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Petrov-Galerkin Spectral Element Method for Mixed Inhomogeneous Boundary Value Problems on Polygons***

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The authors investigate Petrov-Galerkin spectral element method. Some results on Legendre irrational quasi-orthogonal approximations are established, which play important roles in Petrov-Galerkin spectral element method for mixed inhomogeneous boundary value problems of partial differential equations defined on polygons. As examples of applications, spectral element methods for two model problems, with the spectral accuracy in certain Jacobi weighted Sobolev spaces, are proposed. The techniques developed in this paper are also applicable to other higher order methods.

Keywords Legendre quasi-orthogonal approximation, Petrov-Galerkin spectral element method, Mixed inhomogeneous boundary value problems
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1 Introduction

Spectral method has been widely used for scientific computations (see [1–9, 11] and the references therein). The standard spectral method is available for periodic problems and problems defined on rectangular domains. But many practical problems are set on complex domains, for which finite element method are usually used. However, it is also interesting to consider spectral method for non-rectangular domains and unbounded domains (see, e.g., [1, 2, 5, 13–16]).

In this paper, we develop the Petrov-Galerkin spectral element method for polygons, using a family of irrational base functions induced by the Legendre polynomials. The next section is for preliminaries. In Section 3, we establish the basic results on the Legendre irrational quasi-orthogonal approximation on quadrilaterals, which possess the spectral accuracy in certain Jacobi weighted Sobolev spaces. These results form the mathematical foundation of Petrov-Galerkin spectral element method for polygons, and serve as an important tool for numerical

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treatment of mixed inhomogeneous boundary conditions. In Section 4, we propose the Petrov-Galerkin spectral method for a mixed inhomogeneous Dirichlet-Neumann-Robin boundary value problem on quadrilaterals, with the error estimate of numerical solution. In Section 5, we consider the Petrov-Galerkin spectral element method for polygons. The final section is for some concluding remarks.

It is noted that Guo and Jia [10] developed Legendre irrational orthogonal approximation by using other kinds of base functions, which are suitable for numerical solutions of parabolic equations on quadrilaterals. However, it is simpler to use the Legendre irrational orthogonal approximation of this paper for partial differential equations defined on polygons with mixed inhomogeneous boundary conditions. Indeed, the second result of Theorem 3.2 of his paper was used in [10, Section 7] without proof. But, we now consider more general mixed inhomogeneous Dirichlet-Neumann-Robin boundary conditions and weak the restriction on partitions of polygons. On the other hand, pseudospectral method for polygons was also developed recently, which is also called as spectral element method in many literatures (see, e.g., [2, 5, 17]).

2 Preliminaries

Let Ω be a convex quadrilateral with the edges L_j, the vertices $Q_j = (x_j, y_j)$ and the angles θ_j $(1 \le j \le 4)$ (see Figure 1). We make the variable transformation (see [2, 5, 10, 17]) as follows:

$$x = a_0 + a_1\xi + a_2\eta + a_3\xi\eta, \quad y = b_0 + b_1\xi + b_2\eta + b_3\xi\eta, \tag{2.1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{4}(x_1 + x_2 + x_3 + x_4), & b_0 &= \frac{1}{4}(y_1 + y_2 + y_3 + y_4),\\ a_1 &= \frac{1}{4}(-x_1 + x_2 + x_3 - x_4), & b_1 &= \frac{1}{4}(-y_1 + y_2 + y_3 - y_4),\\ a_2 &= \frac{1}{4}(-x_1 - x_2 + x_3 + x_4), & b_2 &= \frac{1}{4}(-y_1 - y_2 + y_3 + y_4),\\ a_3 &= \frac{1}{4}(x_1 - x_2 + x_3 - x_4), & b_3 &= \frac{1}{4}(y_1 - y_2 + y_3 - y_4). \end{aligned} \tag{2.2}$$

The quadrilateral Ω is changed to the square S. If Ω is a parallelogram, then $a_3 = b_3 = 0$. In this case, transformation (2.1) is an affine mapping. Especially, $a_2 = a_3 = b_1 = b_3 = 0$ for any rectangle Ω.

For simplicity, we denote $\frac{\partial x}{\partial \xi}$ by $\partial_\xi x$, etc. The Jacobi matrix of transformation (2.1) is

$$M_\Omega = \begin{pmatrix} \partial_\xi x & \partial_\xi y \\ \partial_\eta x & \partial_\eta y \end{pmatrix} = \begin{pmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{pmatrix}.$$

Its Jacobian determinant is

$$J_\Omega(\xi, \eta) = \begin{vmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{vmatrix}. \tag{2.3}$$

According to [10, (2.7)], there exist positive constants δ_Ω and δ_Ω^*, such that

$$0 < \delta_\Omega \le J_\Omega(\xi, \eta) \le \delta_\Omega^*. \tag{2.4}$$

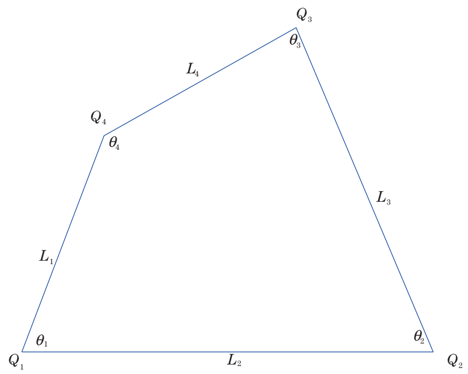

Figure 1 Quadrilateral Ω

The inverse of transformation (2.1) is given by $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. Their explicit presentations were given in the appendix of [10]. They are irrational functions generally. The Jacobi matrix of the above inverse transformation is

$$M_S = M_\Omega^{-1} = \begin{pmatrix} \partial_x \xi & \partial_x \eta \\ \partial_y \xi & \partial_y \eta \end{pmatrix} = \frac{1}{J_\Omega(\xi,\eta)} \begin{pmatrix} b_2 + b_3\xi & -b_1 - b_3\eta \\ -a_2 - a_3\xi & a_1 + a_3\eta \end{pmatrix}. \tag{2.5}$$

Thanks to (2.4), we have

$$0 < \frac{1}{\delta_\Omega^*} \le J_S(x,y) = J_\Omega^{-1}(\xi,\eta) \le \frac{1}{\delta_\Omega}. \tag{2.6}$$

3 Legendre Irrational Quasi-orthogonal Approximations

In this section, we consider the Legendre irrational quasi-orthogonal approximations on quadrilaterals, which are the mathematical foundation of related spectral element method.

3.1 Legendre orthogonal approximation in one dimension

We recall the recent results on the one-dimensional Legendre orthogonal approximation.

Let $\Lambda_\xi = \{\xi \mid |\xi| < 1\}$ and $\chi^{(\alpha,\beta)}(\xi) = (1-\xi)^\alpha(1+\xi)^\beta,\ \alpha,\beta > -1$. We define the Jacobian weighted space $L^2_{\chi^{(\alpha,\beta)}}(\Lambda_\xi)$ in the usual way, with the following inner product and norm:

$$(u,v)_{\chi^{(\alpha,\beta)},\Lambda_\xi} = \int_\Lambda u(\xi)v(\xi)\chi^{(\alpha,\beta)}(\xi)\mathrm{d}\xi, \quad \|v\|_{\chi^{(\alpha,\beta)},\Lambda_\xi} = (v,v)^{\frac{1}{2}}_{\chi^{(\alpha,\beta)},\Lambda_\xi}.$$

We omit the subscript $\chi^{(\alpha,\beta)}$ in notations whenever $\alpha = \beta = 0$.

The Legendre polynomial of degree l is defined by

$$L_l(\xi) = \frac{(-1)^l}{2^l l!}\partial_\xi^l(1-\xi^2)^l, \quad l \ge 0.$$

The set of all Legendre polynomials is a complete $L^2(\Lambda_\xi)$-orthogonal system. Moreover,

$$\|L_l\|^2_{\Lambda_\xi} = \Big(l + \frac{1}{2}\Big)^{-1}. \tag{3.1}$$

Let ${}^0H^1(\Lambda_\xi) = H^1(\Lambda_\xi) \cap \{v \mid v(1) = 0\}$. For any positive integer N, we denote by $\mathcal{P}_N(\Lambda_\xi)$ the set of all polynomials of degree at most N. Furthermore, $\mathcal{P}_N^0(\Lambda_\xi) = \mathcal{P}_N(\Lambda_\xi) \cap H_0^1(\Lambda_\xi)$ and ${}^0\mathcal{P}_N(\Lambda_\xi) = \mathcal{P}_N(\Lambda_\xi) \cap {}^0H^1(\Lambda_\xi)$.

The $L^2(\Lambda_\xi)$-orthogonal projection $P_{N,\Lambda_\xi} : L^2(\Lambda_\xi) \to \mathcal{P}_N(\Lambda_\xi)$ is defined by

$$(P_{N,\Lambda_\xi} v - v, \phi)_{\Lambda_\xi} = 0, \quad \forall \phi \in \mathcal{P}_N(\Lambda_\xi). \tag{3.2}$$

According to [12, Theorem 2.1], we know that if $v \in L^2(\Lambda_\xi)$, $\partial_\xi^r v \in L^2_{\chi^{(r,r)}}(\Lambda_\xi)$ and integers $0 \le r \le N+1$, then

$$\|P_{N,\Lambda_\xi} v - v\|_{\Lambda_\xi} \le cN^{-r} \|\partial_\xi^r v\|_{\chi^{(r,r)},\Lambda_\xi}. \tag{3.3}$$

Hereafter, c denotes a generic positive constant independent of N and any function.

Next, the orthogonal projection $P^{1,0}_{N,\Lambda_\xi} : H_0^1(\Lambda_\xi) \to P_N^0(\Lambda_\xi)$ is defined by

$$(\partial_\xi(P^{1,0}_{N,\Lambda_\xi} v - v), \partial_\xi \phi)_{\Lambda_\xi} = 0, \quad \forall \phi \in \mathcal{P}_N^0(\Lambda_\xi). \tag{3.4}$$

As a special case of [12, Theorem 3.4], we have that if $v \in H_0^1(\Lambda_\xi)$, $\partial_\xi^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi)$ and integers $1 \le r \le N+1$, then

$$\|\partial_\xi^\mu(P^{1,0}_{N,\Lambda_\xi} v - v)\|_{\Lambda_\xi} \le cN^{\mu-r} \|\partial_\xi^r v\|_{\chi^{(r-1,r-1)},\Lambda_\xi}, \quad \mu = 0, 1. \tag{3.5}$$

The orthogonal projection ${}^0P^1_{N,\Lambda_\xi} : {}^0H^1(\Lambda_\xi) \to {}^0P_N(\Lambda_\xi)$ is defined by

$$(\partial_\xi({}^0P^1_{N,\Lambda_\xi} v - v), \partial_\xi \phi)_{\Lambda_\xi} = 0, \quad \forall \phi \in {}^0P_N(\Lambda_\xi). \tag{3.6}$$

We have from a slight modification of [12, Theorem 3.2] that if $v \in {}^0H^1(\Lambda_\xi)$, $\partial_\xi^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi)$ and integers $1 \le r \le N+1$, then

$$\|\partial_\xi^\mu({}^0P^1_{N,\Lambda_\xi} v - v)\|_{\Lambda_\xi} \le cN^{\mu-r} \|\partial_\xi^r v\|_{\chi^{(r-1,r-1)},\Lambda_\xi}, \quad \mu = 0, 1. \tag{3.7}$$

3.2 $L^2(\Omega)$-Legendre irrational orthogonal approximation on quadrilaterals

We now study the $L^2(\Omega)$-Legendre irrational orthogonal approximation. We denote the inner product and norm of $L^2(\Omega)$ by $(u, v)_\Omega$ and $\|v\|_\Omega$, respectively.

