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Decay of Solutions to a Linear Viscous Asymptotic Model for Water Waves***

Olivier GOUBET* Guillaume WARNAULT**

(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The authors discuss a linear viscous asymptotic model for water waves and the decay rate of solutions towards the equilibrium.

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1 Introduction

Modeling the effect of viscosity on the propagation of long waves in shallow water has received much renewed interest in the last decade. Without viscosity effects, it is now a standard procedure to derive asymptotic models for one-way wave propagation. The most encountered models in the literature are Boussinesq systems and Korteweg-de Vries equation, whose derivation was first performed in the 19th century. More general models for two-way waves were introduced in [2]. The derivation starts from Euler equation and proceeds through fine asymptotic analysis to obtain an equation for the horizontal velocity at the top of the fluid, or a system of equations for this velocity and the height of the wave. Taking into account viscosity effects is a challenging issue, since we have to deal with Navier-Stokes equations that provides the flow with a viscous layer at the bottom of the fluid. Some finer asymptotic analysis has to be performed.

The pioneering work for this issue is due to T. Kakutani and K. Matsuuchi [9] who have pointed out that the asymptotic model for viscous water waves is a dispersive PDE supplemented with a diffusion and a nonlocal pseudo-differential operator that features both a dispersive and a diffusive effect. For the physics, this means that the viscous layer in the fluid provides diffusion (this was expected), but also dispersion. Independently, P. Liu and T. Orfila [11], and D. Dutykh and F. Dias [7], have recently derived viscous asymptotical models that feature also nonlocal operators, and that possess the same dispersive properties of those in [9], but with different mathematical properties. These models are Boussinesq type systems with viscous terms. A one-way reduction of these models was addressed in [6].

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*LAMFA UMR 6140 CNRS UPJV, 33 rue Saint-Leu, Amiens 80039 cedex, France.
E-mail: olivier.goubet@u-picardie.fr
**LMPT UMR CNRS 6083, Université François Rabelais Parc Grandmont, 37000 Tours, France.
E-mail: guillaume.warnault@u-picardie.fr
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In a previous work [4], we were concerned with computing both theoretically and numerically the decay rate of solutions to a water wave model with a nonlocal viscous dispersive term. This model reads as follows

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} \mathrm{d}s + u u_x = \gamma u_{xx}, \tag{1.1}$$

where u is the horizontal velocity of the fluid. This equation requires some comments: the usual diffusion is $-\gamma u_{xx}$, while βu_{xxx} is the geometric dispersion and $\frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} \mathrm{d}s$ stands for the nonlocal diffusive-dispersive term. Here β and $\nu, \gamma \geq 0$ are parameters dedicated to balance or unbalance the effects of viscosity and dispersion against nonlinear effects. The dispersion analysis for the linear part of this equation was also addressed in [4]. In the same work, assuming that the effect of the geometric dispersion is less important that viscosity effects (i.e., considering $\beta = 0$ in the equation), we were able to prove that for small initial data, the decay rates of solutions compare to those of solutions to KdV-Burgers equations.

Computing the decay rate for solutions to dispersive-diffusive equations has a long history too. The pioneering work is due to C. Amick, J. Bona and M. Schonbek [1] where the authors handle the decay rate of solutions of KdV-Burgers solutions for any initial data, i.e., without assuming any smallness assumption on the initial data. For a large review of methods for computing the decay rates for solutions to dissipative evolution PDEs, we refer to [8]. Among recent works concerned with dissipative Boussinesq systems, we mention [3], [5] and [12]. This list is by no mean exhaustive.

In the present article, we are interested in computing theoretically the decay rate for solutions to an asymptotic linear viscous model for water waves similar to (1.1), but without the main diffusive term, i.e., considering $\gamma = 0$. In the linear case, our equation reads then (normalizing the other constants)

$$u_t + \frac{1}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{(t-s)^{\frac{1}{2}}} \mathrm{d}s + u_{xxx} + u_x = 0, \tag{1.2}$$

supplemented with initial data u_0 in $L^1(\mathbb{R})$. For this purpose, we follow the method advocated in [4]. Our result compares for large times with the corresponding result for the heat equation and state as follows.

Theorem 1.1 *There exists a constant C such that the following estimate holds true*:

$$\min(t^{\frac{1}{4}}, t^{\frac{1}{2}}) \|u(t)\|_{L^2(\mathbb{R})} + t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R})} \leq C \|u_0\|_{L^1(\mathbb{R})}. \tag{1.3}$$

Actually, the method is to compute a representation of the kernel $K(t,x)$ defined as $u(t, \cdot) = K(t, \cdot) * u_0$ if and only if u solves (1.2) with initial data u_0. This representation is indeed an oscillatory integral. As we will see in the sequel, the estimates on this kernel are much more involved than those for the kernel corresponding to the heat equation or the nonlocal viscous equation as in [4]; it turns out that since we do not have the diffusion term $-u_{xx}$ in the equation, the diffusion is weaker since high frequencies are not exponentially damped. We are concerned here with a viscosity that vanishes for high frequencies. We will come back to this point and discuss the drawbacks of this fact in the sequel. It is worth also to point out that, due to the presence of the non-local term, we do not have any energy that decays along the trajectories and that the famous Schonbek's splitting method does not apply.

We do believe that these inconveniences are only due to mathematical technicalities. For the physics, the validity of the model holds true for long waves, i.e., for initial data whose energy is concentrated for small frequencies. Hence only small frequencies monitor the flow of solutions and the drawback of the vanishing viscosity for high eddies do not account. These topics will be discussed, both from the theoretical and from the numerical point of view in a forthcoming work.

This article is organized as follows. In Section 2, we compute a representation for the kernel $K(t,x)$ as an oscillatory integral. The idea is to solve the equation in Laplace-Fourier variables and then to come back to the (t,x) variables. For this purpose, we need some estimates on solutions to a polynomial equation whose proof will appear in an annex in the last section. In Section 3, we provide some decay estimates on the kernel using van der Corput lemma; statement and short proofs of this well-known result are postponed to the annex in Section 5. Hence we complete the proof of Theorem 1.1 and we provide the reader with a short conclusion.

