = \alpha \left[\frac{\partial \theta}{\partial x'}\right].$$

Let us consider the perturbations of temperature u and enthalpy v:

$$\theta = \overline{\theta} + u, \quad S = \overline{S} + v.$$

Writing for simplicity x instead of x', the problem for the triplet (u, v, φ) reads

$$\begin{aligned} u_t + (1 - \varphi_t)u_x - \Delta_\varphi u - \varphi_t \overline{\theta}_x &= (\Delta_\varphi - \Delta)\overline{\theta}, && x < 0, \\ u &= 0, && x > 0, \\ v_t + (1 - \varphi_t)v_x - \Delta_\varphi(v - \alpha u) - \varphi_t \overline{S}_x &= (\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}), && x \neq 0, \end{aligned}$$

where

$$(\Delta_\varphi - \Delta)(\overline{\theta}) = ((\varphi_y)^2 - \varphi_{yy})\overline{\theta}_x, \quad (\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}) = \alpha((\varphi_y)^2\overline{S}_x - \varphi_{yy}\overline{S}).$$

As in [3–5], we introduce some simplifications: we keep only linear and second-order terms for the perturbation of the front φ, and first-order terms for the perturbations of temperature u and enthalpy v. This leads to the equations

$$\begin{aligned} u_t + u_x - \Delta u - \varphi_t\overline{\theta}_x &= (\Delta_\varphi - \Delta)\overline{\theta}, & x < 0, \\ u &= 0, & x \geq 0, \\ v_t + v_x - \Delta(v - \alpha u) - \varphi_t\overline{S}_x &= (\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}), & x \neq 0. \end{aligned}$$

At $x = 0$ there are several conditions. First

$$[u] = [v] = 0,$$

however, since $u(x) = 0$ for $x > 0$, this is equivalent to

$$u(0^-) = [v] = 0.$$

Second,

$$\sqrt{1 + (\varphi_y)^2}\ [\overline{\theta}_x + u_x] = -\exp(\overline{S} + v),$$

hence up to the second-order,

$$-1 + [u_x] = -(1 + (\varphi_y)^2)^{-\frac{1}{2}}\mathrm{e}^v \sim \Big(1 - \frac{1}{2}(\varphi_y)^2\Big)\Big(1 + v(0) + \frac{1}{2}(v(0))^2\Big),$$

and keeping only the first-order for v yields

$$\begin{aligned} -u_x(0) + v(0) &= \frac{1}{2}(\varphi_y)^2, \\ [v_x] &= -\alpha u_x(0). \end{aligned}$$

Therefore, the final system reads

$$\begin{cases} u_t + u_x - \Delta u - \varphi_t\overline{\theta}_x = ((\varphi_y)^2 - \varphi_{yy})\overline{\theta}_x, & x < 0, \\ v_t + v_x - \Delta(v - \alpha u) - \varphi_t\overline{S}_x = (\varphi_y)^2\overline{S}_x - \varphi_{yy}\overline{S}, & x \neq 0, \\ u(0) = [v] = 0, \\ v(0) - u_x(0) = \dfrac{1}{2}(\varphi_y)^2, \\ [v_x] = -\alpha u_x(0). \end{cases} \tag{A.1}$$

We remark that the equation for u associated with the boundary condition $u(0) = 0$ entirely determines u when φ is given. Therefore, it can be viewed as a kind of pseudo-differential Stefan condition.

The aim of this appendix is the derivation of a self-consistent equation for the front φ, both in Fourier (as in [5]) and in Laplace variables. For this purpose, we rewrite Problem (A.1), making $\overline{\theta}$ and $\overline{S}$ explicit and assuming that u and v vanish at $t=0$, whereas $\varphi(0,\cdot)=\varphi_0$ for some prescribed (and smooth enough) function φ_0. We get

$$\begin{cases} u_t+u_x-\Delta u=(\varphi_t+(\varphi_y)^2-\varphi_{yy})\mathrm{e}^x, & x<0,\\ v_t+v_x-\Delta(v-\alpha u)=\alpha(\varphi_t+(\varphi_y)^2)(x+1)\mathrm{e}^x-\alpha\varphi_{yy}x\mathrm{e}^x, & x<0,\\ v_t+v_x-\Delta v=0, & x>0,\\ u(\cdot,0,\cdot)=[v]=0,\\ v(\cdot,0,\cdot)-u_x(\cdot,0,\cdot)=\dfrac{1}{2}(\varphi_y)^2,\\ [v_x]=-\alpha u_x(\cdot,0,\cdot),\\ u(0,\cdot)=v(0,\cdot)=0,\\ \varphi(0,\cdot)=\varphi_0. \end{cases} \tag{A.2}$$