The irrational functions on the quadrilateral Ω are given by

$$\psi_{l,m}(x, y) = L_l(\xi(x, y))L_m(\eta(x, y)), \quad l, m \ge 0. \tag{3.8}$$

Let

$$V_N(\Omega) = \mathrm{span}\{\psi_{l,m}(x, y) \mid 0 \le l, m \le N\}.$$

The $L^2(\Omega)$-orthogonal projection $P_{N,\Omega} : L^2(\Omega) \to V_N(\Omega)$ is defined by

$$(P_{N,\Omega} v - v, \phi)_\Omega = 0, \quad \forall \phi \in V_N(\Omega). \tag{3.9}$$

For simplicity of statements, we introduce the quantity

$$\begin{aligned} A_{r,\Omega}(v) = \sum_{j=0}^{r} (&\|(1-\xi^2)^{\frac{r}{2}}(a_1 + a_3\eta)^j (b_1 + b_3\eta)^{r-j} \partial_x^j \partial_y^{r-j} v\|_\Omega \\ &+ \|(1-\eta^2)^{\frac{r}{2}}(a_2 + a_3\xi)^j (b_2 + b_3\xi)^{r-j} \partial_x^j \partial_y^{r-j} v\|_\Omega). \end{aligned}$$

Theorem 3.1 *If $v \in L^2(\Omega)$, and $A_{r,\Omega}(v)$ is finite for integers $0 \le r \le N+1$, then*

$$\|P_{N,\Omega}v - v\|_\Omega \le c{\delta^*_\Omega}^{\frac{1}{2}}{\delta_\Omega}^{-\frac{1}{2}}N^{-r}A_{r,\Omega}(v). \tag{3.10}$$

Proof By projection theorem, we get

$$\|P_{N,\Omega}v - v\|^2_\Omega \le \|\phi - v\|^2_\Omega, \quad \forall \phi \in V_N(\Omega).$$

Let $\widehat{v}(\xi,\eta) = v(x(\xi,\eta), y(\xi,\eta))$ and

$$\psi(\xi,\eta) = P_{N,\Lambda_\xi} \bullet P_{N,\Lambda_\eta}\widehat{v}, \quad \phi(x,y) = \psi(\xi(x,y),\eta(x,y)) \in V_N(\Omega). \tag{3.11}$$

Clearly, $S = \Lambda_\xi \times \Lambda_\eta$. By using (2.4) and (3.3), we verify that

$$\begin{aligned}
\|v-\phi\|^2_\Omega &= \iint_S (\widehat{v} - P_{N,\Lambda_\xi} \bullet P_{N,\Lambda_\eta}\widehat{v})^2 J_\Omega(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \\
&\le 2\delta^*_\Omega \iint_S (\widehat{v} - P_{N,\Lambda_\xi}\widehat{v})^2\mathrm{d}\xi\mathrm{d}\eta + 2\delta^*_\Omega \iint_S (P_{N,\Lambda_\xi}(\widehat{v} - P_{N,\Lambda_\eta}\widehat{v}))^2\mathrm{d}\xi\mathrm{d}\eta \\
&\le c\delta^*_\Omega N^{-2r}\|\partial^r_\xi \widehat{v}\|^2_{L^2_{\chi^{(r,r)}}(\Lambda_\xi, L^2(\Lambda_\eta))} + c\delta^*_\Omega \|P_{N,\Lambda_\eta}\widehat{v} - \widehat{v}\|^2_{L^2(\Lambda_\xi, L^2(\Lambda_\eta))} \\
&\le c\delta^*_\Omega N^{-2r}(\|\partial^r_\xi \widehat{v}\|^2_{L^2_{\chi^{(r,r)}}(\Lambda_\xi, L^2(\Lambda_\eta))} + \|\partial^r_\eta \widehat{v}\|^2_{L^2(\Lambda_\xi, L^2_{\chi^{(r,r)}}(\Lambda_\eta))}).
\end{aligned} \tag{3.12}$$

By virtue of (2.1), a direct calculation yields that

$$\partial^r_\xi \widehat{v} = \sum_{j=0}^{r} C^j_r (a_1 + a_3\eta)^j (b_1 + b_3\eta)^{r-j}\partial^j_x \partial^{r-j}_y v, \tag{3.13}$$

$$\partial^r_\eta \widehat{v} = \sum_{j=0}^{r} C^j_r (a_2 + a_3\xi)^j (b_2 + b_3\xi)^{r-j}\partial^j_x \partial^{r-j}_y v. \tag{3.14}$$

Therefore,

$$\|\partial^r_\xi \widehat{v}\|_{L^2_{\chi^{(r,r)}(\Lambda_\xi, L^2(\Lambda_\eta))}} \le c{\delta_\Omega}^{-\frac{1}{2}} \sum_{j=0}^{r} \|(1-\xi^2)^{\frac{r}{2}}(a_1 + a_3\eta)^j (b_1 + b_3\eta)^{r-j}\partial^j_x \partial^{r-j}_y v\|_\Omega, \tag{3.15}$$

$$\|\partial^r_\eta \widehat{v}\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r,r)}}(\Lambda_\eta))} \le c{\delta_\Omega}^{-\frac{1}{2}} \sum_{j=0}^{r} \|(1-\eta^2)^{\frac{r}{2}}(a_2 + a_3\xi)^j (b_2 + b_3\xi)^{r-j}\partial^j_x \partial^{r-j}_y v\|_\Omega. \tag{3.16}$$

Finally, the desired result (3.10) follows from a combination of (3.12), (3.15) and (3.16).

Remark 3.1 In the norms of derivatives $\partial^j_x \partial^{r-j}_y v$ involved in the quantity $A_{r,\Omega}(v)$, there exist the weight functions $(1-\xi^2)^{\frac{r}{2}}$ or $(1-\eta^2)^{\frac{r}{2}}$ respectively, which tend to zero simultaneously as the point $Q(x,y)$ goes to the vertices of Ω. As a result, $\|P_N v - v\|_\Omega$ still keeps the order N^{-r}, even if the approximated function has certain weak singularity at the vertices.

Remark 3.2 If Ω is the rectangle $S_{a,b} = \{(x,y) \mid |x| < a,\ |y| < b,\ a,b > 0\}$, then $a_1 = a,\ b_2 = b,\ a_2 = a_3 = b_1 = b_3 = 0$ and $J_\Omega = ab$. Therefore,

$$\|P_{N,\Omega}v - v\|_\Omega \le cN^{-r}(\|(a^2-x^2)^{\frac{r}{2}}\partial^r_x v\|_\Omega + \|(b^2-y^2)^{\frac{r}{2}}\partial^r_y v\|_\Omega).$$

Obviously, the $L^2(\Omega)$-orthogonal approximation keeps the same spectral accuracy, even if the considered function possesses certain singularity at the edges of quadrilateral.

3.3 Legendre irrational orthogonal approximation in $H_0^1(\Omega)$

We now turn to the $H_0^1(\Omega)$-Legendre irrational orthogonal approximation.

According to the Poincaré inequality, there exists a positive constant c_Ω such that

$$\|w\|_\Omega \le c_\Omega \|\nabla w\|_\Omega, \quad \forall w \in H_0^1(\Omega). \tag{3.17}$$

Let $x_5 = x_1$ and $y_5 = y_1$. We set $\gamma_\Omega = \max(|a_3|, |b_3|)$ and

$$\begin{aligned}\sigma_\Omega &= \max_{(\xi,\eta)\in S}(|b_2 + b_3\xi|, |b_1 + b_3\eta|, |a_2 + a_3\xi|, |a_1 + a_3\eta|)\\ &= \frac{1}{2}\max_{1\le j\le 4}(|x_j - x_{j+1}|, |y_j - y_{j+1}|).\end{aligned}$$

Due to (2.2), we have $\gamma_\Omega \le \sigma_\Omega$.

Let $V_N^0(\Omega) = H_0^1(\Omega) \cap V_N(\Omega)$. The $H_0^1(\Omega)$-orthogonal projection $P_{N,\Omega}^{1,0} : H_0^1(\Omega) \to V_N^0(\Omega)$ is defined by

$$(\nabla(P_{N,\Omega}^{1,0} v - v), \nabla\phi)_\Omega = 0, \quad \forall \phi \in V_N^0(\Omega). \tag{3.18}$$

In order to describe the approximation error, we introduce the quantity

$$B_{r,\Omega}(v) = \sum_{j=1}^{3} B_{r,\Omega}^{(j)}(v)$$

with

$$\begin{aligned}B_{r,\Omega}^{(1)}(v) = \sum_{j=0}^{r}(&\|(1-\xi^2)^{\frac{r-1}{2}}(a_1 + a_3\eta)^j (b_1 + b_3\eta)^{r-j}\partial_x^j\partial_y^{r-j} v\|_\Omega\\ &+ \|(1-\eta^2)^{\frac{r-1}{2}}(a_2 + a_3\xi)^j(b_2 + b_3\xi)^{r-j}\partial_x^j\partial_y^{r-j}v\|_\Omega),\\ B_{r,\Omega}^{(2)}(v) = \sum_{j=0}^{r-1}(&\|(1-\xi^2)^{\frac{r-2}{2}}(a_1 + a_3\eta)^j(b_1 + b_3\eta)^{r-1-j}(a_2 + a_3\xi)\partial_x^{j+1}\partial_y^{r-1-j}v\|_\Omega\\ &+ \|(1-\xi^2)^{\frac{r-2}{2}}(a_1 + a_3\eta)^j(b_1 + b_3\eta)^{r-1-j}(b_2 + b_3\xi)\partial_x^j\partial_y^{r-j}v\|_\Omega\\ &+ \|(1-\eta^2)^{\frac{r-2}{2}}(a_2 + a_3\xi)^j(b_2 + b_3\xi)^{r-1-j}(a_1 + a_3\eta)\partial_x^{j+1}\partial_y^{r-1-j}v\|_\Omega\\ &+ \|(1-\eta^2)^{\frac{r-2}{2}}(a_2 + a_3\xi)^j(b_2 + b_3\xi)^{r-1-j}(b_1 + b_3\eta)\partial_x^j\partial_y^{r-j}v\|_\Omega),\\ B_{r,\Omega}^{(3)}(v) = \sum_{j=0}^{r-2}(&\|(1-\xi^2)^{\frac{r-2}{2}}a_3(a_1 + a_3\eta)^j(b_1 + b_3\eta)^{r-2-j}\partial_x^{j+1}\partial_y^{r-2-j}v\|_\Omega\\ &+ \|(1-\xi^2)^{\frac{r-2}{2}}b_3(a_1 + a_3\eta)^j(b_1 + b_3\eta)^{r-2-j}\partial_x^j\partial_y^{r-1-j}v\|_\Omega\\ &+ \|(1-\eta^2)^{\frac{r-2}{2}}a_3(a_2 + a_3\xi)^j(b_2 + b_3\xi)^{r-2-j}\partial_x^{j+1}\partial_y^{r-2-j}v\|_\Omega\\ &+ \|(1-\eta^2)^{\frac{r-2}{2}}b_3(a_2 + a_3\xi)^j(b_2 + b_3\xi)^{r-2-j}\partial_x^j\partial_y^{r-1-j}v\|_\Omega).\end{aligned}$$

Theorem 3.2 *If $v \in H_0^1(\Omega)$ and $B_{r,\Omega}(v)$ is finite for integers $2 \le r \le N+1$, then*

$$\begin{aligned}\|\nabla(P_{N,\Omega}^{1,0}v - v)\|_\Omega &\le c\sigma_\Omega\delta_\Omega^{-1}N^{1-r}B_{r,\Omega}(v),\\ \|P_{N,\Omega}^{1,0}v - v\|_\Omega &\le c\bar{c}_\Omega(c_\Omega^2 + 1)\sigma_\Omega^{r+1}(\sigma_\Omega + 1)\delta_\Omega^{-2}N^{-r}B_{r,\Omega}(v),\end{aligned} \tag{3.19}$$

where $\bar{c}_\Omega$ *is a positive constant determined in* (3.29) *of this paper.*

Proof By projection theorem, we have

$$\|\nabla(P^{1,0}_{N,\Omega}v - v)\|_\Omega \le \|\nabla(\phi - v)\|_\Omega, \quad \forall \phi \in V^0_N(\Omega). \tag{3.20}$$

Let $\widehat{v}(\xi,\eta) = v(x(\xi,\eta), y(\xi,\eta))$ as before, and

$$\psi(\xi,\eta) = P^{1,0}_{N,\Lambda_\xi} \bullet P^{1,0}_{N,\Lambda_\eta}\widehat{v}, \quad \phi(x,y) = \psi(\xi(x,y), \eta(x,y)) \in V^0_N(\Omega).$$

We denote by $\|w\|_S$ the norm of the space $L^2(S)$, and $\nabla_S w = (\partial_\xi w, \partial_\eta w)^{\mathrm{T}}$. It can be shown that $\nabla_S w = M_\Omega \nabla w$. Thus by (2.5), we have $\nabla w = M_S \nabla_S w$. Hence,

$$\partial_x(\phi - v) = (b_2 + b_3\xi)J^{-1}_\Omega \partial_\xi(\psi - \widehat{v}) - (b_1 + b_3\eta)J^{-1}_\Omega \partial_\eta(\psi - \widehat{v}).$$

With the aid of (2.6), a direct calculation gives

$$\|\partial_x(\phi - v)\|_\Omega \le c\sigma_\Omega \delta_\Omega^{-\frac{1}{2}} \|\nabla_S(\psi - \widehat{v})\|_S.$$

We can estimate $\|\partial_y(\psi - \widehat{v})\|_\Omega$ in the same manner. Consequently,

$$\|\nabla(\phi - v)\|_\Omega \le c\sigma_\Omega \delta_\Omega^{-\frac{1}{2}} \|\nabla_S(\psi - \widehat{v})\|_S. \tag{3.21}$$

We now estimate $\|\nabla_S(\psi - \widehat{v})\|_S$. Clearly,

$$\|\partial_\xi(\psi - \widehat{v})\|^2_S \le \|\partial_\xi(P^{1,0}_{N,\Lambda_\xi}\widehat{v} - \widehat{v})\|^2_S + \|\partial_\xi P^{1,0}_{N,\Lambda_\xi}(P^{1,0}_{N,\Lambda_\eta}\widehat{v} - \widehat{v})\|^2_S.$$

Using (3.5) with $\mu = 1$ gives

$$\|\partial_\xi(P^{1,0}_{N,\Lambda_\xi}\widehat{v} - \widehat{v})\|^2_S \le cN^{2-2r}\|\partial^r_\xi \widehat{v}\|^2_{L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi, L^2(\Lambda_\eta))}.$$