We complete this introduction by outlining some notations. Consider two numerical functions $h(t,x), g(t,x)$ which take values in $\mathbb{R}$. Hence we write $h(t,x) \lesssim g(t,x)$ if there exists a numerical constant c that does not depend on t and x such that $h(t,x) \leq cg(t,x)$. We also write $g(t,x) \approx h(t,x)$ if and only if $h(t,x) \lesssim g(t,x)$ and $g(t,x) \lesssim h(t,x)$. For complex valued functions, we write $g(t,x) \approx h(t,x)$ if their moduli compare, i.e., $|g(t,x)| \approx |h(t,x)|$. The Fourier transform of a function u in $L^1(\mathbb{R})$ is defined as $\widehat{u}(\xi) = \int_{\mathbb{R}} u(x)\exp(-\mathrm{i}x\xi)\mathrm{d}x$ and the Laplace transform of a bounded function v is defined, for any complex number τ such that $\mathrm{Re}\,\tau > 0$ as $\widetilde{v}(\tau) = \int_0^{+\infty} v(t)\exp(-t\tau)\mathrm{d}t$. We set $\langle x\rangle = \sqrt{1+x^2}$. For any complex number τ which does not belong to $\mathbb{R}^-$, we define $\sqrt{\tau}$ as $\sqrt{\tau} = |\tau|^{\frac{1}{2}}\exp(\frac{\mathrm{i}}{2}\arg\tau)$, where $\arg\tau$ belongs to $(-\pi,\pi)$; this function $\tau \mapsto \sqrt{\tau}$ is analytic.

2 Computing $K(t,x)$ as an Oscillatory Integral

2.1 Fourier-Laplace transform

We follow here [4]. Introduce the Fourier-Laplace transform of a function u as

$$\widehat{u}(\tau,\xi) = \int_0^{+\infty}\Big(\int_{\mathbb{R}} u(t,x)\exp(-\mathrm{i}x\xi - t\tau)\mathrm{d}x\Big)\mathrm{d}t. \tag{2.1}$$

We apply the Fourier-Laplace transform to (1.2) and obtain

$$(\tau + \sqrt{\tau} + \mathrm{i}\xi - \mathrm{i}\xi^3)\widehat{u}(\tau,\xi) = \Big(1 + \frac{1}{\sqrt{\tau}}\Big)\widehat{u}_0(\xi), \tag{2.2}$$

where $\widehat{u}_0(\xi)$ is the Fourier transform of the initial data $u(0) = u_0$. Solving for $\widehat{u}$, we have

$$\widehat{u}(\tau,\xi) = \widehat{K}(\tau,\xi)\widehat{u}_0(\xi)$$

with

$$\widehat{K}(\tau,\xi) = \Big(1 + \frac{1}{\sqrt{\tau}}\Big)\frac{1}{(\tau + \sqrt{\tau} + \mathrm{i}\xi - \mathrm{i}\xi^3)}. \tag{2.3}$$

At this stage, to invert the Fourier transform in the space variable we do need the following lemma.

Lemma 2.1 *Let $\Omega = \{z \in \mathbb{C}$ such that $\operatorname{Re} z > 0\}$. For $z \in \Omega$, the equation $X^3 + X = z$ admits three branches of solutions $a(z), \alpha_1(z), \alpha_2(z)$, which vary analytically with respect to z, such that $\operatorname{Re} a(z) > 0$, $\operatorname{Re} \alpha_i(z) < 0$ for $i = 1, 2$, $\operatorname{Im} \alpha_2(z) < 0 < \operatorname{Im} \alpha_1(z)$.*

Proof For $z = 2$, the equation admits a unique positive solution $X = 1$. The derivative $3X^2 + 1$ does not vanish on the set $X^3 + X = z$. Hence the equation owns three different solutions that we can follow continuously with the Implicit Function Theorem. On the other hand, the equation $X^3 + X = z$ cannot have a solution that belongs either to the imaginary axis or the axis $\{x < 0\}$. Then the result is proved.

Remark 2.1 Actually this result is true for $z \neq 0$ belonging to $\{z \in \mathbb{C}; \operatorname{Re} z > 0\} \cup \{z \in \mathbb{C}; |z| \leq \frac{1}{3}\}$ which contains 0 in its interior.

More results about the behavior of these solutions will appear in Annex (see Section 5). We now proceed to the inverse Fourier transform. We set

$$\mathcal{F}_x(\widetilde{F})(\tau, \xi) = \widehat{F}(\tau, \xi) = \frac{\mathrm{i}}{\mathrm{i}(\tau + \sqrt{\tau}) - \xi + \xi^3}.$$

Lemma 2.2 *Let $a(\tau)$ be the unique solution with a positive real part and $\alpha_1(\tau)$, $\alpha_2(\tau)$ the solutions with a negative real part of $X^3 + X = \tau + \sqrt{\tau}$. Then*

$$\widetilde{F}(\tau, x) = \frac{\mathrm{e}^{-a(\tau)x}}{1 + 3a^2(\tau)}, \quad \forall x \in \mathbb{R}^+, \tag{2.4}$$

$$\widetilde{F}(\tau, x) = -\sum_{j=1}^{2} \frac{\mathrm{e}^{-\alpha_j(\tau)x}}{1 + 3\alpha_j^2(\tau)}, \quad \forall x \in \mathbb{R}^-. \tag{2.5}$$

Proof For any positive x, we have

$$\widetilde{F}(\tau, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{F}(\tau, \xi) \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}x\xi}}{\mathrm{i}(\tau + \sqrt{\tau}) - \xi + \xi^3} \mathrm{d}\xi. \tag{2.6}$$

Note that if $\tau \in \Omega$, then $\tau + \sqrt{\tau} \in \Omega$. We apply Lemma 2.1 with $\xi = \mathrm{i}X$, then $\mathrm{i}a(\tau)$ is the unique solution of $\mathrm{i}(\tau + \sqrt{\tau}) - \xi + \xi^3 = 0$ whose imaginary part is positive.

Let γ be a lace who is constituted by the segment $[-R, R]$ and the semicircle $z = R\mathrm{e}^{\mathrm{i}\theta}$ where $\theta \in [0, \pi]$. We apply the Residue Theorem to $f(\xi) = \widehat{F}(\tau, \xi)$ and we obtain

$$\begin{aligned} \int_\gamma f(\xi) \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}\xi &= \int_{-R}^{R} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}x\xi}}{\mathrm{i}(\tau + \sqrt{\tau}) - \xi + \xi^3} \mathrm{d}\xi - \int_0^\pi \frac{R\mathrm{e}^{\mathrm{i}\theta} \mathrm{e}^{\mathrm{i}xR\mathrm{e}^{\mathrm{i}\theta}}}{\mathrm{i}(\tau + \sqrt{\tau}) - R\mathrm{e}^{\mathrm{i}\theta} + R^3 \mathrm{e}^{3\mathrm{i}\theta}} \mathrm{d}\theta \\ &= 2\mathrm{i}\pi \operatorname{Res}\Big(\mathrm{i}a(\tau), \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}x\xi}}{\mathrm{i}(\tau + \sqrt{\tau}) - \xi + \xi^3}\Big). \end{aligned}$$

Since $x > 0$, we have $|\mathrm{e}^{\mathrm{i}xR\mathrm{e}^{\mathrm{i}\theta}}| \leq 1$, hence when $R \to \infty$,

$$\int_0^\pi \frac{R\mathrm{e}^{\mathrm{i}\theta} \mathrm{e}^{\mathrm{i}xR\mathrm{e}^{\mathrm{i}\theta}}}{\mathrm{i}(\tau + \sqrt{\tau}) - R\mathrm{e}^{\mathrm{i}\theta} + R^3 \mathrm{e}^{3\mathrm{i}\theta}} \mathrm{d}\theta \to 0.$$

It follows (2.4) by computing the residue. We use the same method to prove (2.5) choosing a suitable lace.