In what follows, we assume that (u,v,φ) is a sufficiently smooth solution to Problem (A.2). As in [5], we use the first equation in (A.2) and the boundary condition $u(\cdot,0,\cdot)=0$ as a pseudo-differential Stefan condition. We solve the problem for u via both discrete Fourier transform and Laplace transform. For notational convenience, we denote by $\widehat{f}(t,k)$ (resp, $\widehat{f}(t,x,k)$) the kth Fourier coefficient of the function $f(t,\cdot)$ (resp. $f(t,x,\cdot)$). Applying the discrete Fourier transform to both the sides of the equation for u, we are led to the infinitely many equations

$$\widehat{u}_t(t,x,k)+\widehat{u}_x(t,x,k)-\widehat{u}_{xx}(t,x,k)+\lambda_k\widehat{u}(t,x,k)=(\widehat{\varphi}_t(t,k)+\widehat{(\varphi_y)^2}(t,k)+\lambda_k\widehat{\varphi}(t,k))\mathrm{e}^x \tag{A.3}$$

for $k=0,1,2,\cdots$, where we recall that $-\lambda_k=-\lambda_k(\ell)$ is the kth eigenvalue of the realization of the operator D_{yy} in L^2. If we now apply the Laplace transform $\mathscr{L}$ to both the sides of (A.3) and take into account that $u(0,\cdot)=0$, we get the infinitely many equations

$$\begin{aligned} &\lambda(\mathscr{L}\widehat{u})(\lambda,x,k)+(\mathscr{L}\widehat{u}_x)(\lambda,x,k)-(\mathscr{L}\widehat{u}_{xx})(\lambda,x,k)+\lambda_k(\mathscr{L}\widehat{u})(\lambda,x,k)\\ &=((\mathscr{L}\widehat{\varphi}_t(\lambda,k))+(\mathscr{L}\widehat{(\varphi_y)^2})(\lambda,k)+\lambda_k(\mathscr{L}\widehat{\varphi})(\lambda,k))\mathrm{e}^x. \end{aligned} \tag{A.4}$$

To avoid cumbersome notation, in what follows we simply write $\widehat{\psi}(\lambda,x,k)$ (resp. $\widehat{\psi}(\lambda,k)$) for $(\mathscr{L}\widehat{\psi})(\lambda,x,k)$ (resp. $(\mathscr{L}\widehat{\psi})(\lambda,k)$).

A straightforward computation reveals that the solution to (A.4), which vanishes at $x=0$ and tends to 0 as $x\to-\infty$ not slower than $\mathrm{e}^{\frac{x}{2}}$, is given by

$$\widehat{u}(\lambda,x,k)=\frac{1}{\lambda+\lambda_k}(\widehat{\varphi}_t(\lambda,k)+\widehat{(\varphi_y)^2}(\lambda,k)+\lambda_k\widehat{\varphi}(\lambda,k))(\mathrm{e}^x-\mathrm{e}^{\nu_{k,\lambda}x}),\quad x\le 0$$

for any $k=0,1,2,\cdots$ and any $\lambda>0$. For notational convenience, here, and throughout the paper, we set $\nu_{k,\lambda}=\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_k+4\lambda}$ for any $k=0,1,\cdots$ and any $\lambda>0$.

Let us now consider the problem for v, where we disregard (for the moment) the condition $v(\cdot,0,\cdot)-u_x(\cdot,0,\cdot)=\frac{1}{2}(\varphi_y)^2$. Taking the Fourier transform (with respect to the variable y) and then the Laplace transform (with respect to t), we get the Cauchy problems

$$\begin{cases} \widehat{v}_x(\lambda,x,k)-\widehat{v}_{xx}(\lambda,x,k)+(\lambda+\lambda_k)\widehat{v}(\lambda,x,k)\\ \quad=\alpha\Big(x+1+\dfrac{\lambda_k-1}{\lambda_k+\lambda}\Big)(\widehat{\varphi}_t(\lambda,k)+\widehat{(\varphi_y)^2}(\lambda,k))\mathrm{e}^x+\alpha\lambda_k\Big(x-\dfrac{1-\lambda_k}{\lambda_k+\lambda}\Big)\widehat{\varphi}(\lambda,k)\mathrm{e}^x\\ \qquad+\dfrac{\alpha(\nu_{k,\lambda}+\lambda)}{\lambda_k+\lambda}(\widehat{\varphi}_t(\lambda,k)+\widehat{(\varphi_y)^2}(\lambda,k)+\lambda_k\widehat{\varphi}(\lambda,k))\mathrm{e}^{\nu_{k,\lambda}x},\quad x<0,\\ \widehat{v}_t(\lambda,x,k)+\widehat{v}_x(\lambda,x,k)-\widehat{v}_{xx}(\lambda,x,k)+\lambda_k\widehat{v}(\lambda,x,k)=0,\quad x>0,\\ [\widehat{v}(\lambda,\cdot,k)]=0,\\ [\widehat{v}_x(\lambda,\cdot,k)]=-\alpha\widehat{u}_x(\lambda,0,k)=\alpha\nu_{k,\lambda}^{-1}(\widehat{\varphi}_t(\lambda,k)+\widehat{(\varphi_y)^2}(\lambda,k)+\lambda_k\widehat{\varphi}(\lambda,k)) \end{cases}$$

for $k = 0, 1, 2, \cdots$ and $\lambda > 0$.