By virtue of (3.5) with $r = \mu = 1$, we have

$$\|\partial_\xi P^{1,0}_{N,\Lambda_\xi}(P^{1,0}_{N,\Lambda_\eta}\widehat{v} - \widehat{v})\|^2_S \le \|\partial_\xi(P^{1,0}_{N,\Lambda_\eta}\widehat{v} - \widehat{v})\|^2_{L^2(\Lambda_\xi, L^2(\Lambda_\eta))}.$$

Thereby, we use (3.5) with $\mu = 0$ again to assert that

$$\|\partial_\xi P^{1,0}_{N,\Lambda_\xi}(P^{1,0}_{N,\Lambda_\eta}\widehat{v} - \widehat{v})\|^2_S \le cN^{2-2r}\|\partial_\xi \partial^{r-1}_\eta \widehat{v}\|^2_{L^2(\Lambda_\xi, L^2_{\chi^{(r-2,r-2)}}(\Lambda_\eta))}.$$

We can estimate $\|\partial_\eta(\psi - \widehat{v})\|^2_S$ similarly. As a result, we obtain

$$\|\nabla_S(\psi - \widehat{v})\|_S \le cN^{1-r}D_{r,S}(\widehat{v}), \tag{3.22}$$

where $D_{r,S}(w) = D^{(1)}_{r,S}(w) + D^{(2)}_{r,S}(w)$ and

$$D^{(1)}_{r,S}(w) = (\|\partial^r_\xi w\|^2_{L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi, L^2(\Lambda_\eta))} + \|\partial_\xi \partial^{r-1}_\eta w\|^2_{L^2(\Lambda_\xi, L^2_{\chi^{(r-2,r-2)}}(\Lambda_\eta))})^{\frac{1}{2}},$$

$$D^{(2)}_{r,S}(w) = (\|\partial^r_\eta w\|^2_{L^2(\Lambda_\xi, L^2_{\chi^{(r-1,r-1)}}(\Lambda_\eta))} + \|\partial^{r-1}_\xi \partial_\eta w\|^2_{L^2_{\chi^{(r-2,r-2)}}(\Lambda_\xi, L^2(\Lambda_\eta))})^{\frac{1}{2}}.$$

We next estimate the right-hand side of (3.22). By the same argument as in the derivations of (3.15) and (3.16), we verify

$$\begin{aligned}
&\|\partial_\xi^r \widehat{v}\|_{L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi, L^2(\Lambda_\eta))} \\
&\le c\delta_\Omega{}^{-\frac{1}{2}} \sum_{j=0}^{r} \|(1-\xi^2)^{\frac{r-1}{2}} (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-j} \partial_x^j \partial_y^{r-j} v\|_\Omega, \qquad (3.23)\\
&\|\partial_\eta^r \widehat{v}\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r-1,r-1)}}(\Lambda_\eta))} \\
&\le c\delta_\Omega{}^{-\frac{1}{2}} \sum_{j=0}^{r} \|(1-\eta^2)^{\frac{r-1}{2}} (a_2+a_3\xi)^j (b_2+b_3\xi)^{r-j} \partial_x^j \partial_y^{r-j} v\|_\Omega. \qquad (3.24)
\end{aligned}$$

Moreover, by differentiating (3.13) with respect to η, we use (2.1) to obtain

$$\begin{aligned}
\partial_\xi^{r-1}\partial_\eta \widehat{v} =& \sum_{j=0}^{r-1} C_{r-1}^j (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-1-j} ((a_2+a_3\xi)\partial_x^{j+1}\partial_y^{r-1-j} v + (b_2+b_3\xi)\partial_x^j \partial_y^{r-j} v)\\
&+ (r-1)\sum_{j=0}^{r-2} C_{r-2}^j (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-2-j} (a_3 \partial_x^{j+1}\partial_y^{r-2-j} v + b_3 \partial_x^j \partial_y^{r-1-j} v).
\end{aligned}$$

Then, following the same line as in the derivations of (3.15) and (3.16), we obtain

$$\begin{aligned}
&\|\partial_\xi^{r-1}\partial_\eta \widehat{v}\|_{L^2_{\chi^{(r-2,r-2)}}(\Lambda_\xi, L^2(\Lambda_\eta))}\\
&\le c\delta_\Omega{}^{-\frac{1}{2}} \sum_{j=0}^{r-1} (\|(1-\xi^2)^{\frac{r-2}{2}} (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-1-j} (a_2+a_3\xi)\partial_x^{j+1}\partial_y^{r-1-j} v\|_\Omega\\
&+ \|(1-\xi^2)^{\frac{r-2}{2}} (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-1-j} (b_2+b_3\xi)\partial_x^j \partial_y^{r-j} v\|_\Omega)\\
&+ c\sum_{j=0}^{r-2} (\|(1-\xi^2)^{\frac{r-2}{2}} a_3 (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-2-j} \partial_x^{j+1}\partial_y^{r-2-j} v\|_\Omega\\
&+ \|(1-\xi^2)^{\frac{r-2}{2}} b_3 (a_1+a_3\eta)^j (b_1+b_3\eta)^{r-2-j} \partial_x^j \partial_y^{r-1-j} v\|_\Omega). \qquad (3.25)
\end{aligned}$$

Similarly, we use (3.14) and (2.1) to obtain

$$\begin{aligned}
&\|\partial_\xi\partial_\eta^{r-1} \widehat{v}\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r-2,r-2)}}(\Lambda_\eta))}\\
&\le c\delta_\Omega^{-\frac{1}{2}} \sum_{j=0}^{r-1} (\|(1-\eta^2)^{\frac{r-2}{2}} (a_2+a_3\xi)^j (b_2+b_3\xi)^{r-1-j} (a_1+a_3\eta)\partial_x^{j+1}\partial_y^{r-1-j} v\|_\Omega\\
&+ \|(1-\eta^2)^{\frac{r-2}{2}} (a_2+a_3\xi)^j (b_2+b_3\xi)^{r-1-j} (b_1+b_3\eta)\partial_x^j \partial_y^{r-j} v\|_\Omega)\\
&+ c\sum_{j=0}^{r-2} (\|(1-\eta^2)^{\frac{r-2}{2}} a_3 (a_2+a_3\xi)^j (b_2+b_3\xi)^{r-2-j} \partial_x^{j+1}\partial_y^{r-2-j} v\|_\Omega\\
&+ \|(1-\eta^2)^{\frac{r-2}{2}} b_3 (a_2+a_3\xi)^j (b_2+b_3\xi)^{r-2-j} \partial_x^j \partial_y^{r-1-j} v\|_\Omega). \qquad (3.26)
\end{aligned}$$

Finally, we use (3.20)–(3.26) successively to verify

$$\|\nabla(P_{N,\Omega}^{1,0} v - v)\|_\Omega \le c\sigma_\Omega \delta_\Omega^{-\frac{1}{2}} N^{1-r} D_{r,S}(\widehat{v}) \le c\sigma_\Omega \delta_\Omega^{-1} N^{1-r} B_{r,\Omega}(v). \qquad (3.27)$$

We now prove the second result of (3.19). Let $g \in L^2(\Omega)$ and consider an auxiliary problem. It is to find $w \in H_0^1(\Omega)$ such that

$$(\nabla w, \nabla z)_\Omega = (g, z)_\Omega, \quad \forall z \in H_0^1(\Omega). \tag{3.28}$$

Taking $z = w$ in (3.28) and using (3.17), we obtain $\|\nabla w\|_\Omega \le c_\Omega \|g\|_\Omega$. Moreover, by the property of elliptic equation with the homogeneous boundary condition, there exists a positive constant $\overline{c}_\Omega$ such that

$$\|w\|_{H^2(\Omega)} \le \overline{c}_\Omega(\|w\|_\Omega + \|g\|_\Omega) \le \overline{c}_\Omega(c_\Omega\|\nabla w\|_\Omega + \|g\|_\Omega) \le \overline{c}_\Omega(c_\Omega^2 + 1)\|g\|_\Omega. \tag{3.29}$$

We now take $z = P_N^{1,0}v - v$ in (3.28). Then we use (3.18) and (3.27) to verify that

$$\begin{aligned} |(P_N^{1,0}v - v, g)_\Omega| &= |(\nabla w, \nabla(P_N^{1,0}v - v))_\Omega| \\ &= |(\nabla(P_N^{1,0}w - w), \nabla(P_N^{1,0}v - v))_\Omega| \\ &\le \|\nabla(P_N^{1,0}w - w)\|_\Omega \|\nabla(P_N^{1,0}v - v)\|_\Omega \\ &\le c\sigma_\Omega^2\delta_\Omega^{-2}N^{-r}B_{r,\Omega}(v)B_{2,\Omega}(w). \end{aligned} \tag{3.30}$$

Since $r \ge 2$, we have $B_{2,\Omega}(w) \le \sigma_\Omega^{r-1}(\sigma_\Omega + 1)\|w\|_{H^2(\Omega)}$. Finally, we use (3.29) and (3.30) to deduce that

$$\begin{aligned} \|P_N^{1,0}v - v\|_\Omega &= \sup_{\substack{g \in L^2(\Omega) \\ g \ne 0}} \frac{|(P_N^{1,0}v - v, g)_\Omega|}{\|g\|_\Omega} \\ &\le c\sigma_\Omega^2\delta_\Omega^{-2}N^{-r}\frac{B_{r,\Omega}(v)B_{2,\Omega}(w)}{\|g\|_\Omega} \\ &\le c\sigma_\Omega^{r+1}(\sigma_\Omega + 1)\delta_\Omega^{-2}N^{-r}\frac{B_{r,\Omega}(v)\|w\|_{H^2(\Omega)}}{\|g\|_\Omega} \\ &\le c\overline{c}_\Omega(c_\Omega^2 + 1)\sigma_\Omega^{r+1}(\sigma_\Omega + 1)\delta_\Omega^{-2}N^{-r}B_{r,\Omega}(v). \end{aligned}$$

This ends the proof.

Remark 3.3 If $\Omega = S_{a,b}$ as in Remark 3.2, then we could improve the results in Theorem 3.2. To do this, let

$$D_{r,S,a,b}^*(\widehat{v}) = \Big(\frac{b}{a}(D_{r,S}^{(1)}(\widehat{v}))^2 + \frac{a}{b}(D_{r,S}^{(2)}(\widehat{v}))^2\Big)^{\frac{1}{2}}.$$

A direct calculation with (3.20) leads to

$$\|\nabla(P_{N,\Omega}^{1,0}v - v)\|_\Omega \le \|\nabla(\phi - v)\|_\Omega = cN^{1-r}D_{r,S,a,b}^*(\widehat{v}) \le cN^{1-r}B_{r,\Omega}^*(v), \tag{3.31}$$

where $B_{r,\Omega}^*(v) = B_{r,\Omega}^{*,1}(v) + B_{r,\Omega}^{*,2}(v)$, and

$$\begin{aligned} B_{r,\Omega}^{*,1}(v) &= (\|(a^2 - x^2)^{\frac{r-1}{2}}\partial_x^r v\|_\Omega^2 + \|(b^2 - y^2)^{\frac{r-1}{2}}\partial_y^r v\|_\Omega^2)^{\frac{1}{2}}, \\ B_{r,\Omega}^{*,2}(v) &= (b^2\|(b^2 - y^2)^{\frac{r-2}{2}}\partial_x\partial_y^{r-1}v\|_\Omega^2 + a^2\|(a^2 - x^2)^{\frac{r-2}{2}}\partial_x^{r-1}\partial_y v\|_\Omega^2)^{\frac{1}{2}}. \end{aligned}$$

Next, like (3.30), we have

$$|(P_N^{1,0}v - v, g)_\Omega| \le cN^{-r}B_{r,\Omega}^*(v)B_{2,\Omega}^*(w).$$

It is easy to show

$$B^*_{2,\Omega}(w) \le (\|\partial_x^2 v\|^2_\Omega + \|\partial_y^2 v\|^2_\Omega + (a^2+b^2)\|\partial_x\partial_y v\|^2_\Omega)^{\frac{1}{2}}.$$

Moreover, we obtain from (3.28) that $|v|_{H^2(\Omega)} \le 2\|g\|_\Omega$. Finally, from an argument as in the last part of the proof of Theorem 3.2, we derive

$$\|P_N^{1,0}v - v\|_\Omega \le cN^{-r}\sqrt{\max(a^2+b^2,1)}\, B^*_{r,\Omega}(v). \tag{3.32}$$

Remark 3.4 Recently, Guo and Jia [10] considered $L^2(\Omega)$-orthogonal approximation and $H_0^1(\Omega)$-orthogonal approximation by taking the base functions

$$\widetilde{\psi}(x,y) = L_l(\xi(x,y))L_m(\eta(x,y))J_\Omega^{-\frac{1}{2}}(\xi(x,y),\eta(x,y)), \quad l,m \ge 0.$$

But it is simpler to use the results of (3.19) of this paper for numerical solutions of partial differential equations defined on polygons with mixed inhomogeneous boundary conditions.

