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Some Abstract Critical Point Theorems for Self-adjoint Operator Equations and Applications*

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Abstract By using the index theory for linear bounded self-adjoint operators in a Hilbert space related to a fixed self-adjoint operator A with compact resolvent, the authors discuss the existence and multiplicity of solutions for (nonlinear) operator equations, and give some applications to some boundary value problems of first order Hamiltonian systems and second order Hamiltonian systems.

Keywords Self-adjoint operator equations, Index theory, Relative Morse index, Dual variational method, Morse theory, Hamiltonian systems
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1 Introduction and Main Results

Let X be an infinite-dimensional separable Hilbert space with inner product $(\cdot,\cdot)$, and norm $\|\cdot\|$. Let $Y \subset X$ be a Banach space with norm $\|\cdot\|_Y$, and the embedding $Y \hookrightarrow X$ is compact. Let $A : Y \to X$ be continuous, self-adjoint, i.e., $(Ax, y) = (x, Ay)$ for any $x, y \in Y$ with the inner product of X, $\mathrm{Im}(A)$ is a closed subspace of X and $\mathrm{Im}(A) \oplus \ker(A) = X$. In this paper, by an index theory of the following linear operator equation

$$Ax + Bx = 0, \tag{1.1}$$

we consider the existence and multiplicity of solutions of the following nonlinear operator equation:

$$Ax + \Phi'(x) = 0, \tag{1.2}$$

where $B \in \mathcal{L}_s(X)$ (the set of bounded self-adjoint operator), and $\Phi : X \to R$ is differentiable.

In 1980, Amann and Zehnder [1] discussed equation (1.2) under the assumption that $A : \mathrm{dom}(A) \subseteq X \to X$ is a unbounded self-adjoint operator. By the saddle point reduction methods, they obtained some existence results for nontrivial solutions. They also discussed semilinear elliptic boundary value problems, periodic solutions of semilinear wave equations, and periodic solutions of Hamiltonian systems as special cases of the abstract equation. In 1981, Chang [3] extended their results by a simpler and unified approach. Especially, Chang obtained an existence result yielding three distinct solutions. Chang [4] also discussed equation (1.2) by assuming that $A \in \mathcal{L}_s(X)$ has a finite Morse index and Φ' is compact. This framework can

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be used to discuss elliptic partial differential equations. In 1990, Ekeland [8] discussed (1.2) by the dual variational methods and convex analysis theory. He assumed that $A : X \to X^*$ is closed and self-adjoint. As applications, he mainly focussed on second order and first order Hamiltonian systems satisfying various boundary conditions.

As far as authors know, an index theory for convex linear Hamiltonian systems was established first by Ekeland [9] in 1984. By the works of Conley, Zehnder and Long [5, 20, 21, 22], an index theory for symplectic paths was introduced. These index theories have important applications (see e.g., [7, 10, 11, 16, 23]). In [26, 24], Long and Zhu defined spectral flows for paths of linear operators and relative Morse index, and redefined Maslov index for symplectic paths. In the study of the L-solutions (the solutions starting and ending at the same Lagrangian subspace L) of Hamiltonian systems, the first author of this paper introduced in [15] an index theory for symplectic paths using the algebraic methods and given some applications in [14, 15]. And this index was generalized by the authors of this paper and Lin in [17]. As in paper [6], we introduce the index $(i_A(B), \nu_A(B))$ for equation (1.1). With this index, we receive the existence and multiplicity of solutions for (1.2). As applications, we consider the existence and multiplicity of solutions for first order Hamiltonian systems and second order Hamiltonian systems. First we give a brief introduction to the index theory, for some details, we refer to [6].

Definition 1.1 (see [6, Definition 3.1.1]) *For any $B \in \mathcal{L}_s(X)$, we define*

$$\nu_A(B) = \dim\ker(A + B).$$

It was proved in [6] that the nullity $\nu_A(B)$ is finite.