It is easy to show that

$$\begin{aligned}
\widehat{v}(\lambda, x, k) &= c_{1,k}\mathrm{e}^{\nu_{k,\lambda}x} + \frac{\alpha}{\lambda_k+\lambda}(\widehat{\varphi}_t(\lambda,k) + \widehat{(\varphi_y)^2}(\lambda,k))\Big(x+1+\frac{\lambda_k}{\lambda_k+\lambda}\Big)\mathrm{e}^x \\
&\quad + \frac{\alpha}{\lambda_k+\lambda}\frac{\nu_{k,\lambda}+\lambda}{1-2\nu_{k,\lambda}}(\widehat{\varphi}_t(\lambda,k) + \widehat{(\varphi_y)^2}(\lambda,k) + \lambda_k\widehat{\varphi}(\lambda,k))x\mathrm{e}^{\nu_{k,\lambda}x} \\
&\quad + \frac{\alpha\lambda_k}{\lambda_k+\lambda}\widehat{\varphi}(\lambda,k)\Big(x+\frac{\lambda_k}{\lambda_k+\lambda}\Big)\mathrm{e}^x, \quad x<0, \\
\widehat{v}(\lambda, x, k) &= c_{2,k}\mathrm{e}^{(1-\nu_{k,\lambda})x}, \quad x \geq 0,
\end{aligned}$$

where

$$\begin{aligned}
c_{1,k} &= \frac{\alpha\lambda_k}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)}\Big(\nu_{k,\lambda} + \frac{\lambda_k\nu_{k,\lambda}}{\lambda_k+\lambda} + \frac{\nu_{k,\lambda}+\lambda}{1-2\nu_{k,\lambda}}\Big)\widehat{\varphi}(\lambda,k) \\
&\quad + \frac{\alpha}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)}\Big(\frac{\nu_{k,\lambda}+\lambda}{1-2\nu_{k,\lambda}} + 2\nu_{k,\lambda} + \frac{\lambda_k\nu_{k,\lambda}}{\lambda_k+\lambda}\Big)(\widehat{\varphi}_t(\lambda,k) + \widehat{(\varphi_y)^2}(\lambda,k)), \\
c_{2,k} &= \frac{\alpha\lambda_k}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)}\Big(\nu_{k,\lambda} - \frac{\lambda_k\nu_{k,\lambda}}{\lambda_k+\lambda} + \frac{\nu_{k,\lambda}+\lambda}{1-2\nu_{k,\lambda}} + \frac{\lambda_k}{\lambda_k+\lambda}\Big)\widehat{\varphi}(\lambda,k) \\
&\quad + \frac{\alpha}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)}\Big(\frac{\nu_{k,\lambda}+\lambda}{1-2\nu_{k,\lambda}} - \frac{\lambda_k\nu_{k,\lambda}}{\lambda_k+\lambda} + \frac{\lambda_k}{\lambda_k+\lambda} + 1\Big)(\widehat{\varphi}_t(\lambda,k) + \widehat{(\varphi_y)^2}(\lambda,k)).
\end{aligned}$$

Now, we are in a position to determine the equation for the front. Indeed, rewriting the boundary condition

$$v(\cdot, 0, \cdot\,) - u_x(\cdot, 0, \cdot\,) = \frac{1}{2}(\varphi_y)^2,$$

in Fourier and Laplace variables, and using the above results, we get to the following equations for the front (in the Fourier coordinates):

$$\begin{aligned}
&\Big\{\frac{\alpha}{1-2\nu_{k,\lambda}}\Big(-\frac{\lambda_k(2\lambda_k+\lambda)}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)} + \frac{\lambda_k^2(1-\nu_{k,\lambda})}{(\lambda_k+\lambda)^2}\Big) + \frac{\lambda_k}{\nu_{k,\lambda}}\Big\}\widehat{\varphi}(\lambda,k) \\
&+ \Big\{\frac{\alpha}{1-2\nu_{k,\lambda}}\Big(\frac{1-\nu_{k,\lambda}+\lambda}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)} + \frac{\lambda_k(1-\nu_{k,\lambda})}{(\lambda_k+\lambda)^2}\Big) + \frac{1}{\nu_{k,\lambda}}\Big\}\widehat{\varphi}_t(\lambda,k) \\
&+ \Big\{\frac{\alpha}{1-2\nu_{k,\lambda}}\Big(\frac{1-\nu_{k,\lambda}+\lambda}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)} + \frac{\lambda_k(1-\nu_{k,\lambda})}{(\lambda_k+\lambda)^2}\Big) + \frac{1}{\nu_{k,\lambda}} - \frac{1}{2}\Big\}\widehat{(\varphi_y)^2}(\lambda,k) = 0.
\end{aligned}$$