3.4 Legendre irrational quasi-orthogonal approximation on quadrilaterals

We are now in position to study the Legendre irrational quasi-orthogonal approximation on quadrilaterals. Let $\widehat{v}(\xi,\eta) = v(x(\xi,\eta),y(\xi,\eta))$, and (see [10])

$$\begin{aligned}
\widehat{v}_{b,L_1}(\eta) &= \frac{1}{2}((1-\eta)\widehat{v}(-1,-1) + (1+\eta)\widehat{v}(-1,1)),\\
\widehat{v}_{b,L_2}(\xi) &= \frac{1}{2}((1-\xi)\widehat{v}(-1,-1) + (1+\xi)\widehat{v}(1,-1)),\\
\widehat{v}_{b,L_3}(\eta) &= \frac{1}{2}((1-\eta)\widehat{v}(1,-1) + (1+\eta)\widehat{v}(1,1)),\\
\widehat{v}_{b,L_4}(\xi) &= \frac{1}{2}((1-\xi)\widehat{v}(-1,1) + (1+\xi)\widehat{v}(1,1)).
\end{aligned} \tag{3.33}$$

Next, we set

$$\begin{aligned}
&\widehat{v}^0_{b,L_1}(\eta) = \widehat{v}(-1,\eta) - \widehat{v}_{b,L_1}(\eta), \quad \widehat{v}^0_{b,L_2}(\xi) = \widehat{v}(\xi,-1) - \widehat{v}_{b,L_2}(\xi),\\
&\widehat{v}^0_{b,L_3}(\eta) = \widehat{v}(1,\eta) - \widehat{v}_{b,L_3}(\eta), \qquad \widehat{v}^0_{b,L_4}(\xi) = \widehat{v}(\xi,1) - \widehat{v}_{b,L_4}(\xi).
\end{aligned} \tag{3.34}$$

The above four functions vanish at the endpoints of Λ_ξ or Λ_η, respectively. Further, we set

$$\begin{aligned}
\widehat{v}_{b,\partial\Omega}(\xi,\eta) &= \widehat{v}^{(1)}_{b,\partial\Omega}(\xi,\eta) + \widehat{v}^{(2)}_{b,\partial\Omega}(\xi,\eta),\\
\widehat{v}^{(1)}_{b,\partial\Omega}(\xi,\eta) &= \frac{1}{2}((1-\xi)\widehat{v}(-1,\eta) + (1-\eta)\widehat{v}(\xi,-1) + (1+\xi)\widehat{v}(1,\eta) + (1+\eta)\widehat{v}(\xi,1)),\\
\widehat{v}^{(2)}_{b,\partial\Omega}(\xi,\eta) &= -\frac{1}{4}((1-\xi)(1-\eta)\widehat{v}(-1,-1) + (1+\xi)(1-\eta)\widehat{v}(1,-1)\\
&\quad +(1+\xi)(1+\eta)\widehat{v}(1,1) + (1-\xi)(1+\eta)\widehat{v}(-1,1)),
\end{aligned} \tag{3.35}$$

or equivalently,

$$\begin{aligned}
\widehat{v}_{b,\partial\Omega}(\xi,\eta) = {}& \frac{1}{2}((1-\xi)\widehat{v}^0_{b,L_1}(\eta) + (1-\eta)\widehat{v}^0_{b,L_2}(\xi) + (1+\xi)\widehat{v}^0_{b,L_3}(\eta) + (1+\eta)\widehat{v}^0_{b,L_4}(\xi))\\
&+ \frac{1}{4}((1-\xi)(1-\eta)\widehat{v}(-1,-1) + (1+\xi)(1-\eta)\widehat{v}(1,-1)\\
&+ (1+\xi)(1+\eta)\widehat{v}(1,1) + (1-\xi)(1+\eta)\widehat{v}(-1,1)).
\end{aligned}$$

Let

$$\widehat{v}_\Omega^0(\xi,\eta) = \widehat{v}(\xi,\eta) - \widehat{v}_{b,\partial\Omega}(\xi,\eta). \tag{3.36}$$

Obviously, $\widehat{v}_\Omega^0(\xi,\eta)$ vanishes on $\partial\Omega$. Accordingly, for any positive integer N_b, we introduce the projection corresponding to $\partial\Omega$, by

$$\begin{aligned} {}_*\widehat{P}^1_{N_b,\partial\Omega}\widehat{v}_{b,\partial\Omega}(\xi,\eta) = &\frac{1}{2}((1-\xi)P^{1,0}_{N_b,\Lambda_\eta}\widehat{v}^0_{b,L_1}(\eta) + (1-\eta)P^{1,0}_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_2}(\xi) \\ &+ (1+\xi)P^{1,0}_{N_b,\Lambda_\eta}\widehat{v}^0_{b,L_3}(\eta) + (1+\eta)P^{1,0}_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_4}(\xi)) \\ &+ \frac{1}{4}((1-\xi)(1-\eta)\widehat{v}(-1,-1) + (1+\xi)(1-\eta)\widehat{v}(1,-1) \\ &+ (1+\xi)(1+\eta)\widehat{v}(1,1) + (1-\xi)(1+\eta)\widehat{v}(-1,1)). \end{aligned}$$

We now set

$$\begin{aligned} &v_\Omega^0(x,y) = \widehat{v}_\Omega^0(\xi,\eta)|_{\xi=\xi(x,y),\eta=\eta(x,y)}, \quad v_{b,\partial\Omega}(x,y) = \widehat{v}_{b,\partial\Omega}(\xi,\eta)|_{\xi=\xi(x,y),\eta=\eta(x,y)}, \\ &{}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y) = {}_*\widehat{P}^1_{N_b,\partial\Omega}\widehat{v}_{b,\partial\Omega}(\xi,\eta)|_{\xi=\xi(x,y),\eta=\eta(x,y)}. \end{aligned} \tag{3.37}$$

Then, we define the Legendre quasi-orthogonal projection ${}_*P^1_{N,N_b,\Omega}v$ as

$${}_*P^1_{N,N_b,\Omega}v(x,y) = P^{1,0}_{N,\Omega}v^0_\Omega(x,y) + {}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y). \tag{3.38}$$

It can be checked that ${}_*P^1_{N,N_b,\Omega}v(x,y) = v(x,y)$ at the four vertices of Ω. Since

$${}_*P^1_{N,N_b,\Omega}v - v = P^{1,0}_{N,\Omega}v^0_\Omega(x,y) - v^0_\Omega(x,y) + {}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y) - v_{b,\partial\Omega}(x,y),$$

we have

$$\|\nabla({}_*P^1_{N,N_b,\Omega}v - v)\|_\Omega \le \|\nabla(P^{1,0}_{N,\Omega}v^0_\Omega - v^0_\Omega)\|_\Omega + \|\nabla({}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega} - v_{b,\partial\Omega})\|_\Omega. \tag{3.39}$$

We are going to estimate the first term of the right-hand side of (3.39). We use (3.27), (3.36) and (3.35) successively to obtain

$$\begin{aligned} \|\nabla(P^{1,0}_{N,\Omega}v^0_\Omega - v^0_\Omega)\|_\Omega &\le c\sigma_\Omega\delta_\Omega^{-\frac{1}{2}}N^{1-r}D_{r,S}(\widehat{v}^0_\Omega) \\ &\le c\sigma_\Omega\delta_\Omega^{-\frac{1}{2}}N^{1-r}(D_{r,S}(\widehat{v}) + D_{r,S}(\widehat{v}^{(1)}_{b,\partial\Omega}) + D_{r,S}(\widehat{v}^{(2)}_{b,\partial\Omega})). \end{aligned} \tag{3.40}$$

Thus, it suffices to estimate the right-hand side of (3.40). Firstly, by using (3.27) again, we have

$$D_{r,S}(\widehat{v}) \le c\delta_\Omega^{-\frac{1}{2}}\sum_{j=1}^{3}B^{(j)}_{r,\Omega}(v). \tag{3.41}$$

We next estimate $D_{r,S}(\widehat{v}^{(1)}_{b,\partial\Omega})$. We have from (3.35) that for $r \ge 2$,

$$\begin{aligned} \partial^r_\xi\widehat{v}^{(1)}_{b,\partial\Omega}(\xi,\eta) &= \frac{1}{2}\partial^r_\xi((1-\eta)\widehat{v}(\xi,-1) + (1+\eta)\widehat{v}(\xi,1)), \\ \partial^r_\eta\widehat{v}^{(1)}_{b,\partial\Omega}(\xi,\eta) &= \frac{1}{2}\partial^r_\eta((1-\xi)\widehat{v}(-1,\eta) + (1+\xi)\widehat{v}(1,\eta)). \end{aligned}$$

On the other hand, for $r \geq 3$,

$$\partial_\xi^{r-1}\partial_\eta \widehat{v}_{b,\partial\Omega}^{(1)}(\xi,\eta) = \frac{1}{2}\partial_\xi^{r-1}(-\widehat{v}(\xi,-1)+\widehat{v}(\xi,1)) = \frac{1}{2}\int_{\Lambda_\eta}\partial_\xi^{r-1}\partial_\eta\widehat{v}(\xi,\eta)\mathrm{d}\eta,$$

$$\partial_\xi\partial_\eta^{r-1}\widehat{v}_{b,\partial\Omega}^{(1)}(\xi,\eta) = \frac{1}{2}\partial_\eta^{r-1}(-\widehat{v}(-1,\eta)+\widehat{v}(1,\eta)) = \frac{1}{2}\int_{\Lambda_\xi}\partial_\xi\partial_\eta^{r-1}\widehat{v}(\xi,\eta)\mathrm{d}\xi.$$

Besides,

$$\begin{aligned}\partial_\xi\partial_\eta\widehat{v}_{b,\partial\Omega}^{(1)}(\xi,\eta) &= \frac{1}{2}(-\partial_\eta\widehat{v}(-1,\eta)-\partial_\xi\widehat{v}(\xi,-1)+\partial_\eta\widehat{v}(1,\eta)+\partial_\xi\widehat{v}(\xi,1))\\ &= \frac{1}{2}\int_{\Lambda_\eta}\partial_\xi\partial_\eta\widehat{v}(\xi,\eta)\mathrm{d}\eta + \frac{1}{2}\int_{\Lambda_\xi}\partial_\xi\partial_\eta\widehat{v}(\xi,\eta)\mathrm{d}\xi.\end{aligned}$$

With the aid of the previous equalities, a direct calculation yields

$$D_{r,S}(\widehat{v}_{b,\partial\Omega}^{(1)}) \leq c\delta_\Omega^{-\frac{1}{2}}(B_{r,\Omega}^{(2)}(v)+B_{r,\Omega}^{(3)}(v)+B_{r,\Omega}^{(4)}(v)), \tag{3.42}$$

where

$$\begin{aligned}B_{r,\Omega}^{(4)}(v) = \sum_{k=0}^{r}\Big(&\sum_{\nu=2,4}\|(1-\xi^2)^{\frac{r-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r-k}\partial_x^k\partial_y^{r-k}v\|_{L^2(L_\nu)}\\ &+\sum_{\nu=1,3}\|(1-\eta^2)^{\frac{r-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r-k}\partial_x^k\partial_y^{r-k}v\|_{L^2(L_\nu)}\Big).\end{aligned}$$

Finally, we estimate $D_{r,S}(\widehat{v}_{b,\partial\Omega}^{(2)})$. In fact,

$$\begin{aligned}\partial_\xi^r\widehat{v}_{b,\partial\Omega}^{(2)}(\xi,\eta) &= \partial_\eta^r\widehat{v}_{b,\partial\Omega}^{(2)}(\xi,\eta) = 0 &&\text{for } r\geq 2,\\ \partial_\xi^{r-1}\partial_\eta\widehat{v}_{b,\partial\Omega}^{(2)}(\xi,\eta) &= \partial_\xi\partial_\eta^{r-1}\widehat{v}_{b,\partial\Omega}^{(2)}(\xi,\eta) = 0 &&\text{for } r\geq 3.\end{aligned}$$

Moreover,

$$\partial_\xi\partial_\eta\widehat{v}_{b,\partial\Omega}^{(2)}(\xi,\eta) = -\frac{1}{4}\int_S\partial_\xi\partial_\eta\widehat{v}(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta.$$

As a result, $D_{r,S}(\widehat{v}_{b,\partial\Omega}^{(2)}) = 0$ for $r \geq 3$, and

$$\begin{aligned}D_{2,S}(\widehat{v}_{b,\partial\Omega}^{(2)}) \leq\ &\|(a_1+a_3\eta)(a_2+a_3\xi)\partial_x^2 v\|_\Omega + \|(b_1+b_3\eta)(b_2+b_3\xi)\partial_y^2 v\|_\Omega\\ &+\|(a_1+a_3\eta)(b_2+b_3\xi)\partial_x\partial_y v\|_\Omega + \|(b_1+b_3\eta)(a_2+a_3\xi)\partial_x\partial_y v\|_\Omega\\ &+\|a_3\partial_x v\|_\Omega + \|b_3\partial_y v\|_\Omega\\ \leq\ &c\delta_\Omega^{-\frac{1}{2}}(B_{r,\Omega}^{(2)}(v)+B_{r,\Omega}^{(3)}(v)).\end{aligned} \tag{3.43}$$