We now proceed to the final estimate for $x > 0$. By the inverse Laplace transform, we have, for any positive ε,

$$K(t,x) = \frac{1}{2\mathrm{i}\pi}\int_{\varepsilon-\mathrm{i}\infty}^{\varepsilon+\mathrm{i}\infty}\Big(1+\frac{1}{\sqrt{\tau}}\Big)\frac{\mathrm{e}^{t\tau-a(\tau)x}}{1+3a^2(\tau)}\mathrm{d}\tau. \tag{2.7}$$

Since the singularity is integrable in 0, we pass to the limit $\varepsilon \to 0$ and obtain

$$K(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Big(1+\frac{1}{\sqrt{\mathrm{i}s}}\Big)\frac{\mathrm{e}^{\mathrm{i}ts-a(\mathrm{i}s)x}}{1+3a^2(\mathrm{i}s)}\mathrm{d}s. \tag{2.8}$$

We cut (2.8) in four parts:

$$K_1^+(t,x) = \frac{1}{2\pi}\int_0^{+\infty}\frac{1}{\sqrt{\mathrm{i}s}}\frac{\mathrm{e}^{\mathrm{i}ts-a(\mathrm{i}s)x}}{1+3a^2(\mathrm{i}s)}\mathrm{d}s, \quad K_1^-(t,x) = \frac{1}{2\pi}\int_{-\infty}^{0}\frac{1}{\sqrt{\mathrm{i}s}}\frac{\mathrm{e}^{\mathrm{i}ts-a(\mathrm{i}s)x}}{1+3a^2(\mathrm{i}s)}\mathrm{d}s,$$
$$K_2^+(t,x) = \frac{1}{2\pi}\int_0^{+\infty}\frac{\mathrm{e}^{\mathrm{i}ts-a(\mathrm{i}s)x}}{1+3a^2(\mathrm{i}s)}\mathrm{d}s, \qquad K_2^-(t,x) = \frac{1}{2\pi}\int_{-\infty}^{0}\frac{\mathrm{e}^{\mathrm{i}ts-a(\mathrm{i}s)x}}{1+3a^2(\mathrm{i}s)}\mathrm{d}s.$$

Remark 2.2 Analogous formulae hold true for $x < 0$, substituting (2.5) to (2.4) in the computations; in this case we have eight integrals to handle.

3 Proof of Theorem 1.1

3.1 Strategy for the proof

Let us describe our strategy on the simple example of the heat equation whose kernel is $K_{\mathrm{heat}}(t,x) \approx \frac{1}{\sqrt{t}}\exp(-\frac{x^2}{4t})$. It is an exercise to prove that $|K_{\mathrm{heat}}(t,x)| \lesssim \min(t^{-\frac{1}{2}}, x^{-1})$. Therefore, $\|K_{\mathrm{heat}}(t,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{2}}$ and

$$\|K_{\mathrm{heat}}(t,\cdot)\|^2_{L^2(\mathbb{R})} \lesssim \int_0^{+\infty}\min(t^{-1},x^{-2})\mathrm{d}x \approx \int_0^{\sqrt{t}}\frac{\mathrm{d}x}{t} + \int_{\sqrt{t}}^{+\infty}\frac{\mathrm{d}x}{x^2} \approx t^{-\frac{1}{2}}.$$

Here some extra difficulty occurs. To begin with, due to the drift $u_x + u_{xxx}$ the kernel is not symmetric with respect to $x = 0$; we expect, as for the Airy equation, the kernel to have better decay properties for positive x.

On the other hand, we are interested in the long time behavior of solutions. By a rule of thumbs, we know that the large time behavior of the kernel K relies on the behavior of the oscillatory integral for small s. Hence we surmise that the kernel K_1 will monitor the decay rate for solutions. It transpires from the computations below that the other part of the kernel K_2 is more difficult to handle; we do believe that this drawback is only due to the mathematical setting and is not relevant for the physics.

Let us go a little further. Let α be any solution to $X^3 + X = \mathrm{i}s + \sqrt{\mathrm{i}s}$. Then the modulus of the integrand in the very definition of $K_2^+(t,x)$ is $\frac{\exp(-|\mathrm{Re}\,\alpha||x|)}{|1+3\alpha^2|}$. As stated in Annex, it turns out that for large s, $|\mathrm{Re}\,\alpha_1| \approx |\mathrm{Re}\,a| \approx s^{\frac{1}{3}}$ while $|\mathrm{Re}\,\alpha_2| \approx s^{-\frac{1}{6}}$. In the former case, we have an exponential decay which smashes down the high frequencies s, in the latter case, we have a vanishing viscosity at the infinity. This explains why mathematical difficulties will occur for $x < 0$.

Remark 3.1 For the sake of comparison, it is worth to point out that the heat kernel reads as the sum of oscillatory integral (for positive x) as

$$K^{+}_{\text{heat}}(t,x) \approx \int_0^{+\infty} \exp\Big(\mathrm{i}\Big(st - \frac{\sqrt{s}}{\sqrt{2}}x\Big)\Big)\frac{\exp(-\frac{\sqrt{s}}{\sqrt{2}}x)}{\sqrt{s}}\mathrm{d}s.$$

3.2 Estimates on $K_1(t,x)$

We begin with the term which monitors the decay rate for large times. We now state and prove a result that asserts a heat kernel decay rate for this term.

Theorem 3.1 *Consider K_1 defined as above. For any $t>0$, for any x in $\mathbb{R}$ the following inequality holds true*:

$$|K_1(t,x)| \lesssim \min\Big(\frac{1}{t^{\frac{1}{2}}}, \frac{1}{|x|}\Big).$$

Proof We just focus on $K_1^+(t,x)$, the other integral being similar. We perform the change of variable $s \mapsto s^2$ in the integral and we bound $\int_0^{+\infty} \mathrm{e}^{\mathrm{i}s^2t-\beta x}\frac{\mathrm{d}s}{1+3\beta^2}$ where β is any root of $X^3+X=\mathrm{i}s^2+s\sqrt{\mathrm{i}}$. For this purpose, we apply van der Corput Lemma 5.5 (see Annex for the precise statement) with the phase $\psi(s)=s^2t$ and $A(s)=\mathrm{e}^{-\beta x}\frac{\mathrm{d}s}{1+3\beta^2}$, with $b=0$ and $d=+\infty$. We then have, observing $\psi''(0)=2t$,

$$\Big|\int_0^{+\infty} \mathrm{e}^{\mathrm{i}s^2t-\beta x}\frac{\mathrm{d}s}{1+3\beta^2}\Big| \lesssim t^{-\frac{1}{2}}(\|A\|_{L^\infty}+\|A'\|_{L^1}). \tag{3.1}$$