Definition 1.2 (see [6, 14]) *For any $B_1, B_2 \in \mathcal{L}_s(X)$ with $B_1 < B_2$, we define*

$$I_A(B_1, B_2) = \sum_{\lambda \in [0,1)} \nu_A((1-\lambda)B_1 + \lambda B_2);$$

and for any $B_1, B_2 \in \mathcal{L}_s(X)$, we define

$$I_A(B_1, B_2) = I_A(B_1, K \cdot \mathrm{Id}) - I_A(B_2, K \cdot \mathrm{Id}),$$

where $\mathrm{Id} : X \to X$ *is the identity map and* $K \cdot \mathrm{Id} > B_1, K \cdot \mathrm{Id} > B_2$ *for some real number* $K > 0$.

Let $0 \in \mathcal{L}_s(X)$ be the zero operator. We give the following definition for related index.

Definition 1.3 *For any $B \in \mathcal{L}_s(X)$, we define*

$$i_A(B) = I_A(0, B).$$

We call $i_A(B)$ index of B related to A. If $A = -J\frac{\mathrm{d}}{\mathrm{d}t}$, $B = B(t)$ is a symmetric continuous matrix function and $Y = W_L = \{z = (x,y)^{\mathrm{T}} \in W^{1,2}([0,1], \mathbb{R}^{2n}) \mid z(0), z(1) \in L\}$, $X = L^2([0,1], \mathbb{R}^{2n})$, in [14] it was proved that $I_A(0, B)$ is the index $i_L(B)$ up to a constant when considering the L-boundary value problems for some Lagrangian subspace $L \subset \mathbb{R}^{2n}$.

By [6, Proposition 3.1.5], we have the following result.

Proposition 1.1 *The following statements hold:*
(1) *For any $B_1, B_2 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2)$ and $i_A(B)$ are well-defined and finite;*
(2) *For any $B_1, B_2, B_3 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2) + I_A(B_2, B_3) = I_A(B_1, B_3)$;*
(3) *For any $B_1, B_2 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2) = i_A(B_2) - i_A(B_1)$;*
(4) *For any $B_1, B_2 \in \mathcal{L}_s(X)$ with $B_1 < B_2$, $\nu_A(B_1) + i_A(B_1) \le i_A(B_2)$.*

Next for any given $\widehat{B} \in \mathcal{L}_s(X)$, with $\nu_A(\widehat{B}) = 0$, the operator $\Lambda := (A + \widehat{B}) : Y \to X$ is invertible and the inverse $\Lambda^{-1} : X \to X$ is compact. For any $B \in \mathcal{L}_s(X)$ with $B - \widehat{B} \geq \epsilon \cdot \mathrm{Id}$ for some constant $\epsilon > 0$, we define a bilinear form:

$$\phi_{A,B|\widehat{B}}(x, y) = (\Lambda^{-1}x, y) + ((B - \widehat{B})^{-1}x, y), \quad \forall\, x, y \in X. \tag{1.3}$$

We have

$$X = E_A^+(B \mid \widehat{B}) \oplus E_A^0(B \mid \widehat{B}) \oplus E_A^-(B \mid \widehat{B}), \tag{1.4}$$

such that $\phi_{A,B|\widehat{B}}$ is positive definite, null and negative definite on $E_A^+(B \mid \widehat{B})$, $E_A^0(B \mid \widehat{B})$ and $E_A^-(B \mid \widehat{B})$ respectively. Moreover, $E_A^0(B \mid \widehat{B})$ and $E_A^-(B \mid \widehat{B})$ are finitely dimensional.

Definition 1.4 (see [6, Definition 3.2.3]) *For any $B \in \mathcal{L}_s(X)$ with $B - \widehat{B} \geq \epsilon \cdot \mathrm{Id}$ for some constant $\epsilon > 0$, we define*

$$i_A(B \mid \widehat{B}) = \dim E_A^-(B \mid \widehat{B}), \quad \nu_A(B \mid \widehat{B}) = \dim E_A^0(B \mid \widehat{B}).$$

This relative index is a kind of Morse index. It plays an important role in the relationship between Morse Theory and the index $(i_A(B), \nu_A(B))$.