We now estimate the second term of the right-hand side of (3.39). By virtue of (3.35), we have

$$\begin{aligned}&{}_*\widehat{P}^1_{N_b,\partial\Omega}\widehat{v}_{b,\partial\Omega} - \widehat{v}_{b,\partial\Omega}\\ &= \frac{1}{2}((1-\xi)(P^{1,0}_{N_b,\Lambda_\eta}\widehat{v}^0_{b,L_1}(\eta)-\widehat{v}^0_{b,L_1}(\eta)) + (1-\eta)(P^{1,0}_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_2}(\xi)-\widehat{v}^0_{b,L_2}(\xi))\\ &\quad+(1+\xi)(P^{1,0}_{N_b,\Lambda_\eta}\widehat{v}^0_{b,L_3}(\eta)-\widehat{v}^0_{b,L_3}(\eta)) + (1+\eta)(P^{1,0}_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_4}(\xi)-\widehat{v}^0_{b,L_4}(\xi))).\end{aligned}$$

With the aid of (3.5), we verify that for $r_b \geq 2$,

$$\begin{aligned}&\|\nabla_S({}_*\widehat{P}^1_{N_b,\partial\Omega}\widehat{v}_{b,\partial\Omega}-\widehat{v}_{b,\partial\Omega})\|_S\\ &\leq cN_b^{1-r_b}(\|(1-\eta^2)^{\frac{r_b-1}{2}}\partial_\eta^{r_b}\widehat{v}^0_{b,L_1}\|_S+\|(1-\xi^2)^{\frac{r_b-1}{2}}\partial_\xi^{r_b}\widehat{v}^0_{b,L_2}\|_S\\ &\quad+\|(1-\eta^2)^{\frac{r_b-1}{2}}\partial_\eta^{r_b}\widehat{v}^0_{b,L_3}\|_S+\|(1-\xi^2)^{\frac{r_b-1}{2}}\partial_\xi^{r_b}\widehat{v}^0_{b,L_4}\|_S)\\ &=cN_b^{1-r_b}(\|(1-\eta^2)^{\frac{r_b-1}{2}}\partial_\eta^{r_b}\widehat{v}(-1,\eta)\|_S+\|(1-\xi^2)^{\frac{r_b-1}{2}}\partial_\xi^{r_b}\widehat{v}(\xi,-1)\|_S\\ &\quad+\|(1-\eta^2)^{\frac{r_b-1}{2}}\partial_\eta^{r_b}\widehat{v}(1,\eta)\|_S+\|(1-\xi^2)^{\frac{r_b-1}{2}}\partial_\xi^{r_b}\widehat{v}(\xi,1)\|_S).\end{aligned}$$

The above inequality with (2.6) implies

$$\|\nabla({}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}-v_{b,\partial\Omega})\|_\Omega\leq c\sigma_\Omega\delta_\Omega^{-1}N_b^{1-r_b}B^{(4)}_{r_b,\Omega}(v). \tag{3.44}$$

A combination of (3.39)–(3.44) gives

$$\|\nabla({}_*P^1_{N,N_b,\Omega}v-v)\|_\Omega\leq c\sigma_\Omega\delta_\Omega^{-1}\Big(N^{1-r}\sum_{j=1}^4B^{(j)}_{r,\Omega}(v)+N_b^{1-r_b}B^{(4)}_{r_b,\Omega}(v)\Big). \tag{3.45}$$

Remark 3.5 If $\Omega=S_{a,b}$ as in Remarks 3.2 and 3.3, then by virtue of (3.31), we obtain

$$\|\nabla(P^{1,0}_{N,\Omega}v^0_\Omega-v^0_\Omega)\|_\Omega\leq cN^{1-r}(B^*_{r,\Omega}(v)+D^*_{r,\Omega,a,b}(\widehat{v}^{(1)}_{b,\partial\Omega})+D^*_{r,\Omega,a,b}(\widehat{v}^{(2)}_{b,\partial\Omega})). \tag{3.46}$$

Like (3.42), we have

$$D^*_{r,\Omega,a,b}(\widehat{v}^{(1)}_{b,\partial\Omega})\leq c(B^{*,2}_{r,\Omega}(v)+B^{*,3}_{r,\Omega}(v)), \tag{3.47}$$

where

$$B^{*,3}_{r,\Omega}(v)=\Big(\sum_{\nu=2,4}\|(a^2-x^2)^{\frac{r-1}{2}}\partial_x^rv\|^2_{L^2(L_\nu)}+\sum_{\nu=1,3}\|(b^2-y^2)^{\frac{r-1}{2}}\partial_y^rv\|^2_{L^2(L_\nu)}\Big)^{\frac12}.$$

Similarly,

$$D^*_{r,\Omega,a,b}(\widehat{v}^{(2)}_{b,\partial\Omega})\leq cB^{*,2}_{r,\Omega}(v). \tag{3.48}$$

On the other hand, it can be checked that

$$\|\nabla({}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}-v_{b,\partial\Omega})\|_\Omega\leq cB^{*,3}_{r_b,\Omega}(v). \tag{3.49}$$

Finally, a combination of (3.46)–(3.49) leads to

$$\|\nabla({}_*P^1_{N,N_b,\Omega}v-v)\|_\Omega\leq cN^{1-r}\Big(\sum_{j=1}^3B^{*,j}_{r,\Omega}(v)+B^{*,3}_{r_b,\Omega}(v)\Big). \tag{3.50}$$

We find that in the estimate (3.50), there is no the term corresponding to $B^{(3)}_{r,\Omega}(v)$, which appears in (3.45). This fact leads to the super-convergence of spectral element method with rectangular elements.

3.5 Other Legendre irrational quasi-orthogonal approximations

We now turn to several Legendre irrational quasi-orthogonal approximations corresponding to Neumann or Robin boundary conditions imposed on certain parts of the boundary. For fixedness, we assume that certain Neumann or Robin boundary conditions are given on $L_1 \cup L_2$ (see Figure 1). Let $\partial^*\Omega = L_3 \cup L_4$, and

$$^0H^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial^*\Omega\}, \quad ^0V_N(\Omega) = {}^0H^1(\Omega) \cap V_N(\Omega).$$

The orthogonal projection $^0P^1_{N,\Omega} : {}^0H^1(\Omega) \to {}^0V_N(\Omega)$ is defined by

$$(\nabla({}^0P^1_{N,\Omega}v - v), \nabla\phi)_\Omega = 0, \quad \forall \phi \in {}^0V_N(\Omega). \tag{3.51}$$

Next, let

$$\widehat{v}^0_{b,L_3}(\eta) = \widehat{v}(1,\eta) - \frac{1}{2}(1+\eta)\widehat{v}(1,1), \quad \widehat{v}^0_{b,L_4}(\xi) = \widehat{v}(\xi,1) - \frac{1}{2}(1+\xi)\widehat{v}(1,1).$$

Evidently, $\widehat{v}^0_{b,L_3}(1) = \widehat{v}^0_{b,L_4}(1) = 0$. Further, we set $\widehat{v}^0_\Omega(\xi,\eta) = \widehat{v}(\xi,\eta) - \widehat{v}_{b,\partial\Omega}(\xi,\eta)$, with

$$\widehat{v}_{b,\partial\Omega}(\xi,\eta) = \frac{1}{2}(1+\xi)\widehat{v}^0_{b,L_3}(\eta) + \frac{1}{2}(1+\eta)\widehat{v}^0_{b,L_4}(\xi) + \frac{1}{4}(1+\xi)(1+\eta)\widehat{v}(1,1). \tag{3.52}$$

The function $\widehat{v}^0_\Omega(\xi,\eta)$ vanishes on $\partial^*\Omega$.

We also introduce the projection

$${}_*\widehat{P}^1_{N_b,\partial\Omega}\widehat{v}_{b,\partial\Omega}(\xi,\eta) = \frac{1}{2}(1+\xi){}^0P^1_{N_b,\Lambda_\eta}\widehat{v}^0_{b,L_3}(\eta) + \frac{1}{2}(1+\eta){}^0P^1_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_4}(\xi) + \frac{1}{4}(1+\xi)(1+\eta)\widehat{v}(1,1).$$

Finally, we introduce the quantities $v^0_\Omega(x,y), v_{b,\partial\Omega}(x,y)$ and ${}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y)$ in the same way as in (3.37), and define the quasi-Legendre orthogonal projection ${}_*P^1_{N,N_b,\Omega}v$ by

$${}_*P^1_{N,N_b,\Omega}v(x,y) = {}^0P^1_{N,\Omega}v^0_\Omega(x,y) + {}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y). \tag{3.53}$$

It can be shown that ${}_*P^1_{N,N_b,\Omega}v(x,y) = v(x,y)$ at the vertex $Q_4 = (1,1)$.

Following the same line as in the derivation of (3.45), we could prove that

$$\|\nabla({}_*P^1_{N,N_b,\Omega}v - v)\|_\Omega \le c\sigma_\Omega\delta_\Omega^{-1}\Big(N^{1-r}\sum_{j=1}^4 B^{(j)}_{r,\Omega}(v) + N_b^{1-r_b}B^{(5)}_{r_b,\Omega}(v)\Big), \tag{3.54}$$

where

$$B^{(5)}_{r,\Omega}(v) = \sum_{k=0}^r (\|(1-\eta^2)^{\frac{r-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r-k}\partial_x^k\partial_y^{r-k}v\|_{L^2(L_3)} + \|(1-\xi^2)^{\frac{r-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r-k}\partial_x^k\partial_y^{r-k}v\|_{L^2(L_4)}).$$

For designing and analyzing Petrov-Galerkin spectral element method for polygons, we need other Legendre quasi-orthogonal projections. For simplicity, we suppose that certain Neumann or Robin boundary conditions are given on L_1. Let $\partial^*\Omega = L_2 \cup L_3 \cup L_4$, and

$$^0\overline{H}^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial^*\Omega\}, \quad ^0V_N(\Omega) = {}^0\overline{H}^1(\Omega) \cap V_N(\Omega).$$

The orthogonal projection ${}^0\overline{P}^1_{N,\Omega} : {}^0\overline{H}^1(\Omega) \to {}^0\overline{V}_N(\Omega)$ is defined by

$$(\nabla({}^0\overline{P}^1_{N,\Omega}v - v), \nabla\phi)_\Omega = 0, \quad \forall \phi \in {}^0\overline{V}_N(\Omega). \tag{3.55}$$

Next, let

$$\widehat{v}^0_{b,L_2}(\xi) = \widehat{v}(\xi,-1) - \frac{1}{2}(1+\xi)\widehat{v}(1,-1), \quad \widehat{v}^0_{b,L_4}(\xi) = \widehat{v}(\xi,1) - \frac{1}{2}(1+\xi)\widehat{v}(1,1),$$
$$\widehat{v}^0_{b,L_3}(\eta) = \widehat{v}(1,\eta) - \frac{1}{2}(1-\eta)\widehat{v}(1,-1) - \frac{1}{2}(1+\eta)\widehat{v}(1,1).$$

Clearly, $\widehat{v}^0_{b,L_2}(1) = \widehat{v}^0_{b,L_3}(\pm 1) = \widehat{v}^0_{b,L_4}(1) = 0$. Further, we set $\widehat{v}^0_\Omega(\xi,\eta) = \widehat{v}(\xi,\eta) - \widehat{v}_{b,\partial\Omega}(\xi,\eta)$ with

$$\begin{aligned}\widehat{v}_{b,\partial\Omega}(\xi,\eta) = {} & \frac{1}{2}((1-\eta)\widehat{v}^0_{b,L_2}(\xi) + (1+\xi)\widehat{v}^0_{b,L_3}(\eta) + (1+\eta)\widehat{v}^0_{b,L_4}(\xi)) \\ & + \frac{1}{4}((1+\xi)(1-\eta)\widehat{v}(1,-1) + (1+\xi)(1+\eta)\widehat{v}(1,1)).\end{aligned} \tag{3.56}$$

The function $\widehat{v}^0_\Omega(\xi,\eta)$ vanishes on $\partial^*\Omega$.

We also introduce the projection

$$\begin{aligned}& {}_*\widehat{P}^1_{N_b,\partial\Omega}\widehat{v}_{b,\partial\Omega}(\xi,\eta) \\ & = \frac{1}{2}((1-\eta){}^0P^1_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_2}(\xi) + (1+\xi)P^{1,0}_{N_b,\Lambda_\eta}\widehat{v}^0_{b,L_3}(\eta) + (1+\eta){}^0P^1_{N_b,\Lambda_\xi}\widehat{v}^0_{b,L_4}(\xi)) \\ & \quad + \frac{1}{4}((1+\xi)(1-\eta)\widehat{v}(1,-1) + (1+\xi)(1+\eta)\widehat{v}(1,1)).\end{aligned}$$

Finally, we introduce the quantities $v^0_\Omega(x,y), v_{b,\partial\Omega}(x,y)$ and ${}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y)$ in the same way as in (3.37), and define the quasi-Legendre orthogonal projection ${}_*P^1_{N,N_b,\Omega}v$ by

$${}_*P^1_{N,N_b,\Omega}v(x,y) = {}^0\overline{P}^1_{N,\Omega}v^0_\Omega(x,y) + {}_*P^1_{N_b,\partial\Omega}v_{b,\partial\Omega}(x,y). \tag{3.57}$$

It is easy to show that ${}_*P^1_{N,N_b,\Omega}v(x,y) = v(x,y)$ at the vertices $(x,y) = (1,-1)$ and $(1,1)$.