On one hand $|A(s)| \lesssim \frac{\exp(-|\mathrm{Re}\,\beta||x|)}{|1+3\beta^2|} \lesssim 1$. On the other hand,

$$|A'(s)| \lesssim \Big(\frac{|\beta'||x|}{|1+3\beta^2|} + \frac{6|\beta'||\beta|}{|1+3\beta^2|^2}\Big)\exp(-|\mathrm{Re}\,\beta||x|) \lesssim \frac{|\beta'|\langle x\rangle}{|1+3\beta^2|}\exp(-|\mathrm{Re}\,\beta||x|). \tag{3.2}$$

We now divide the computation according to the case $s<1$ or $s>1$. For small frequencies, we then have, due to Remark 5.1 below

$$\int_0^1 \frac{|\beta'|\langle x\rangle}{|1+3\beta^2|}\exp(-|\mathrm{Re}\,\beta||x|)\mathrm{d}s \lesssim \int_0^1 \langle x\rangle \exp(-cs|x|)\mathrm{d}s \lesssim 1; \tag{3.3}$$

indeed for $|x|\le 1$, we just use $\exp(-cs|x|)\le 1$, while for $|x|\ge 1$, we perform the change of variable $y=cxs$. For high frequencies, and for the worst case that is $x<0$ and for $\beta=\alpha_2(\mathrm{i}s^2)$ the root that has a vanishing real part for large s, we have

$$\int_1^{+\infty} \mathrm{e}^{-\mathrm{Re}\,\beta x}\frac{|\beta'|\langle x\rangle}{|1+3\beta^2|}\mathrm{d}s \lesssim \int_1^{+\infty} \exp(-c|x|s^{-\frac{1}{3}})\frac{\langle x\rangle}{s^{\frac{5}{3}}}\mathrm{d}s \lesssim \int_1^{+\infty}\frac{\mathrm{d}s}{s^{\frac{4}{3}}} \lesssim 1. \tag{3.4}$$

Hence the inequality (3.1) provides the good decay estimate with respect to t.

We now proceed to the decay rate with respect to x. We focus once again on the worst case, i.e., for $x<0$ and for $\beta=\alpha_2(\mathrm{i}s^2)$. We divide the integral into two parts according to the case $s<1$ or $s>1$. For the former case,

$$\Big|\int_0^1 \mathrm{e}^{\mathrm{i}s^2t-\beta x}\frac{\mathrm{d}s}{1+3\beta^2}\Big| \lesssim \int_0^1 \exp(-cs|x|)\mathrm{d}s \approx \frac{1}{|x|}. \tag{3.5}$$

For the latter

$$\Big|\int_1^{+\infty} \mathrm{e}^{\mathrm{i}s^2t-\beta x}\frac{\mathrm{d}s}{1+3\beta^2}\Big| \lesssim \int_1^{+\infty} \exp(-c|x|s^{-\frac{1}{3}})\frac{\mathrm{d}s}{s^{\frac{4}{3}}} \approx \frac{1}{|x|}. \tag{3.6}$$

The proof of the theorem is then completed.

3.3 Estimates on $K_2(t,x)$, $x<0$

Here we encounter the main mathematical difficulties, for small t. We state and prove the following result.

Theorem 3.2 *Consider K_2 defined as above. For any $t>0$, for any $x<0$ the following inequality holds true*:

$$|K_2(t,x)| \lesssim \min(t^{-\frac{1}{2}}, |x|^{-1}) + \min(|tx|^{-1}, t^{-1}, t^{-\frac{1}{3}}, |tx|^{-\frac{1}{4}}).$$

Proof It is worth to point out that K_2 is actually the sum of many integrals, some with α_1, the other ones with α_2. Since the first case is similar to the case $x>0$, and then since these integrals can be bounded as in Theorem 3.3 below, we skip the details for the sake of conciseness. We focus here on the integrals with α_2, which have a vanishing viscosity at the infinity. Set $\alpha = \alpha_2(\mathrm{i}s)$ for the sake of convenience. By symmetry we only consider $s>0$ that is $K_2^+(t,x)$.

To begin with, we divide the integral into two parts, according to the case $s<1$ or $s>1$. The former stands for small frequencies, the latter for high ones. Set $K_2^+(t,x) = L(t,x) + H(t,x)$, where

$$L(t,x) = \frac{1}{2\pi}\int_0^1 \frac{\mathrm{e}^{\mathrm{i}ts-\alpha x}}{1+3\alpha^2}\mathrm{d}s. \tag{3.7}$$

We first have, for small frequencies,

Lemma 3.1

$$|L(t,x)| \lesssim \min(1, |x|^{-2}, t^{-1}) \lesssim \min(t^{-\frac{1}{2}}, |x|^{-1}). \tag{3.8}$$

Proof On one hand, due to Lemma 5.1 there exists a constant c that is independent of x and t such that

$$|L(t,x)| \lesssim \int_0^1 \exp(-cs^{\frac{1}{2}}|x|)\mathrm{d}s \lesssim \min\Big(1, \frac{1}{x^2}\Big). \tag{3.9}$$

On the other hand, using Lemma 5.4 with $\psi(s) = st$ and $A(s) = \frac{\mathrm{e}^{\mathrm{i}ts-\alpha x}}{1+3\alpha^2}$, we have

$$|L(t,x)| \lesssim \frac{1}{t}(\|A\|_{L^\infty} + \|A'\|_{L^1}). \tag{3.10}$$

Then using once again Lemma 5.1, we obtain $|A(s)| \lesssim 1$ and

$$\begin{aligned}\int_0^1 |A'(s)|\mathrm{d}s &\lesssim \int_0^1 \frac{\exp(-|\mathrm{Re}\,\alpha||x|)}{|1+3\alpha^2|}\Big(|\alpha' x| + \frac{6|\alpha\alpha'|}{|1+3\alpha^2|}\Big)\mathrm{d}s \\ &\lesssim \int_0^1 \exp(-cs^{\frac{1}{2}}|x|)\Big(\frac{\langle x\rangle}{s^{\frac{1}{2}}}\Big)\mathrm{d}s \lesssim 1;\end{aligned} \tag{3.11}$$

actually for $|x|\le 1$ we have used $\exp(-cs^{\frac{1}{2}}|x|)(\frac{\langle x\rangle}{s^{\frac{1}{2}}}) \lesssim s^{-\frac{1}{2}}$ while for $|x|\ge 1$ we have performed the change of variable $y = c|x|s^{\frac{1}{2}}$.

We gather the previous computations in the following estimate $|L(t,x)| \lesssim \min(1, \frac{1}{t}, \frac{1}{x^2})$. The proof of the lemma is then completed by interpolation.