Theorem 1.1 (see [6, Theorem 3.2.4]) *We have the following statements:*
(1) *For any $B > \widehat{B}$, we have*

$$\nu_A(B \mid \widehat{B}) = \nu_A(B).$$

(2) *Assume $B_2 > B_1 > \widehat{B}$. Then*

$$i_A(B_2|\widehat{B}) \geq i_A(B_1 \mid \widehat{B}) + \nu_A(B_1 \mid \widehat{B}).$$

(3) *Assume $B_2, B_1 > \widehat{B}$. Then*

$$i_A(B_2|\widehat{B}) - i_A(B_1 \mid \widehat{B}) = I_A(B_1, B_2) = i_A(B_2) - i_A(B_1).$$

Now we use the index $(i_A(B), \nu_A(B))$ to reach our main results.

Theorem 1.2 *Assume that $\Phi \in C^2(X, \mathbb{R})$ satisfies*
(P) *There exists $C > 0$ and $M \in \mathbb{R}$ such that*

$$\|\Phi'(z)\| < C\|z\|, \quad \Phi''(z) \geq M \cdot \mathrm{Id}, \quad \forall z \in X.$$

(P_0) $\Phi'(\theta) = \theta$, $\Phi''(\theta) = B_0 \in \mathcal{L}_s(X)$ *and* $\nu_A(B_0) = 0$.
(P_∞) *There exist a $B_\infty \in \mathcal{L}_s(X)$ with $\nu_A(B_\infty) = 0$ and $K > 0$, such that*

$$\Phi''(z) \geq B_\infty, \quad \|z\| \geq K. \tag{1.5}$$

(P_t) $i_A(B_\infty) > i_A(B_0) + 1$.
Then (1.2) *has at least one nontrivial solution.*

Remark 1.1 In the condition (P_∞), the requirement $\nu_A(B_\infty) = 0$ is not essential since if $\nu_A(B_\infty) \neq 0$, we can perturb the operator B_∞ slightly to the operator $\widetilde{B}_\infty$ such that $\nu_A(\widetilde{B}_\infty) = 0$, $i_A(B_\infty) = i_A(\widetilde{B}_\infty)$ and $\Phi''(z) \geq \widetilde{B}_\infty$, $\|z\| \geq K$. Up to the authors known, some similar conditions as (1.5) in (P_∞) were introduced in [14, 18].

Theorem 1.3 *Assume that the conditions in Theorem* 1.2 *are all satisfied and further more* Φ *is even, then* (1.2) *has at least* $i_A(B_\infty) - i_A(B_0) - 1$ *pairs of nontrivial solutions.*

Remark 1.2 Comparing with [6, Theorem 3.1.7], the functional Φ is more restricted at infinity than that in our Theorem 1.2, where it is required essentially that $\Phi''(x)$ is pinched by two linear self-adjoint bounded operators B_1 and B_2, that is $B_1 \leq \Phi''(x) \leq B_2$, and with the conditions $i_A(B_1) + \nu_A(B_1) = i_A(B_2) + \nu_A(B_2)$, $\nu_A(B_2) = 0$. Namely, in [6, Theorem 3.1.7], the functional Φ behaves as a quadratic functional at infinity but in our Theorem 1.2 it is only required that the functional Φ is estimated from below by a quadratic functional.

2 Proofs of the Main Results

The following lemma is similar to [18, Lemma 3.3].

Lemma 2.1 *Assume that* $\Phi \in C^2(X, \mathbb{R})$ *satisfies* (P), (P$_\infty$). *Then there exists a sequence of functions* $\Phi_m \in C^2(X, \mathbb{R})$, $m \in \mathbb{N}$ *satisfying*

(1) *there exists an increasing sequence of real numbers* $R_m \to \infty$ $(m \to \infty)$ *such that*

$$\Phi_m(z) \equiv \Phi(z), \quad \forall \|z\| \leq R_m, \tag{2.1}$$

(2) *for each* $m \in \mathbb{N}$,

$$\Phi''_m(z) \geq B_\infty, \quad \forall \|z\| \geq K, \tag{2.2}$$

(3) *there exists* $\widetilde{C} > 0$ *and* $\widetilde{M} \in \mathbb{R}$, *such that*

$$\|\Phi'_m(z)\| < \widetilde{C}\|z\|, \quad \Phi''_m \geq \widetilde{M} \cdot \mathrm{Id}, \quad \forall\, z \in X,\ m \in \mathbb{N}, \tag{2.3}$$