Following the same line as in the derivation of (3.45), we could prove that

$$\|\nabla({}_*P^1_{N,N_b,\Omega}v - v)\|_\Omega \le c\sigma_\Omega\delta_\Omega^{-1}\Big(N^{1-r}\sum_{j=1}^{4}B^{(j)}_{r,\Omega}(v) + N_b^{1-r_b}B^{(6)}_{r_b,\Omega}(v)\Big), \tag{3.58}$$

where

$$\begin{aligned}B^{(6)}_{r,\Omega}(v) = \sum_{k=0}^{r}(&\|(1-\eta^2)^{\frac{r-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r-k}\partial_x^k\partial_y^{r-k}v\|_{L^2(L_3)} \\ & + \|(1-\xi^2)^{\frac{r-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r-k}\partial_x^k\partial_y^{r-k}v\|_{L^2(L_2\cup L_4)}).\end{aligned}$$

In the same manner, we can define various Legendre quasi-orthogonal projections ${}_*P^1_{N,N_b,\Omega}v$ corresponding to Neumann or Robin boundary conditions imposed on some edges of Ω, and derive the error estimates similar to (3.45), (3.54) and (3.58), respectively.

Remark 3.6 If $\Omega = S_{a,b}$ as in Remarks 3.2 and 3.5, then we could derive the error estimates of the above Legendre quasi-orthogonal projections, which are similar to (3.50).

4 Petrov-Galerkin Spectral Method for Mixed Inhomogeneous Boundary Value Problems

In this section, we propose the Petrov-Galerkin spectral method for quadrilaterals.

Let $\beta(x,y)$ be a non-negative and uniformly bounded function, $\partial^{**}\Omega = L_1 \cup L_2$ and $\partial^*\Omega = L_3 \cup L_4$ (see Figure 1). We consider the following mixed inhomogeneous boundary value problem:

$$\begin{cases} -\Delta U(x,y) = f(x,y), & (x,y) \in \Omega, \\ \partial_n U(x,y) + \beta(x,y)U(x,y) = g_2(x,y), & (x,y) \in \partial^{**}\Omega, \\ U(x,y) = g_1(x,y), & (x,y) \in \partial^*\Omega. \end{cases} \tag{4.1}$$

We set

$$V_{g_1}(\Omega) = \{v \in H^1(\Omega) \cap C(\overline{\Omega}) \mid v = g_1 \text{ on } \partial^*\Omega\},$$
$$\overline{V}(\Omega) = \{v \in H^1(\Omega) \cap C(\overline{\Omega}) \mid v = 0 \text{ on } \partial^*\Omega\}$$

and

$$a_\beta(u,v) = (\nabla U, \nabla v)_\Omega + \int_{\partial^{**}\Omega} \beta(x,y)U(x,y)v(x,y)\mathrm{d}s, \quad \forall u \in V_{g_1}(\Omega),\ v \in \overline{V}(\Omega).$$

The weak formulation of (4.1) is to find $U \in V_{g_1}(\Omega)$ such that

$$a_\beta(U,v) - \int_{\partial^{**}\Omega} g_2(x,y)v(x,y)\mathrm{d}s = (f,v)_\Omega, \quad \forall v \in \overline{V}(\Omega). \tag{4.2}$$

For solving the above problem properly, we first consider an auxiliary problem. For this purpose, we set $\widehat{g}_1(\xi,\eta) = g_1(x(\xi,\eta), y(\xi,\eta))$ and define the projection ${}_*P^1_{N_b,\partial^*\Omega} g_1(x,y)$ by

$${}_*P^1_{N_b,\partial^*\Omega} g_1(x,y) = {}^0P^1_{N_b,\Lambda_\eta}\Big(\widehat{g}_1(1,\eta) - \frac{1}{2}\widehat{g}_1(1,1)(1+\eta)\Big) + \frac{1}{2}\widehat{g}_1(1,1)(1+\eta)\Big|_{\eta=\eta(x,y)}, \quad \text{on } L_3,$$
$${}_*P^1_{N_b,\partial^*\Omega} g_1(x,y) = {}^0P^1_{N_b,\Lambda_\xi}\Big(\widehat{g}_1(\xi,1) - \frac{1}{2}\widehat{g}_1(1,1)(1+\xi)\Big) + \frac{1}{2}\widehat{g}_1(1,1)(1+\xi)\Big|_{\xi=\xi(x,y)}, \quad \text{on } L_4.$$

The auxiliary problem is to seek the solution $W \in V_{{}_*P^1_{N_b,\partial^*\Omega} g_1}(\Omega)$ such that

$$a_\beta(W,v) - \int_{\partial^{**}\Omega} g_2(x,y)v(x,y)\mathrm{d}s = (f,v)_\Omega, \quad \forall v \in \overline{V}(\Omega). \tag{4.3}$$

Obviously, we have from (4.2) and (4.3) that

$$\begin{cases} -\Delta(U(x,y) - W(x,y)) = 0, & (x,y) \in \Omega, \\ \partial_n(U(x,y) - W(x,y)) + \beta(x,y)(U(x,y) - W(x,y)) = 0, & (x,y) \in \partial^{**}\Omega, \\ U(x,y) - W(x,y) = g_1(x,y) - {}_*P^1_{N_b,\partial^*\Omega} g_1(x,y), & (x,y) \in \partial^*\Omega. \end{cases}$$

According to the properties of elliptic equation and the error estimates for the Legendre quasi-orthogonal approximation, we verify that

$$\begin{aligned} \|U - W\|_{H^1(\Omega)} &\le c\|g_1 - {}_*P^1_{N_b,\partial^*\Omega} g_1\|_{H^{\frac{1}{2}}(\partial^*\Omega)} \\ &\le c\|g_1 - {}_*P^1_{N_b,\partial^*\Omega} g_1\|^{\frac{1}{2}}_{H^1(\partial^*\Omega)} \|g_1 - {}_*P^1_{N_b,\partial^*\Omega} g_1\|^{\frac{1}{2}}_{L^2(\partial^*\Omega)} \\ &\le c\sigma_\Omega \delta_\Omega^{-1} N_b^{\frac{1}{2}-r_b} K_{r_b,\partial^*\Omega}(g_1), \end{aligned} \tag{4.4}$$

where

$$K_{r_b,\partial^*\Omega}(g_1)=\sum_{k=0}^{r_b}(\|(1-\eta^2)^{\frac{r_b-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r_b-k}\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_3)}$$
$$+\|(1-\xi^2)^{\frac{r_b-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r_b-k}\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_4)}).$$

We are going to design the Petrov-Galerkin spectral scheme for solving (4.3). We need three kinds of base functions. Let $L_l(\xi)$ be the Legendre polynomial of degree l, as before. The base functions, corresponding to the interior domain Ω, are given by

$$\psi^0_{\Omega,l,m}(x,y)=\frac{1}{\sqrt{(4l+6)(4m+6)}}(L_l(\xi)-L_{l+2}(\xi))(L_m(\eta)-L_{m+2}(\eta))\Big|_{\xi=\xi(x,y),\eta=\eta(x,y)}.$$

The base functions, corresponding to the edges L_3 and L_4, are defined as

$$\psi^0_{L_3,l_3}(x,y)=\frac{1}{2\sqrt{4l_3+6}}(1+\xi)(L_{l_3}(\eta)-L_{l_3+2}(\eta))\Big|_{\xi=\xi(x,y),\eta=\eta(x,y)},$$
$$\psi^0_{L_4,l_4}(x,y)=\frac{1}{2\sqrt{4l_4+6}}(1+\eta)(L_{l_4}(\xi)-L_{l_4+2}(\xi))\Big|_{\xi=\xi(x,y),\eta=\eta(x,y)}.$$

The base function, corresponding to the vertex Q_4, is

$$\psi_{Q_4}(x,y)=\frac{1}{4}(1+\xi)(1+\eta)\Big|_{\xi=\xi(x,y),\eta=\eta(x,y)}.$$

Now, let $W_{N,N_b}(\Omega)$ be the finite-dimensional set spanned by all $\psi^0_{\Omega,l,m}(x,y),\psi^0_{L_3,l_3}(x,y)$, $\psi^0_{L_4,l_4}(x,y)$ and $\psi_{Q_4}(x,y)$, $0\le l,m\le N-2,\ 0\le l_3,l_4\le N_b$. Clearly, $W_{N,N_b}(\Omega)\subset H^1(\Omega)\cap C(\overline{\Omega})$. Further, we let

$$V_{N,N_b}(\Omega)=\{\phi\in W_{\mathbf{N},N_b}(\Omega)\mid \phi={}_*P^1_{N_b,\partial^*\Omega}g_1\ \text{on}\ \partial^*\Omega\},$$
$$V^*_{N,N_b}(\Omega)=\{\phi\in W_{\mathbf{N},N_b}(\Omega)\mid \phi=0\ \text{on}\ \partial^*\Omega\}.$$

The spectral method for (4.3) is to find $w_{N,N_b}\in V_{N,N_b}(\Omega)$ such that

$$a_\beta(w_{N,N_b},\phi)-\int_{\partial^{**}\Omega}g_2(x,y)\phi(x,y)\mathrm{d}s=(f,\phi)_\Omega,\quad\forall\phi\in V^*_{N,N_b}(\Omega).\tag{4.5}$$

For derivation of error estimate of numerical solution, we introduce the auxiliary orthogonal projection $P^1_{N,N_b,\Omega}v:H^1(\Omega)\cap C(\overline{\Omega})\to V_{N,N_b}(\Omega)$, such that

$$a_\beta(P^1_{N,N_b,\Omega}v-v,\phi)=0,\quad\forall\phi\in V^*_{N,N_b}(\Omega).\tag{4.6}$$

This, together with (4.3), leads to

$$a_\beta(P^1_{N,N_b,\Omega}W,\phi)-\int_{\partial^{**}\Omega}g_2(x,y)\phi(x,y)\mathrm{d}s=(f,\phi)_\Omega,\quad\forall\phi\in V^*_{N,N_b}(\Omega).\tag{4.7}$$

Subtracting (4.7) from (4.5) yields

$$a_\beta(w_{N,N_b}-P^1_{N,N_b,\Omega}W,\phi)=0,\quad\forall\phi\in V^*_{N,N_b}(\Omega).$$

This implies $w_{N,N_b}=P^1_{N,N_b,\Omega}W$, and thus

$$a_\beta(w_{N,N_b}-W,w_{N,N_b}-W)=a_\beta(P^1_{N,N_b,\Omega}W-W,P^1_{N,N_b,\Omega}W-W). \tag{4.8}$$

We shall use the following lemma.

Lemma 4.1 *For any $v\in V_{g_1}(\Omega)$ and $z\in W_{N,N_b}(\Omega)$,*

$$a_\beta(v-P^1_{N,N_b,\Omega}v,v-P^1_{N,N_b,\Omega}v)\le a_\beta(v-z,v-z). \tag{4.9}$$

Proof Clearly, $P^1_{N,N_b,\Omega}v-z\in V^*_{N,N_b}(\Omega)$. Thereby, a direct calculation with (4.6) gives

$$\begin{aligned}a_\beta(v-z,v-z)&=a_\beta(v-P^1_{N,N_b,\Omega}v,v-P^1_{N,N_b,\Omega}v)+a_\beta(z-P^1_{N,N_b,\Omega}v,z-P^1_{N,N_b,\Omega}v)\\&\quad+2a_\beta(v-P^1_{N,N_b,\Omega}v,P^1_{N,N_b,\Omega}v-z)\\&\ge a_\beta(v-P^1_{N,N_b,\Omega}v,v-P^1_{N,N_b,\Omega}v).\end{aligned}$$

This ends the proof.

Let ${}_*P^1_{N,N_b,\Omega}v$ be the projection defined by (3.53). By using (4.9) with $v=W$ and $z={}_*P^1_{N,N_b,\Omega}U$, we obtain from (4.8) that

$$\begin{aligned}a_\beta(w_{N,N_b}-W,\ w_{N,N_b}-W)&\le a_\beta({}_*P^1_{N,N_b,\Omega}U-W,\ {}_*P^1_{N,N_b,\Omega}U-W)\\&\le 2a_\beta({}_*P^1_{N,N_b,\Omega}U-U,\ {}_*P^1_{N,N_b,\Omega}U-U)+2a_\beta(U-W,\ U-W).\end{aligned}$$

Finally, we use (4.4), (3.54) and triangle inequality to reach

$$\begin{aligned}&a_\beta(w_{N,N_b}-U,w_{N,N_b}-U)\\&\le c\sigma_\Omega\delta_\Omega^{-1}\Big(N^{1-r}\sum_{j=1}^{4}B^{(j)}_{r,\Omega}(U)+N_b^{1-r_b}B^{(5)}_{r_b,\Omega}(U)+N_b^{\frac12-r_b}K_{r_b,\partial^*\Omega}(g_1)\Big).\end{aligned} \tag{4.10}$$

Remark 4.1 In actual computation, we evaluate the terms

$$\int_{\partial^{**}\Omega}\beta(x,y)g_2(x,y)\phi(x,y)\mathrm{d}s$$

and $(f,\phi)_\Omega$ approximately. Thus, in general, there exist two additional errors depending on the accuracy of numerical quadratures and the smoothness of f and g_2.

5 Petrov-Galerkin Spectral Element Method for Polygons

We are now in position to study Petrov-Galerkin spectral element method for mixed inhomogeneous boundary value problems defined on polygons.