For high frequencies, we state and prove the following lemma.

Lemma 3.2

$$|H(t,x)| \lesssim \min(|tx|^{-1}, t^{-1}, t^{-\frac{1}{3}}, |tx|^{-\frac{1}{4}}). \tag{3.12}$$

Proof The proof consists in establishing precise estimates on different regions of the quarter plane $t>0$, $x<0$.

First step $t \geq 1$ or $t|x| \geq 1$.

Consider $S \geq 1$ to be precise in the sequel. On one hand, due to Lemma 5.3,

$$\Big|\int_1^S \frac{\mathrm{e}^{its-\alpha x}}{1+3\alpha^2}\mathrm{d}s\Big| \lesssim \int_1^S \exp(-c|x|s^{-\frac{1}{6}})\frac{\mathrm{d}s}{s^{\frac{2}{3}}} \lesssim \min\Big((S^{\frac{1}{3}}-1), \frac{S}{x^4}\Big); \tag{3.13}$$

actually $\exp(-c|x|s^{-\frac{1}{6}}) \leq 1$ provides us with the upper bound $S^{\frac{1}{3}}-1$, while the estimate $y^4\exp(-y) \lesssim 1$ for $y = c|x|s^{-\frac{1}{6}}$ provides us with $\frac{S}{x^4}$.

On the other hand, using Lemma 5.4 with $\psi(s) = st$ and $A(s) = \frac{\mathrm{e}^{-\alpha x}}{1+3\alpha^2}$, we have

$$\Big|\int_S^{+\infty} \frac{\mathrm{e}^{its-\alpha x}}{1+3\alpha^2}\mathrm{d}s\Big| \lesssim \frac{1}{t}(\|A\|_{L^\infty(S,+\infty)} + \|A'\|_{L^1(S,+\infty)}). \tag{3.14}$$

Using Lemma 5.3, we first have

$$|A(s)| \lesssim \exp(-c|x|s^{-\frac{1}{6}})s^{-\frac{2}{3}} \lesssim \min(S^{-\frac{2}{3}}, x^{-4}).$$

We also infer from Lemma 5.3

$$|A'(s)| \lesssim \Big(\frac{|\alpha' x|}{|1+3\alpha^2|} + \frac{|\alpha\alpha'|}{|1+3\alpha^2|^2}\Big)\exp(-c|\mathrm{Re}\,\alpha x|) \lesssim \Big(\frac{|x|}{s^{\frac{4}{3}}} + \frac{1}{s^{\frac{5}{3}}}\Big)\exp(-c|x|s^{-\frac{1}{6}}). \tag{3.15}$$

On one hand, performing the change of variable $y = c|x|s^{-\frac{1}{6}}$,

$$\begin{aligned}\int_S^{+\infty} \frac{|x|}{s^{\frac{4}{3}}}\exp(-c|x|s^{-\frac{1}{6}})\mathrm{d}s &\eqsim \frac{1}{|x|}\int_S^{+\infty}\frac{x^2}{s^{\frac{4}{3}}}\exp(-c|x|s^{-\frac{1}{6}})\mathrm{d}s \\ &\eqsim \frac{1}{|x|}\int_0^{c|x|S^{-\frac{1}{6}}} ye^{-y}\mathrm{d}y \lesssim \min\Big(|x|^{-1}, \frac{|x|}{S^{\frac{1}{3}}}\Big).\end{aligned} \tag{3.16}$$

On the other hand,

$$\begin{aligned}\int_S^{+\infty}\exp(-c|x|s^{-\frac{1}{6}})\frac{\mathrm{d}s}{s^{\frac{5}{3}}} &\lesssim \frac{1}{|x|S^{\frac{1}{2}}}\int_S^{+\infty}\exp(-c|x|s^{-\frac{1}{6}})\frac{|x|\mathrm{d}s}{s^{\frac{7}{6}}} \\ &\eqsim \frac{1}{|x|S^{\frac{1}{2}}}\int_S^{+\infty}\partial_s(\exp(-c|x|s^{-\frac{1}{6}}))\mathrm{d}s \\ &\eqsim \frac{1-\exp(-c|x|S^{-\frac{1}{6}})}{|x|S^{\frac{1}{2}}} \lesssim S^{-\frac{2}{3}}.\end{aligned} \tag{3.17}$$

Using the inequality $y^3\exp(-2y) \lesssim \exp(-y)$, we also have

$$\begin{aligned}\int_S^{+\infty}\exp(-c|x|s^{-\frac{1}{6}})\frac{\mathrm{d}s}{s^{\frac{5}{3}}} &\eqsim x^{-4}\int_S^{+\infty}\Big(\frac{|x|}{s^{\frac{1}{6}}}\Big)^3\exp(-c|x|s^{-\frac{1}{6}})\frac{|x|\mathrm{d}s}{s^{\frac{7}{6}}} \\ &\lesssim x^{-4}\int_S^{+\infty}\exp\Big(-\frac{c}{2}|x|s^{-\frac{1}{6}}\Big)\frac{|x|\mathrm{d}s}{s^{\frac{7}{6}}} \\ &\eqsim x^{-4}\int_S^{+\infty}\partial_s\Big(\exp\Big(-\frac{c}{2}|x|s^{-\frac{1}{6}}\Big)\Big)\mathrm{d}s \lesssim x^{-4}.\end{aligned} \tag{3.18}$$

We gather the computations above in the formula: for any $S \geq 1$,

$$|H(t,x)| \lesssim \min\Big(S^{\frac{1}{3}} - 1, \frac{S}{x^4}\Big) + \frac{1}{t}\Big(\min\Big(\frac{1}{S^{\frac{2}{3}}}, \frac{1}{x^4}\Big) + \min\Big(\frac{1}{|x|}, \frac{|x|}{S^{\frac{1}{3}}}\Big)\Big). \tag{3.19}$$

Choosing $S = 1$ provides us with the upper bound $\frac{1}{t}\min(1, \frac{1}{|x|})$. This completes the proof of Lemma 3.2 while $t \geq 1$ or $t|x| \geq 1$. It remains to establish the result for small t.

Second step $t \leq |x| \leq t^{\frac{1}{3}} \leq 1$.

Setting $S = \frac{1}{t} \geq 1$ in inequality (3.19), we obtain

$$|H(t,x)| \lesssim \frac{1}{t^{\frac{1}{3}}} + \frac{|x|}{t^{\frac{2}{3}}}. \tag{3.20}$$

This completes the result if moreover $|x| \lesssim t^{\frac{1}{3}}$. At this stage, we have proved the lemma if either $t \geq 1$ or $|x| \lesssim t^{\frac{1}{3}}$.

Third step $t \leq t^{\frac{1}{3}} \lesssim |x|$.