We observe that

$$\widehat{\varphi}_t(\lambda,k) = \lambda\widehat{\varphi}(\lambda,k) - \widehat{\varphi}_0(k), \quad k = 0, 1, 2, \cdots, \ \lambda > 0,$$

where $\varphi_0 = \varphi(0, \cdot\,)$. Therefore, we can rewrite the previous equations in the following equivalent way:

$$\begin{aligned}
&\Big\{\frac{\alpha}{1-2\nu_{k,\lambda}}\Big(-\frac{2\lambda_k^2+\lambda_k\lambda-\lambda+\lambda\nu_{k,\lambda}-\lambda^2}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)} + \frac{\lambda_k(1-\nu_{k,\lambda})}{\lambda_k+\lambda}\Big) + \frac{\lambda_k+\lambda}{\nu_{k,\lambda}}\Big\}\widehat{\varphi}(\lambda,k) \\
&- \Big\{\frac{\alpha}{1-2\nu_{k,\lambda}}\Big(\frac{1-\nu_{k,\lambda}+\lambda}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)} + \frac{\lambda_k(1-\nu_{k,\lambda})}{(\lambda_k+\lambda)^2}\Big) + \frac{1}{\nu_{k,\lambda}}\Big\}\widehat{\varphi}_0(k) \\
&+ \Big\{\frac{\alpha}{1-2\nu_{k,\lambda}}\Big(\frac{1-\nu_{k,\lambda}+\lambda}{(1-2\nu_{k,\lambda})(\lambda_k+\lambda)} + \frac{\lambda_k(1-\nu_{k,\lambda})}{(\lambda_k+\lambda)^2}\Big) + \frac{1}{\nu_{k,\lambda}} - \frac{1}{2}\Big\}\widehat{(\varphi_y)^2}(\lambda,k) = 0.
\end{aligned}$$

We now multiply both sides of the previous equation by $\nu_{k,\lambda}(1-2\nu_{k,\lambda})^2$. Thus, the coefficient $A(\lambda,k)$ of $\widehat{\varphi}(\lambda,k)$ reads

$$\begin{aligned}A(\lambda,k)=\alpha\nu_{k,\lambda}\Big(&\frac{-2\lambda_k^2-\lambda_k\lambda+\lambda-\lambda\nu_{k,\lambda}+\lambda^2+\lambda_k-3\lambda_k\nu_{k,\lambda}+2\lambda_k\nu_{k,\lambda}^2}{\lambda_k+\lambda}\Big)\\&+(\lambda_k+\lambda)(1-2\nu_{k,\lambda})^2.\end{aligned}$$

We write $A(\lambda,k)=\alpha A_1(\lambda,k)+A_0(\lambda,k)$. Using the formula $(1-2\nu_{k,\lambda})^2=1+4\lambda+4\lambda_k$, we have, on the one hand,

$$A_0(\lambda,k)=(\lambda_k+\lambda)(1+4\lambda+4\lambda_k)=4\lambda^2+4\lambda_k^2+8\lambda\lambda_k+\lambda+\lambda_k,$$

and on the other hand,

$$A_1(\lambda,k)=\nu_{k,\lambda}\Big(-\lambda_k+1-\nu_{k,\lambda}+\frac{-\lambda_k^2+\lambda^2-2\lambda_k\nu_{k,\lambda}+2\lambda_k\nu_{k,\lambda}^2}{\lambda_k+\lambda}\Big).$$

Using the formula $2\lambda_k\nu_{k,\lambda}^2-2\lambda_k\nu_{k,\lambda}-2\lambda_k(\lambda+\lambda_k)=0$, we get

$$A_1(\lambda,k)=-\lambda_k+\frac{\lambda}{2}(\sqrt{1+4\lambda+4\lambda_k}-1).$$

Thus, the coefficient of $\widehat{\varphi}(\lambda,k)$ reads

$$A(\lambda,k)=4(\lambda^2+\lambda_k^2)+\lambda(1+8\lambda_k)-(\alpha-1)\lambda_k+\alpha\frac{\lambda}{2}(\sqrt{1+4\lambda+4\lambda_k}-1).\tag{A.5}$$

Finally, if we drop the fractional term in (A.5) and return to the coordinates t and y, we see the linear operator

$$4(\varphi_{tt}+\varphi_{yyyy})+(I-8D_{yy})\varphi_t+(\alpha-1)\varphi_{yy},$$

which is indeed the linear part of (1.11).

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