5.1 Composite Legendre irrational quasi-orthogonal approximation on polygons

Let Ω be a polygon with the boundary $\partial\Omega=\partial^*\Omega\cup\partial^{**}\Omega$ and $\partial^*\Omega\cap\partial^{**}\Omega=\emptyset$. We may impose Neumann or Robin boundary conditions on $\partial^{**}\Omega$. We divide Ω into convex quadrilaterals Ω_i $(1\le i\le n)$ with the boundary $\partial\Omega_i$, the edges $L_{i,\nu}$, the vertices $Q_{i,\nu}$ and the angles $\theta_{i,\nu}$ $(1\le\nu\le 4)$. Besides, $\partial^*\Omega_i=\partial\Omega_i\cap\partial^*\Omega$ and $\partial^{**}\Omega_i=\partial\Omega_i\cap\partial^{**}\Omega$. The local variable transformation are denoted by $\xi_i = \xi_i(x, y)$ and $\eta_i = \eta_i(x, y)$ $(1 \le i \le n)$. The corresponding quantities $\sigma_{\Omega_i}, \gamma_{\Omega_i}, \delta_{\Omega_i}, d_{i,1}, d_{i,2}, a_{i,1}, a_{i,2}, a_{i,3}, b_{i,1}, b_{i,2}$ and $b_{i,3}$ are defined in the same way as for single quadrilateral (see Section 3). Let $h_i = \operatorname{diam} \Omega_i$. Assume that the partition of Ω satisfies the following hypotheses:

(H_1) $\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\Omega}_i$ and $\Omega_i \cap \Omega_k = \emptyset$ if $i \ne k$,

(H_2) each vertex of Ω_i is also one of vertices of adjacent quadrilaterals,

(H_3) if $\overline{\Omega}_i \cap \partial\Omega \ne \emptyset$, then Ω_i has at most two edges belonging to $\partial^{**}\Omega$,

(H_4) if $L_{i,\nu} \subseteq \partial^*\Omega$, then $L_{i,\nu} \nsubseteq \partial^{**}\Omega$,

(H_5) there are positive constants λ_0 and λ_1 such that $0 < \lambda_0 \le \theta_{i,\nu} \le \lambda_1 < \pi,\ 1 \le \nu \le 4,\ 1 \le i \le n$, and so $0 < \delta_0 h_i^2 \le \delta_{\Omega_i} \le \delta_1 h_i^2,\ 1 \le i \le n$.

Let $N = (N_1, N_2, \cdots, N_n)$ and $r = (r_1, r_2, \cdots, r_n)$. We define the composite Legendre irrational quasi-orthogonal projection ${}_*P^1_{\mathbf{N},N_b,\Omega} v$ by

$$ {}_*P^1_{\mathbf{N},N_b,\Omega} v|_{\Omega_i} = {}_*P^1_{N_i,N_b,\Omega_i} v, \quad 1 \le i \le n, $$

where the local projections ${}_*P^1_{N_i,N_b,\Omega_i} v$ are constructed in such a way that

(A) if $\partial^{**}\Omega_i = \emptyset$, then ${}_*P^1_{N_i,N_b,\Omega_i} v$ is given by (3.38),

(B) if $\partial^{**}\Omega_i \ne \emptyset$, say $\partial^{**}\Omega_i = L_{i,1} \cup L_{i,2}$, then ${}_*P^1_{N_i,N_b,\Omega_i} v$ is similar to (3.53),

(C) if $\partial^{**}\Omega_i \ne \emptyset$, say $\partial^{**}\Omega_i = L_{i,1}$, then ${}_*P^1_{N_i,N_b,\Omega_i} v$ is similar to (3.57).

Clearly, if $L_{i,\nu}$ and $L_{k,\nu}$ are the same segment, say $L_{i,3} = L_{k,1}$, then the coefficients in the expansions of $P^{1,0}_{N_b,\Lambda_{\eta_i}} \widehat{v}^0_{b,L_{i,3}}(\eta_i)$ and $P^{1,0}_{N_b,\Lambda_{\eta_k}} \widehat{v}^0_{b,L_{k,1}}(\eta_k)$ are the same. It can be checked that ${}_*P^1_{\mathbf{N},N_b,\Omega} v \in H^1(\Omega) \cap C(\overline{\Omega})$.

For description of approximation error, we introduce the notations $B^{(j)}_{r_i,\Omega_i}(v)$ $(1 \le j \le 4)$ and $B^{(k)}_{r_b,\Omega_i}(v)$ $(4 \le k \le 6)$, with the quantities $\xi_i, \eta_i, \sigma_{\Omega_i}, \gamma_{\Omega_i}, \delta_{\Omega_i}, d_{i,1}, d_{i,2}, a_{i,1}, a_{i,2}, a_{i,3}, b_{i,1}, b_{i,2}$ and $b_{i,3}$, respectively. Also, we introduce the quantity $D_{r_b,\Omega_i}(v)$ in such a way that

(A) if $\partial^{**}\Omega_i = \emptyset$, then $D_{r_b,\Omega_i}(v) = B^{(4)}_{r_b,\Omega_i}(v)$,

(B) if $\partial^{**}\Omega_i \ne \emptyset$, say $\partial^{**}\Omega_i = L_{i,1} \cup L_{i,2}$, then $D_{r_b,\Omega_i}(v) = B^{(5)}_{r_b,\Omega_i}(v)$,

(C) if $\partial^{**}\Omega_i \ne \emptyset$, say $\partial^{**}\Omega_i = L_{i,1}$, then $D_{r_b,\Omega_i}(v) = B^{(6)}_{r_b,\Omega_i}(v)$.

According to the previous statements and a standard argument as in [2, 5, 15], we observe that if $B^{(j)}_{r_i,\Omega_i}(v)$ and $D_{r_b,\Omega_i}(v)$ are finite for integers $2 \le r_i \le N_i + 1$ $(1 \le i \le n)$ and $2 \le r_b \le N_b + 1$, then

$$ \|\nabla({}_*P^1_{\mathbf{N},N_b,\Omega} v - v)\|_\Omega \le c \sum_{i=1}^{n} \sigma_{\Omega_i} \delta^{-1}_{\Omega_i} \Big(N_i^{1-r_i} \sum_{j=1}^{4} B^{(j)}_{r_i,\Omega_i}(v) + N_b^{1-r_b} D_{r_b,\Omega_i}(v) \Big). \tag{5.1} $$

5.2 Spectral element method for polygons

Let $\beta(x, y)$ be a non-negative and uniformly bounded function. We consider the following problem:

$$ \begin{cases} -\Delta U(x, y) = f(x, y), & (x, y) \in \Omega, \\ \partial_n U(x, y) + \beta(x, y) U(x, y) = g_2(x, y), & (x, y) \in \partial^{**}\Omega, \\ U(x, y) = g_1(x, y), & (x, y) \in \partial^*\Omega. \end{cases} \tag{5.2} $$

If $\partial^*\Omega = \emptyset$ and $\beta(x, y) \equiv 0$, then we require additionally that $(f, 1)_\Omega = 0$ for consistency, and that $(U, 1)_\Omega = 0$ for uniqueness of solution. For simplicity, we suppose that $\partial^*\Omega \neq \emptyset$, or $\beta(x, y)$ is not always null.

Now, we set

$$\begin{aligned} V_{g_1}(\Omega) &= \{v \in H^1(\Omega) \cap C(\overline{\Omega}) \mid v = g_1 \text{ on } \partial^*\Omega\}, \\ \overline{V}(\Omega) &= \{v \in H^1(\Omega) \cap C(\overline{\Omega}) \mid v = 0 \text{ on } \partial^*\Omega\} \end{aligned}$$

and

$$a_\beta(u, v) = (\nabla U, \nabla v)_\Omega + \int_{\partial^{**}\Omega} \beta(x, y)U(x, y)v(x, y)\mathrm{d}s, \quad \forall u \in V_{g_1}(\Omega),\ v \in \overline{V}(\Omega).$$

The weak formulation of (5.2) is to find $U \in V_{g_1}(\Omega)$ such that

$$a_\beta(U, v) - \int_{\partial^{**}\Omega} g_2(x, y)v(x, y)\mathrm{d}s = (f, v)_\Omega, \quad \forall v \in \overline{V}(\Omega). \tag{5.3}$$

For solving (5.3), we first consider an auxiliary problem. To do this, we let

$$\widehat{g}_1(\xi, \eta) = g_1(x(\xi, \eta), y(\xi, \eta)),$$

and introduce the projection

$${}_*P^1_{N_b,\partial^*\Omega} g_1(x, y)|_{\partial^*\Omega_i} = {}_*P^1_{N_b,\partial^*\Omega_i} g_1(x, y),$$

in which the local projection ${}_*P^1_{N_b,\partial^*\Omega_i} g_1(x, y)$ depends on the location of $\partial^*\Omega_i$. For instance, if the edge $L_{i,1} = \partial^*\Omega_i$, then

$$\begin{aligned} {}_*P^1_{N_b,\partial^*\Omega} g_1(x, y) = {} & P^{1,0}_{N_b,\Lambda_{\eta_i}} \Big(\widehat{g}_1(-1, \eta_i) - \frac{1}{2}\widehat{g}_1(-1, -1)(1 - \eta_i) - \frac{1}{2}\widehat{g}_1(-1, 1)(1 + \eta_i)\Big) \\ & + \frac{1}{2}\widehat{g}_1(-1, -1)(1 - \eta_i) + \frac{1}{2}\widehat{g}_1(-1, 1)(1 + \eta_i)\Big|_{\xi_i=\xi_i(x,y),\eta_i=\eta_i(x,y)}. \end{aligned}$$

The auxiliary problem is to seek solution $W \in V_{{}_*P^1_{N_b,\partial^*\Omega} g_1}(\Omega)$ such that

$$a_\beta(W, v) - \int_{\partial^{**}\Omega} g_2(x, y)v(x, y)\mathrm{d}s = (f, v)_\Omega, \quad \forall v \in \overline{V}(\Omega). \tag{5.4}$$

We have from (5.3) and (5.4) that

$$\begin{cases} -\Delta(U(x, y) - W(x, y)) = 0, & (x, y) \in \Omega, \\ \partial_n(U(x, y) - W(x, y)) + \beta(x, y)(U(x, y) - W(x, y)) = 0, & (x, y) \in \partial^{**}\Omega, \\ U(x, y) - W(x, y) = g_1(x, y) - {}_*P^1_{N_b,\partial^*\Omega} g_1(x, y), & (x, y) \in \partial^*\Omega. \end{cases}$$

Like (4.4), we have

$$\|U - W\|_{H^1(\Omega)} \le cN_b^{\frac{1}{2}-r_b} \sum_{i=1}^{n} \sigma_{\Omega_i} \delta^{-1}_{\Omega_i} K_{r_b,\partial^*\Omega_i}(g_1), \tag{5.5}$$

where $K_{r_b,\partial^*\Omega_i}(g_1)=0$ if $\partial^*\Omega_i=\emptyset$, otherwise,

$$K_{r_b,\partial^*\Omega_i}(g_1)=\begin{cases}\sum\limits_{k=0}^{r_b}\|(1-\eta^2)^{\frac{r_b-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r_b-k}\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_{i,1})}, & \text{if } L_{i,1}\subseteq\partial^*\Omega,\\ \sum\limits_{k=0}^{r_b}\|(1-\xi^2)^{\frac{r_b-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r_b-k}\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_{i,2})}, & \text{if } L_{i,2}\subseteq\partial^*\Omega,\\ \sum\limits_{k=0}^{r_b}\|(1-\eta^2)^{\frac{r_b-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r_b-k}\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_{i,3})}, & \text{if } L_{i,3}\subseteq\partial^*\Omega,\\ \sum\limits_{k=0}^{r_b}\|(1-\xi^2)^{\frac{r_b-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r_b-k}\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_{i,4})}, & \text{if } L_{i,4}\subseteq\partial^*\Omega.\end{cases}$$

For solving problem (5.4) numerically, we need three kinds of base functions. We first take the base functions corresponding to Ω_i, as

$$\psi^0_{\Omega_i,l_i,m_i}(x,y)=\begin{cases}\dfrac{1}{\sqrt{(4l_i+6)(4m_i+6)}}\\ \cdot(L_{l_i}(\xi_i)-L_{l_i+2}(\xi_i))(L_{m_i}(\eta_i)-L_{m_i+2}(\eta_i))\Big|_{\xi_i=\xi_i(x,y),\ \eta_i=\eta_i(x,y)}, & \text{on } \overline{\Omega}_i,\\ 0, & \text{otherwise.}\end{cases}$$

Next, we define the base functions corresponding to the edges of quadrilaterals. For instance, if $L_{i,k}=L_{i,1}=L_{k,3}$, then the corresponding base function

$$\psi^0_{L_{i,k},l_{i,k}}(x,y)=\begin{cases}\dfrac{1}{2\sqrt{4l_{i,k}+6}}(1-\xi_i)(L_{l_{i,k}}(\eta_i)-L_{l_{i,k}+2}(\eta_i))\Big|_{\xi_i=\xi_i(x,y),\eta_i=\eta_i(x,y)}, & \text{on } \overline{\Omega}_i,\\ \dfrac{1}{2\sqrt{4l_{i,k}+6}}(1+\xi_k)(L_{l_{i,k}}(\eta_k)-L_{l_{i,k}+2}(\eta_k))\Big|_{\xi_k=\xi_k(x,y),\eta_k=\eta_k(x,y)}, & \text{on } \overline{\Omega}_k,\\ 0, & \text{otherwise.}\end{cases}$$