(4) *there exists* γ, *satisfying* $\gamma \cdot \mathrm{Id} > B_\infty$, $\nu_A(\gamma \cdot \mathrm{Id}) = 0$ *and* $C_m > 0$, *such that*

$$\|\Phi'_m(z) - \gamma z\| < C_m. \tag{2.4}$$

Proof Choose a sequence $\{R_m\}$ of positive numbers such that $K < R_1 < R_2 < \cdots < R_m < \cdots \to \infty$, $m \to \infty$. For each $m \in \mathbb{N}$, define $\phi_m : [R_m, 2R_m] \to R$ as

$$\phi_m(s) = \frac{2}{9R_m^3}(s - R_m)^3 - \frac{1}{9R_m^4}(s - R_m)^4, \quad s \in [R_m, 2R_m]. \tag{2.5}$$

Then define the function

$$\psi_m(s) = 1 - \frac{128R_m^2}{9(12R_m^2 + s^2)}. \tag{2.6}$$

Now for each $m \in \mathbb{N}$, define the function

$$\eta_m(s) = \begin{cases} 0, & 0 \leq s \leq R_m, \\ \phi_m(s), & R_m \leq s \leq 2R_m, \\ \psi_m(s), & 2R_m \leq s \leq \infty. \end{cases} \tag{2.7}$$

Then define Φ_m by

$$\Phi_m(z) = (1 - \eta_m(\|z\|))\Phi(z) + \frac{\gamma}{2}\eta_m(\|z\|)\|z\|^2, \quad m \in \mathbb{N}, \tag{2.8}$$

which satisfies the properties (1)–(4). In fact, we can get the statements (2.1), (2.3), (2.4) by direct computations. In order to check (2.2), we show

$$(\Phi_m''(z)x, x) \geq (B_\infty x, x) \tag{2.9}$$

for all $x \in X$, with $\|z\| \geq K$. The proof is the same as that of Lemma 3.4 in [19]. The only difference is that in [19, Lemma 3.4], it deals with the finite dimensional case, but here we deal with the infinite dimensional case. Since where involved only formal computations, all the estimates are still valid, the proof carries over verbatim.

Then we choose $\alpha \in \mathbb{R}$, with $-\alpha$ large enough, such that

$$\nu_A(\alpha \cdot \mathrm{Id}) = 0, \tag{2.10}$$

$$B_\infty - \alpha \cdot \mathrm{Id} \geq \mathrm{Id}, \quad \widetilde{M} - \alpha > 1, \tag{2.11}$$

$$N_m''(z) \geq \mathrm{Id}, \quad \forall\, z \in X,\ m \in \mathbb{N}, \tag{2.12}$$

$$N_m''(z) \geq B_\infty - \alpha \cdot \mathrm{Id}, \quad \|z\| \geq K, \quad m \in \mathbb{N}, \tag{2.13}$$

where $N_m(z) = \Phi_m(z) - \frac{\alpha}{2}(z,z)$, $m \in \mathbb{N}$. Let $N_\infty(z) = \frac{1}{2}((B_\infty - \alpha\cdot\mathrm{Id})z, z)$, $\widetilde{N}_\gamma = \frac{1}{2}(\gamma - \alpha)(z,z)$, $N(z) = \Phi(z) - \frac{\alpha}{2}(z,z)$. We have $N_m, N_\infty,\ \widetilde{N}_\gamma \in C^2(X,\mathbb{R})$, and $N_m''(z), N_\infty''(z), \widetilde{N}_\gamma''(z) \geq \mathrm{Id}$, $\forall z \in X$. Define

$$\Lambda z = Az + \alpha z, \tag{2.14}$$

$$\Psi_m(z) = \frac{1}{2}(\Lambda^{-1}z, z) + N_m^*(z), \quad m \in \mathbb{N}, \tag{2.15}$$

$$\widetilde{\Psi}_\gamma(z) = \frac{1}{2}(\Lambda^{-1}z, z) + \widetilde{N}_\gamma^*(z), \tag{2.16}$$

$$\Psi_\infty(z) = \frac{1}{2}(\Lambda^{-1}z, z) + N_\infty^*(z), \tag{2.17}$$

where $N_m^*,\ \widetilde{N}_\gamma^*$ and N_∞^* are the Fenchel dual of $N_m,\ \widetilde{N}_\gamma$ and N_∞ (see [8] for the definition and properties). We know $\Psi_m, \Psi_\infty, \widetilde{\Psi}_\gamma \in C^2(X,\mathbb{R})$.