In this region, we shall use the Stationary Phase method to enforce some decay, since in this region the diffusion has not enough strength. Assume in the sequel $t \leq t^{\frac{1}{3}} \lesssim |x|$ and then $\frac{|x|}{t}$ large enough. To begin with, we use the following trick

$$H(t,x) \approx -\frac{1}{x}\int_1^{+\infty} \partial_s(\mathrm{e}^{-\alpha x})\frac{\mathrm{e}^{\mathrm{i}st}}{\alpha'(1+3\alpha^2)}\mathrm{d}s. \tag{3.21}$$

This equality is valid in the sense of oscillatory integrals. At this stage, let us observe that $\alpha'(1+3\alpha^2) = \mathrm{i} + \frac{\sqrt{\mathrm{i}}}{2\sqrt{s}} = \mathrm{i}(1+\varepsilon(s))$ is almost constant. We then have

$$H(t,x) \approx -\frac{1}{\mathrm{i}x}\int_1^{+\infty} \partial_s(\mathrm{e}^{-\alpha x + \mathrm{i}st})\frac{\mathrm{d}s}{1+\varepsilon} + \frac{t}{x}\int_1^{+\infty} \mathrm{e}^{-\alpha x + \mathrm{i}st}\frac{\mathrm{d}s}{1+\varepsilon} = H_1(t,x) + H_2(t,x). \tag{3.22}$$

On one hand, integrating by parts,

$$|H_1(t,x)| \lesssim \frac{1}{|x|}\int_1^{+\infty} \exp(-|\mathrm{Re}\,\alpha||x|)\frac{|\varepsilon'|\mathrm{d}s}{|1+\varepsilon|^2} \lesssim \frac{\|\varepsilon'\|_{L^1}}{|x|} \lesssim \frac{1}{|x|}. \tag{3.23}$$

On the other hand, we estimate the second integral by van der Corput Lemma 5.5. We set the phase to be $\psi(s) = st - \mathrm{Im}\,\alpha(s)x$ and $A(s) = \frac{\exp(-\mathrm{Re}\,\alpha x)}{1+\varepsilon}$. We have $\psi'(s) = t - \mathrm{Im}\,\alpha'(s)x$ which converges towards t when s goes to the infinity and $\psi'(1) = t - |\mathrm{Im}\,\alpha(1)||x|$; we may work on a region such that $\psi'(1) < 0$ (i.e., $\frac{|x|}{t} > \frac{1}{|\mathrm{Im}\,\alpha(1)|}$), the complementary region being handled by the computations above. Moreover, we may assume without loss of generality that $\psi''(s) = -\mathrm{Im}\,\alpha''x$ does not vanish for $s \geq 1$ (if this is not true, we know that there exists θ large enough such that $\mathrm{Im}\,\alpha''$ does not vanish on $(\theta, +\infty)$ and we cut K_2^+ in $L+H$ accordingly to the case $s < \theta$ and $s > \theta$). Therefore, with this assumption, there exists a unique s_* such that $\psi'(s_*) = 0$ and moreover since $-\mathrm{Im}\,\alpha'(s) \approx s^{-\frac{2}{3}}$, we have $s_* \approx (\frac{|x|}{t})^{\frac{3}{2}}$.

We now write $H_2(t,x) = \frac{t}{x}(\int_1^{s_*} + \int_{s_*}^{+\infty})(\mathrm{e}^{\mathrm{i}\psi}A)\mathrm{d}s$ and compute each integral by using Lemma 5.5. We then have

$$\Big|\int_1^{+\infty} \mathrm{e}^{\mathrm{i}\psi}A\mathrm{d}s\Big| \lesssim \frac{\|A\|_{L^\infty} + \|A'\|_{L^1}}{|\psi''(s_*)|^{\frac{1}{2}}}. \tag{3.24}$$

On one hand, $|\psi''(s_*)| \approx (s_*^{-\frac{5}{3}})|x| \approx \frac{t^{\frac{5}{2}}}{|x|^{\frac{3}{2}}}$. On the other hand, $|A(s)| \lesssim 1$ and

$$|A'(s)| \lesssim (|\mathrm{Re}\,\alpha' x| + |\varepsilon'|)\exp(-|\mathrm{Re}\,\alpha x|) \lesssim \Big(\frac{|x|}{s^{\frac{7}{6}}} + \frac{1}{s^{\frac{3}{2}}}\Big)\exp(-c|x|s^{-\frac{1}{6}}). \tag{3.25}$$

Then $\|A'\|_{L^1} \lesssim 1$. We gather the previous computations to write

$$|H(t,x)| \lesssim \frac{1}{|x|} + \frac{t}{|x|}\frac{|x|^{\frac{3}{4}}}{t^{\frac{5}{4}}} \lesssim \frac{1}{(t|x|)^{\frac{1}{4}}}, \tag{3.26}$$

since $t \le t^{\frac{1}{3}} \lesssim |x|$. Then the proof of the lemma is completed.

The proof of Theorem 3.2 follows by gathering the results of Lemmas 3.1 and 3.2.

3.4 Estimates on $K_2(t,x)$, $x>0$

We state and prove the following result.

Theorem 3.3 *Consider K_2 defined as above. For any $t>0$, for any $x>0$ the following inequalities hold true*:

$$|K_2(t,x)| \lesssim \min\Big(\frac{1}{\sqrt{t}}, \frac{1}{x}\Big) + \min\Big(\frac{1}{t}, \frac{1}{t^{\frac{1}{3}}}, \frac{1}{x}\Big) \lesssim \min\Big(\frac{1}{\sqrt{t}}, \frac{1}{x}\Big).$$

Proof We focus on K_2^+, the other integral being similar. As in the case $x<0$ we divide $K_2^+ = L + H$ in two parts, accordingly to $s<1$ or $s>1$. It is worth to point out that $L(t,x)$ satisfies the same estimate than in Lemma 3.1. For the high frequency part, we set $a = a(\mathrm{i}s)$ the root with positive real part of $X^3 + X = \mathrm{i}s + \sqrt{\mathrm{i}s}$. Set $S \ge 1$. On one hand, due to Lemma 5.2,

$$\Big|\int_1^S \frac{\mathrm{e}^{\mathrm{i}ts - ax}}{1+3a^2}\mathrm{d}s\Big| \lesssim \int_1^S \exp(-cxs^{\frac{1}{3}})\frac{\mathrm{d}s}{s^{\frac{2}{3}}} \lesssim \min\Big(S^{\frac{1}{3}} - 1, \frac{1}{x}\Big); \tag{3.27}$$

indeed, the estimate $\exp(-cxs^{\frac{1}{3}}) \le 1$ provides us with the upper bound $S^{\frac{1}{3}} - 1$, while the change of variable $y = cxs^{\frac{1}{3}}$ provides us with x^{-1}. On the other hand, using Lemma 5.4 with $\psi(s) = st$ and $A(s) = \frac{\exp(-ax)}{1+3a^2}$, we have

$$\Big|\int_S^{+\infty} \frac{\mathrm{e}^{\mathrm{i}ts - ax}}{1+3a^2}\mathrm{d}s\Big| \lesssim \frac{1}{t}(\|A\|_{L^\infty(S,+\infty)} + \|A'\|_{L^1(S,+\infty)}). \tag{3.28}$$