The third kind of base functions correspond to the vertices of quadrilaterals. For example, if $Q_{i_1,1}=Q_{i_2,2}=Q_{i_3,3}=Q_{i_4,4}$, then the corresponding base function

$$\psi_{Q_{i_1,i_2,i_3,i_4}}(x,y)=\begin{cases}\frac14(1-\xi_{i_1})(1-\eta_{i_1})\Big|_{\xi_{i_1}=\xi_{i_1}(x,y),\eta_{i_1}=\eta_{i_1}(x,y)}, & \text{on } \overline{\Omega}_{i_1},\\ \frac14(1+\xi_{i_2})(1-\eta_{i_2})\Big|_{\xi_{i_2}=\xi_{i_2}(x,y),\eta_{i_2}=\eta_{i_2}(x,y)}, & \text{on } \overline{\Omega}_{i_2},\\ \frac14(1+\xi_{i_3})(1+\eta_{i_3})\Big|_{\xi_{i_3}=\xi_{i_3}(x,y),\eta_{i_3}=\eta_{i_3}(x,y)}, & \text{on } \overline{\Omega}_{i_3},\\ \frac14(1-\xi_{i_4})(1+\eta_{i_4})\Big|_{\xi_{i_4}=\xi_{i_4}(x,y),\eta_{i_4}=\eta_{i_4}(x,y)}, & \text{on } \overline{\Omega}_{i_4},\\ 0, & \text{otherwise.}\end{cases}$$

Now, let $W_{\mathbf{N},N_b}(\Omega)$ be the set spanned by $\psi^0_{\Omega_i,l_i,m_i}(x,y)$, $0\le l_i,m_i\le N_i-2$, all $\psi^0_{L_{i,k},l_{i,k}}(x,y)$, $0\le l_{i,k}\le N_b$ and all $\psi_{Q_{i_1,i_2,i_3,i_4}}(x,y)$. Clearly, $W_{\mathbf{N},N_b}(\Omega)\subset H^1(\Omega)\cap C(\overline{\Omega})$. Furthermore,

$$V_{\mathbf{N},N_b}(\Omega)=\{\phi\in W_{\mathbf{N},N_b}(\Omega)\mid \phi=P^1_{N_b,\partial^*\Omega}g_1 \text{ on } \partial^*\Omega\},$$
$$V^*_{\mathbf{N},N_b}(\Omega)=\{\phi\in W_{\mathbf{N},N_b}(\Omega)\mid \phi=0 \text{ on } \partial^*\Omega\}.$$

The Petrov-Galerkin spectral element method for (5.4) is to find $w_{\mathbf{N},N_b}\in V_{\mathbf{N},N_b}(\Omega)$ such that

$$a_\beta(w_{\mathbf{N},N_b},\phi)-\int_{\partial^{**}\Omega}g_2(x,y)\phi(x,y)\mathrm{d}s=(f,\phi)_\Omega,\quad \forall\phi\in V^*_{\mathbf{N},N_b}(\Omega). \tag{5.6}$$

For derivation of error estimate of numerical solution, we introduce the orthogonal projection $P^1_{\mathbf{N},N_b,\Omega}v : H^1(\Omega)\cap C(\overline{\Omega}) \to V_{\mathbf{N},N_b}(\Omega)$ such that

$$a_\beta(P^1_{\mathbf{N},N_b,\Omega}v - v, \nabla\phi) = 0, \quad \forall \phi \in V^*_{\mathbf{N},N_b}(\Omega).$$

This, along with (5.4), leads to

$$a_\beta(P^1_{\mathbf{N},N_b,\Omega}W, \phi) - \int_{\partial^{**}\Omega} g_2(x,y)\phi(x,y)\mathrm{d}s = (f,\phi)_\Omega, \quad \forall \phi \in V^*_{\mathbf{N},N_b}(\Omega). \tag{5.7}$$

Subtracting (5.7) from (5.4) yields

$$a_\beta(P^1_{\mathbf{N},N_b,\Omega}W - w_{\mathbf{N},N_b}, \nabla\phi) = 0, \quad \forall \phi \in V^*_{\mathbf{N},N_b}(\Omega).$$

This implies $w_{\mathbf{N},N_b} = P^1_{\mathbf{N},N_b,\Omega}W$.

Following the same line as in the derivation of (4.9), we can prove the following lemma.

Lemma 5.1 *For any $v \in V_{g_1}(\Omega)$ and $z \in W_{N,N_b}(\Omega)$,*

$$a_\beta(v - P^1_{N,N_b,\Omega}v, v - P^1_{N,N_b,\Omega}v) \le a_\beta(v - z, v - z). \tag{5.8}$$

By using (5.8) with $v = W$ and $z = {}_*P^1_{N,N_b,\Omega}U$, we obtain

$$\begin{aligned}
&a_\beta(w_{\mathbf{N},N_b} - W, w_{\mathbf{N},N_b} - W)\\
&= a_\beta(P^1_{\mathbf{N},N_b,\Omega}W - W, P^1_{\mathbf{N},N_b,\Omega}W - W)\\
&\le a_\beta({}_*P^1_{\mathbf{N},N_b,\Omega}U - W, {}_*P^1_{\mathbf{N},N_b,\Omega}U - W)\\
&\le 2a_\beta({}_*P^1_{\mathbf{N},N_b,\Omega}U - U, {}_*P^1_{\mathbf{N},N_b,\Omega}U - U) + 2a_\beta(U - W, U - W).
\end{aligned} \tag{5.9}$$

Finally, a combination of (5.1), (5.5) and (5.9) leads to the following conclusion.

Theorem 5.1 *If the hypotheses* (H$_1$)–(H$_5$) *hold, $U \in H^1(\Omega)\cap C(\overline{\Omega})$, and all $B^{(j)}_{r_i,\Omega_i}(v)$, $D_{r_b,\Omega_i}(v)$ and $K_{r_b,\Omega_i}(v)$ are finite for integers $2 \le r_i \le N_i + 1$, $1 \le i \le n$ and $2 \le r_b \le N_b + 1$, then*

$$\begin{aligned}
&\|U - w_{\mathbf{N},N_b}\|_{H^1(\Omega)}\\
&\le c\sum_{i=1}^n \sigma_{\Omega_i}\delta^{-1}_{\Omega_i}\Big(N_i^{1-r_i}\sum_{j=1}^4 B^{(j)}_{r_i,\Omega_i}(U) + N_b^{1-r_b}D_{r_b,\Omega_i}(U) + N_b^{\frac12 - r_b}K_{r_b,\partial^*\Omega_i}(g_1)\Big).
\end{aligned} \tag{5.10}$$

Remark 5.1 If all elements are rectangles, then

$$\begin{aligned}
&\|U - w_{\mathbf{N},N_b}\|_{H^1(\Omega)}\\
&\le c\sum_{i=1}^n \sigma_{\Omega_i}\delta^{-1}_{\Omega_i}\Big(N_i^{1-r_i}\sum_{j=1}^3 B^{(*,j)}_{r_i,\Omega_i}(U) + N_b^{1-r_b}D^{(*)}_{r_b,\Omega_i}(U) + N_b^{\frac12 - r_b}K_{r_b,\partial^*\Omega_i}(g_1)\Big).
\end{aligned} \tag{5.11}$$

Hereafter, $D^{(*)}_{r_b,\Omega_i}(v) = B^{(*,3)}_{r_b,\Omega_i}(v)$, as long as $\partial^{**}\Omega_i = \emptyset$. If $\partial^{**}\Omega_i = L_{i,1}\cup L_{i,2}$, then

$$D^{(*)}_{r_b,\Omega_i}(v) = (\|(a^2 - x^2)^{\frac{r-1}{2}}\partial_x^r v\|^2_{L^2(L_{i,4})} + \|(b^2 - y^2)^{\frac{r-1}{2}}\partial_y^r v\|^2_{L^2(L_{i,3})})^{\frac12}.$$

If $\partial^{**}\Omega_i = L_{i,1}$, then

$$D^{(*)}_{r_b,\Omega_i}(v) = (\|(a^2-x^2)^{\frac{r-1}{2}}\partial_x^r v\|^2_{L^2(L_{i,2}\cup L_{i,4})} + \|(b^2-y^2)^{\frac{r-1}{2}}\partial_y^r v\|^2_{L^2(L_{i,3})})^{\frac{1}{2}}.$$

Remark 5.2 We have $\max(|a_{i,3}|,|b_{i,3}|,|\sigma_{\Omega_i}|) \le ch_i$. Therefore, the result (5.10) implies

$$\begin{aligned}\|U - w_{\mathbf{N},N_b}\|_{H^1(\Omega)} \le\ & c\sum_{i=1}^{n} N_i^{1-r_i}\Big(\sum_{k=0}^{r_i} h_i^{r_i-1}\|\partial_x^k\partial_y^{r_i-k}U\|_\Omega + \sum_{k=0}^{r_i-1} h_i^{r_i-2}\|\partial_x^k\partial_y^{r_i-k}U\|_\Omega \\ & + \sum_{\nu=1}^{4}\sum_{k=0}^{r_i} h_i^{r_i-1}\|\partial_x^k\partial_y^{r_i-k}U\|_{L^2(L_{i,\nu})}\Big) \\ & + cN_b^{1-r_b}\sum_{i=1}^{n}\sum_{\nu=1}^{4}\sum_{k=0}^{r_b} h_i^{r_b-1}(\|\partial_x^k\partial_y^{r_b-k}U\|_{L^2(L^*_{i,\nu})} \\ & + cN_b^{-\frac{1}{2}}\|\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_{i,\nu}\cap\partial^*\Omega)}).\end{aligned} \tag{5.12}$$

Hereafter, $L^*_{i,\nu} = \emptyset$ if $L^*_{i,\nu} \subseteq \partial^{**}\Omega$. Otherwise $L^*_{i,\nu} = L_{i,\nu}$. The above result is similar to [5, (5.4.16)] and the corresponding result of [17] for multi-domain pseudospectral method of a special problem.

Remark 5.3 If all elements are rectangles, then

$$\begin{aligned}\|U - w_{\mathbf{N},N_b}\|_{H^1(\Omega)} \le\ & c\sum_{i=1}^{n}\sum_{k=0}^{r_i} N_i^{1-r_i}h_i^{r_i-1}\Big(\|\partial_x^k\partial_y^{r_i-k}U\|_\Omega + \sum_{\nu=1}^{4}\|\partial_x^k\partial_y^{r_i-k}U\|_{L^2(L_{i,\nu})}\Big) \\ & + cN_b^{1-r_b}\sum_{i=1}^{n}\sum_{\nu=1}^{4}\sum_{k=0}^{r_b} h_i^{r_b-1}(\|\partial_x^k\partial_y^{r_b-k}U\|_{L^2(L^*_{i,\nu})} \\ & + N_b^{-\frac{1}{2}}\|\partial_x^k\partial_y^{r_b-k}g_1\|_{L^2(L_{i,\nu}\cap\partial^*\Omega)}).\end{aligned} \tag{5.13}$$

Since the error estimate (5.13) does not contain the term of order $N_i^{1-r_i}h_i^{r_i-2}$, which appears in (5.12) for the general case, our new method with rectangular elements provides the global super-convergence automatically. For example, if $h_i = h$, $N_i = N_b = N$ and $U \in H^{N+1}(\Omega)$, then $\|U - w_{\mathbf{N},N_b}\|_{H^1(\Omega)} \le c(\frac{h}{N})^N\|U\|_{H^{N+1}(\Omega)}$.

Remark 5.4 We could regard the above suggested spectral element method as a new $h-p$ version. However, there exist several differences between them. Firstly, unlike the finite element method, we use spectral approximation on each element. Next, there are some Jacobi weights in the piece-wise norms involved in the error estimates, which cover certain weak singularity of solution. Thirdly, our new method with rectangular elements provides the global super-convergence automatically.

Remark 5.5 The Petrov-Galerkin spectral element was also discussed in [10]. Whereas, the Robin boundary condition was not considered in that paper. Next, one supposed in [10] that each element has at most one edge belonging to $\partial^{**}\Omega$. This may bring some difficulties in numerical process at the corners of polygons. By the way, the first result of (3.19) was used in [10], but without the proof.

6 Concluding Remarks

In this paper, we develop the Petrov-Galerkin spectral element method for polygons. We establish the basic results on the Legendre irrational quasi-orthogonal approximation, which still keeps the spectral accuracy, even if the considered function has certain weak singularity. These results play important roles in Petrov-Galerkin spectral and spectral element methods for mixed inhomogeneous boundary value problems of partial differential equations defined on polygons. As examples of applications, we provide the Petrov-Galerkin spectral element schemes for two mixed inhomogeneous boundary value problems, with the spectral accuracy. It is also demonstrated that if all elements are rectangular, then the global super-convergence follows automatically. The approximation results and techniques of this work are also applicable to many other problems with complex geometry, as well as exterior problems with non-rectangular obstacles.

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