Lemma 2.2 *For any $m \in \mathbb{N}$, there is a $\widetilde{C}_m$, such that $\|N_m^{*'}(z) - \widetilde{N}_\gamma^{*''}(0)z\| \leq \widetilde{C}_m$, $\forall m \in \mathbb{N}, z \in X$.*

Proof Otherwise, there are $\{z_n\} \subset X$, such that $N_m^{*'}(z_n) - \widetilde{N}_\gamma^{*''}(0)z_n = y_n$, and $\|y_n\| \to \infty$ $(n \to \infty)$. That is

$$N_m^{*'}(z_n) = \widetilde{N}_\gamma^{*''}(0)z_n + y_n = (\gamma - \alpha)^{-1}z_n + y_n, \tag{2.18}$$

$$N_m'((\gamma - \alpha)^{-1}z_n + y_n) = z_n, \tag{2.19}$$

and from the definition of N_m, we have

$$\Phi_m'((\gamma-\alpha)^{-1}z_n + y_n) - \alpha((\gamma-\alpha)^{-1}z_n + y_n) = z_n, \tag{2.20}$$

and

$$\Phi_m'((\gamma-\alpha)^{-1}z_n + y_n) - \gamma((\gamma-\alpha)^{-1}z_n + y_n) = (\alpha - \gamma)y_n, \tag{2.21}$$

but from the proposition (4) in Lemma 2.1, the left-hand side is bounded. This is a contradiction to the fact that y_n are unbounded.

Lemma 2.3 *For any $m \in \mathbb{N}$, Ψ_m satisfies the* (PS) *condition, and the critical-point set $\mathcal{K}_m = \{z \in X \mid \Psi'_m(z) = 0\}$ is compact set.*

Proof For any $m \in \mathbb{N}$, assume $\{z_n\} \subset X$, and $\Psi'_m(z_n) \to 0$. From Lemma 2.2, we have

$$\|\Psi'_m(z_n) - \widetilde{\Psi}''_\gamma(0)z_n\| = \|N^{*'}_m(z_n) - \widetilde{N}^{*''}_\gamma(0)z_n\| \leq \widetilde{C}_m. \tag{2.22}$$

And since $\nu_A(\gamma \cdot \mathrm{Id}) = 0$, we have that $\widetilde{\Psi}''_\gamma(0)$ has bounded inverse, so $\{z_n\}$ are bounded. Then there exists a subsequence $z_{n_k} \rightharpoonup z_0$ in X, and $\Lambda^{-1}z_{n_k} \to \Lambda^{-1}z_0$ in X. From the definition of Ψ_m, we have

$$\Lambda^{-1}z_{n_k} + N^{*'}_m(z_{n_k}) = \Psi'_m(z_{n_k}), \tag{2.23}$$

and

$$N^{*'}_m(z_{n_k}) = \Psi'_m(z_{n_k}) - \Lambda^{-1}z_{n_k}, \tag{2.24}$$

so $z_{n_k} = N'_m(\Psi'_m(z_{n_k}) - \Lambda^{-1}z_{n_k}) \to N'_m(-\Lambda^{-1}z_0)$, $n_k \to \infty$. The (PS) condition is satisfied. From the similar reason, we have that $\mathcal{K}_m$ is a compact set.

Because $\nu_A(\gamma \cdot \mathrm{Id}) = 0$, we have $X = E^-_\gamma \oplus E^+_\gamma$, where $\widetilde{\Psi}_\gamma$ is negative definite on E^-_γ and positive definite on E^+_γ, and $\dim(E^-_\gamma) = i_A(\gamma \cdot \mathrm{Id} \mid \alpha \cdot \mathrm{Id})$. Similarly to Lemma II.5.1 in [2], we have the following lemma.