Using once again Lemma 5.2, we have

$$\begin{aligned}\int_S^{+\infty} |A'(s)|\mathrm{d}s &\lesssim \int_S^{+\infty} \Big(\frac{|a'||x|}{|1+3a^2|} + \frac{|aa'|}{|1+3a^2|^2}\Big)\mathrm{e}^{-x\mathrm{Re}\,a}\mathrm{d}s \\ &\lesssim \int_S^{+\infty} \Big(\frac{|x|}{s^{\frac{4}{3}}} + \frac{1}{s^{\frac{5}{3}}}\Big)\mathrm{e}^{-cxs^{\frac{1}{3}}}\mathrm{d}s \\ &\lesssim \int_S^{+\infty} \exp(-cxs^{\frac{1}{3}})\frac{\mathrm{d}s}{s^{\frac{5}{3}}} \lesssim \min\Big(\frac{1}{S^{\frac{2}{3}}}, \frac{1}{x^2}\Big);\end{aligned} \tag{3.29}$$

actually, the estimate $\exp(-cxs^{\frac{1}{3}}) \le 1$ leads to the upper bound $S^{-\frac{2}{3}}$, while the change of variable $y = cxs^{\frac{1}{3}}$ leads to x^{-2}. We also have

$$|A(s)| \lesssim \exp(-cxs^{\frac{1}{3}})s^{-\frac{2}{3}} \lesssim \min(S^{-\frac{2}{3}}, x^{-2}). \tag{3.30}$$

We gather the computations above as follows

$$|H(t,x)| \lesssim \min\Big(S^{\frac{1}{3}} - 1, \frac{1}{x}\Big) + \frac{1}{t}\min(S^{-\frac{2}{3}}, x^{-2}). \tag{3.31}$$

Choosing $S = 1$ provides us with the upper bound $\frac{1}{t}$, while choosing for $t \leq 1$, $S = t^{-1}$ provides the bound $t^{-\frac{1}{3}}$. For $S = +\infty$, we have the bound $\frac{1}{x}$ that completes the proof of the theorem.

3.5 Completing the proof of Theorem 1.1

We begin with the L^∞ estimates on $K(t,\cdot) : x \mapsto K(t,x)$. The upper bound $\|K(t)\|_{L^\infty_x} \lesssim t^{-\frac{1}{2}}$ is a straightforward consequence of Theorems 3.1–3.3. For the L^2_x estimate, it is worth to point out that the upper bound $\min(\frac{1}{\sqrt{t}}, \frac{1}{x})$ provides us with a upper bound in L^2_x that behaves as $t^{-\frac{1}{4}}$, as for the heat equation. It remains to consider the high frequencies part of K_2, for $x < 0$ and $t \leq 1$; we focus on $H(t,x)$ which satisfies the estimates in Lemma 3.2. We then have

$$\begin{aligned}
\|H(t,\cdot)\|^2_{L^2_x} &\lesssim \int_0^{+\infty} \min\Big(\frac{1}{t^{\frac{2}{3}}}, \frac{1}{|tx|^{\frac{1}{2}}}, \frac{1}{t^2x^2}\Big)\mathrm{d}x \\
&= \int_0^{t^{\frac{1}{3}}} \frac{\mathrm{d}x}{t^{\frac{2}{3}}} + \int_{t^{\frac{1}{3}}}^{\frac{1}{t}} \frac{\mathrm{d}x}{\sqrt{tx}} + \int_{\frac{1}{t}}^{+\infty} \frac{\mathrm{d}x}{(tx)^2} \lesssim \frac{1}{t}.
\end{aligned} \tag{3.32}$$

Therefore $\|K(t,\cdot)\|_{L^2_x} \lesssim \frac{1}{t^{\frac{1}{4}}} + \frac{1}{t^{\frac{1}{2}}}$. We conclude using classical results about convolution: for any $p \geq 1$, it is well-known that

$$\|K(t,\cdot) * u_0\|_{L^p_x} \leq \|K(t)\|_{L^p_x}\|u_0\|_{L^1_x}. \tag{3.33}$$

This completes the proof of Theorem 1.1.

4 Conclusion

In this article, we have addressed the issue of the decay rate of solutions to a linear viscous asymptotic model for water waves. In a previous work, where the equation features diffusion term as $-u_{xx}$, we have proved that the decay of solutions compare to the decay of solutions to the heat equation. In this article, we have addressed the mathematical challenge of dealing without this diffusion term. Actually, the non-local term in the equation provides us with a much weaker dissipation. Despite this fact, we were able to prove that solutions to the linear equation converges towards the equilibrium when time goes to the infinity, and with a good rate of convergence for large time. Indeed, it turns out that the decay of solutions for $t \leq 1$ and $x < 0$ is much more weaker than for the solutions to the heat equation. At this stage, we do not know how to handle the full nonlinear equation. This will be the purpose of a forthcoming work.

5 Annex: Preliminary Material

5.1 Behavior of the solutions to $X^3 + X = \mathrm{i}s + \sqrt{\mathrm{i}s}$

Here s is a real number and $\sqrt{\mathrm{i}s} = \sqrt{|s|}\exp(\mathrm{i}\frac{s}{|s|}\frac{\pi}{4})$. By symmetry, we will only discuss the case $s > 0$. As stated before, the equation possesses three branches of solutions that vary analytically with respect to s. Singularity will occur either for $s = 0$ or $s = +\infty$. Let $a(\mathrm{i}s)$, $\alpha_1(\mathrm{i}s)$, $\alpha_2(\mathrm{i}s)$ as defined in Lemma 2.1.

For small s, we can easily prove that $a(\mathrm{i}s) \sim \sqrt{\mathrm{i}s}$, $\alpha_1(\mathrm{i}s) \sim \mathrm{i}$, $\alpha_2(\mathrm{i}s) \sim -\mathrm{i}$. Actually, the real part of each root vanishes and the decay rate of this real part monitors the decay rate of the kernel for large x. For s going to the infinity, we have that $a(\mathrm{i}s) \sim \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} s^{\frac{1}{3}}$, $\alpha_1(\mathrm{i}s) \sim \mathrm{e}^{\mathrm{i}\frac{5\pi}{6}} s^{\frac{1}{3}}$, $\alpha_2(\mathrm{i}s) \sim -\mathrm{i}s^{\frac{1}{3}}$. We observe that the two first root provide an exponential damping of the high frequencies (with a speed that is $s^{\frac{1}{3}}$; the same computation for the heat kernel provides damping with speed $s^{\frac{1}{2}}$), while the real part of α_2 vanishes at the infinity; this explains why the proof of the main result is more involved than the similar proof for the heat equation.