Lemma 2.4 *For any $m \in \mathbb{N}$, there is an $a_m \in \mathbb{R}$ with $-a_m$ large enough, such that*

$$H_q(X, (\Psi_m)_{a_m}; \mathbb{R}) = \delta_{qr}\mathbb{R},$$

where $r = \dim(E^-_\gamma) = i_A(\gamma \cdot \mathrm{Id} \mid \alpha \cdot \mathrm{Id})$.

Proof Since $\nu_A(\gamma) = 0$, 0 is a non-degenerate critical point of $\widetilde{\Psi}_\gamma$, we have

$$X = E^-_\gamma \oplus E^+_\gamma, \tag{2.25}$$

such that there is a $c_\gamma > 0$, satisfying

$$\widetilde{\Psi}''_\gamma(0)|_{E^-_\gamma} \leq -c_\gamma \cdot \mathrm{Id}, \quad \text{and} \quad \widetilde{\Psi}''_\gamma(0)|_{E^+_\gamma} \geq c_\gamma \cdot \mathrm{Id}. \tag{2.26}$$

From Lemma 2.2, we have

$$\|\Psi'_m(z) - \widetilde{\Psi}''_\gamma(0)z\| = \|N^{*'}_m(z) - (\gamma - \alpha)^{-1}z\| \leq \widetilde{C}_m, \quad \forall\, m \in \mathbb{N},\ z \in X. \tag{2.27}$$

Let $R^+_m > \widetilde{C}_m \setminus c_\gamma$. Then if $z^+ \in E^+_\gamma$ and $\|z^+\| \geq R^+_m$, we have

$$\begin{aligned}\langle \Psi'_m(z), z^+\rangle &= \langle \widetilde{\Psi}''_\gamma(0)z^+, z^+\rangle + \langle(\Psi'_m(z) - \widetilde{\Psi}''_\gamma(0)z), z^+\rangle \\ &\geq c_\gamma\|z^+\|^2 - \widetilde{C}_m\|z^+\| > 0.\end{aligned} \tag{2.28}$$

Let $\mathcal{M} = (E^+_\gamma \cap B_{R^+_m}) \oplus E^-_\gamma$. We have that Ψ_m has no critical point outside $\mathcal{M}$, and that $-\Psi'(z)$ points inward to $\mathcal{M}$ on $\partial\mathcal{M}$. Further more, we have

$$\begin{aligned}\Psi_m(z) &= \Psi_m(0) + \int_0^1 \langle \Psi'_m(tz), z\rangle \mathrm{d}t \\ &= \Psi_m(0) + \int_0^1 \langle \Psi'_m(tz) - \widetilde{\Psi}''_\gamma(0)tz, z\rangle \mathrm{d}t + \int_0^1 \langle \widetilde{\Psi}''_\gamma(0)tz, z\rangle \mathrm{d}t.\end{aligned} \tag{2.29}$$

That is

$$\begin{aligned}&\Psi_m(0) - \widetilde{C}_m\|z\| - \frac{1}{2}\|\widetilde{\Psi}''_\gamma(0)\|\|z^-\|^2 \\ &\leq \Psi_m(z) \leq \Psi_m(0) + \widetilde{C}_m\|z\| - \frac{c_\gamma}{2}\|z^-\|^2 + \frac{1}{2}\|\widetilde{\Psi}''_\gamma(0)\|\|z^+\|^2. \end{aligned}\tag{2.30}$$

We obtain

$$\Psi_m(z) \to -\infty \Leftrightarrow \|z^-\| \to \infty, \quad \text{uniformly in } \ z^+ \in E_\gamma^+ \cap B_{R_m^+}. \tag{2.31}$$