We now quantify these remarks first for small s, then for large s.

Lemma 5.1 *Let α be one of the roots of $X^3 + X = \mathrm{i}s + \sqrt{\mathrm{i}s}$. Then for $s \lesssim 1$,*

$$1 \lesssim |1 + 3\alpha^2|, \tag{5.1}$$

$$s^{\frac{1}{2}} \lesssim |\mathrm{Re}\,\alpha|, \tag{5.2}$$

$$|\alpha(s)| \lesssim 1, \tag{5.3}$$

$$|\alpha'(s)| \approx s^{-\frac{1}{2}}. \tag{5.4}$$

Proof The proof of the first and the third assertion are easy. For the second assertion, we use the identity

$$\sqrt{2s} = (\alpha + \overline{\alpha})(\alpha^2 + \overline{\alpha}^2 - |\alpha|^2 + 1) = 2\mathrm{Re}\,\alpha(4(\mathrm{Re}\,\alpha)^2 - 3|\alpha|^2 + 1).$$

For the last assertion we use the identity $\alpha'(1 + 3\alpha^2) = \mathrm{i} + \frac{\sqrt{\mathrm{i}}}{2\sqrt{s}}$.

Remark 5.1 We also need some properties of the solutions to $X^3 + X = \mathrm{i}s^2 + s\sqrt{\mathrm{i}}$. Set β for any root of this equation. Then, with exactly the same proof (and no singularity at $s = 0$),

$$1 \lesssim |1 + 3\beta^2|, \tag{5.5}$$

$$s \lesssim |\mathrm{Re}\,\beta|, \tag{5.6}$$

$$|\beta(s)| \lesssim 1, \tag{5.7}$$

$$|\beta'(s)| \approx 1. \tag{5.8}$$

Let us now state the result for high frequencies, that is large s.

Lemma 5.2 *Let a be the root of $X^3 + X = \mathrm{i}s + \sqrt{\mathrm{i}s}$ with positive real part. Then for $1 \lesssim s$,*

$$1 \lesssim |1 + 3a^2|, \tag{5.9}$$

$$s^{\frac{1}{3}} \lesssim |\mathrm{Re}\,a|, \tag{5.10}$$

$$|a(s)| \approx s^{\frac{1}{3}}, \tag{5.11}$$

$$|a'(s)| \approx s^{-\frac{2}{3}}. \tag{5.12}$$

Analogous results hold true for α_1. For the other root the situation is more complicated.

Lemma 5.3 *Let α_2 be the root of $X^3 + X = \mathrm{i}s + \sqrt{\mathrm{i}s}$ with negative real part and imaginary* *part. Then for* $1 \lesssim s$,

$$1 \lesssim |1+3\alpha_2^2|, \tag{5.13}$$

$$s^{-\frac{1}{6}} \approx |\mathrm{Re}\,\alpha_2|, \tag{5.14}$$

$$|\alpha_2(s)| \approx s^{\frac{1}{3}}, \tag{5.15}$$

$$|\alpha_2'(s)| \approx s^{-\frac{2}{3}}. \tag{5.16}$$

Proof The proofs are easy. We omit the proofs for the sake of conciseness; we just point out that the proof of the second statement of this lemma comes from the identity $\sqrt{2s} = 2\mathrm{Re}\,\alpha_2(4(\mathrm{Re}\,\alpha_2)^2 - 3|\alpha|^2 + 1)$.

5.2 Around van der Corput lemma

We recall some preliminary material. See [10] for instance. We recall first some Nonstationary Phase Lemma.

Lemma 5.4 *Consider a positive convex function* $\psi : [b,d] \to \mathbb{R}$. *Assume that* ψ' *is positive on* $[b,d]$. *Consider a complex valued function* A *which is bounded and whose derivative is in* L^1. *Then*

$$\Big|\int_b^d \exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s\Big| \le \frac{4(\|A\|_{L^\infty} + \|A'\|_{L^1})}{|\psi'(b)|}.$$

Proof Write

$$\begin{aligned}
\mathrm{i}\int_b^d \exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s &= -\int_b^d \partial_s(\exp(\mathrm{i}\psi(s)))\frac{A(s)}{\psi'(s)}\mathrm{d}s \\
&= -\Big[\exp(\mathrm{i}\psi(s))\frac{A(s)}{\psi'(s)}\Big]_b^d \\
&\quad + \int_b^d \exp(\mathrm{i}\psi(s))\Big(\frac{A(s)}{\psi'(s)} + A(s)\partial_s\Big(\frac{1}{\psi'(s)}\Big)\Big)\mathrm{d}s.
\end{aligned} \tag{5.17}$$

Therefore, using that $\partial_s(\frac{1}{\psi'(s)})$ is positive, we have

$$\Big|\int_b^d \exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s\Big| \le 2\frac{\|A\|_{L^\infty}}{\psi'(b)} + \frac{\|A'\|_{L^1}}{\psi'(b)} + \|A\|_{L^\infty}\Big|\int_b^d \partial_s\Big(\frac{1}{\psi'(s)}\Big)\mathrm{d}s\Big|, \tag{5.18}$$

and the proof is over.

It is worth to point out that the bounds do not depend on b, d except through A. Let us now state the van der Corput lemma.

Lemma 5.5 *Consider a positive strictly convex function* $\psi : (b,d] \to \mathbb{R}$. *Assume* $\psi'(b) = 0$ *and* $\psi''(b) > 0$. *Consider a complex valued function* A *which is bounded and whose derivative is in* L^1. *Then*

$$\Big|\int_b^d \exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s\Big| \le \frac{8(\|A\|_{L^\infty} + \|A'\|_{L^1})}{|\psi''(b)|^{\frac{1}{2}}}.$$

Proof Introduce $\delta > 0$ to be specified in the sequel. On one hand

$$\Big|\int_b^{b+\varepsilon} \exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s\Big| \le \|A\|_{L^\infty}\delta. \tag{5.19}$$

On the other hand, using the previous Lemma 5.4, we have

$$\Big|\int_{b+\varepsilon}^{d}\exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s\Big| \le \frac{4(\|A\|_{L^\infty}+\|A'\|_{L^1})}{|\psi'(b+\delta)|}. \tag{5.20}$$

Using the estimate $\psi'(b+\delta)\ge\psi''(b)\delta$, we then have

$$\Big|\int_{b}^{d}\exp(\mathrm{i}\psi(s))A(s)\mathrm{d}s\Big| \le 4(\|A\|_{L^\infty}+\|A'\|_{L^1})\Big(\delta+\frac{1}{\psi''(b)\delta}\Big). \tag{5.21}$$

Choosing $\delta=\frac{1}{\psi''(b)^{\frac{1}{2}}}$ completes the proof.

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