Thus, $\forall T > 0,\ \exists a'_m < a_m < -T,\ r_1 > r_2 > 0$ such that

$$(E_\gamma^+ \cap B_{R_m^+}) \oplus (E_\gamma^- \setminus B_{r_1}) \subset (\Psi_m)_{a'_m} \cap \mathcal{M} \subset (E_\gamma^+ \cap B_{R_m^+}) \oplus (E_\gamma^- \setminus B_{r_2}) \subset (\Psi_m)_{a_m} \cap \mathcal{M}. \tag{2.32}$$

And from Lemma 2.3 we choose T large enough such that $\mathcal{K}_m \cap (\Psi_m)_{-T} = \emptyset$. The negative gradient flow of Ψ_m defines a strong deformation retract

$$\tau_1 : (\Psi_m)_{a_m} \cap \mathcal{M} \to (\Psi_m)_{a'_m} \cap \mathcal{M}. \tag{2.33}$$

Another strong deformation retract in $(\Psi_m)_{a_m} \cap \mathcal{M}$

$$\tau_2 : (E_\gamma^+ \cap B_{R_m^+}) \oplus (E_\gamma^- \setminus\ B_{r_2}) \to (E_\gamma^+ \cap B_{R_m^+}) \oplus (E_\gamma^- \setminus\ B_{r_1}) \tag{2.34}$$

is defined by $\tau_2 = \xi(1, \cdot)$, where

$$\xi(t; z^+ + z^-) = \begin{cases} z^+ + z^-, & \|z^-\| \geq r_1, \\ z^+ + \dfrac{z^-}{\|z^-\|}(tr_1 + (1-t)\|z^-\|), & \|z^-\| \leq r_1. \end{cases} \tag{2.35}$$

We compose these two strong deformation retracts, $\tau = \tau_2 \circ \tau_1$, and then obtain a strong deformation retract

$$\tau : (\Psi_m)_{a_m} \cap \mathcal{M} \to (E_\gamma^+ \cap B_{R_m^+}) \oplus E_\gamma^- \setminus\ B_{r_1}, \tag{2.36}$$

and the following deformation

$$\eta(t; z^+ + z^-) = \begin{cases} z^+ + z^-, & \|z^+\| \leq R_m^+, \\ z^- + \dfrac{z^+}{\|z^+\|}(tR_m^+ + (1-t)\|z^+\|), & \|z^-\| \geq R_m^+ \end{cases} \tag{2.37}$$

is a strong deformation retract of the topological pair from $(X, (\Psi_m)_{a_m})$ to $(\mathcal{M}, \mathcal{M} \cap (\Psi_m)_{a_m})$. Finally, we have

$$\begin{aligned} H_q(X, (\Psi_m)_{a_m}) &\cong H_q(\mathcal{M}, \mathcal{M} \cap (\Psi_m)_{a_m}) \\ &\cong H_q((E_\gamma^+ \cap B_{R_m^+}) \oplus E_\gamma^-, (E_\gamma^+ \cap B_{R_m^+}) \oplus E_\gamma^- \setminus B_{r_1}) \\ &\cong H_q(E_\gamma^-, E_\gamma^- \setminus B_{r_1}) \\ &\cong H_q(E_\gamma^- \cap B_{r_1}, \partial(E_\gamma^- \cap B_{r_1})) \\ &\cong \delta_{qr}\mathbb{R}. \end{aligned}$$

Let $\mathcal{K}_m^* = \mathcal{K}_m \setminus \{\theta\}$. From Definition 1.4 and Theorem 1.1, we have that θ is an isolate critical point of Ψ_m. And since $\mathcal{K}_m$ is compact for every $m \in \mathbb{N}$, we have $\mathcal{K}_m^*$ is also compact. Then we have the next lemma.

Lemma 2.5 *For any $\varepsilon, \mu > 0$ small enough there exists a functional $\widehat{\Psi}_m$, such that*
(1) $\|\Psi_m - \widehat{\Psi}_m\|_{C^2} < \epsilon$,
(2) $\Psi(z) = \widehat{\Psi}_m$, $z \notin N_{2\mu}(\mathcal{K}_m^*)$,
(3) $\Psi''_m(z) = \widehat{\Psi}''_m(z)$, $z \in N_\mu(\mathcal{K}_m^*)$,
where $N_\mu(\mathcal{K}_m^*) = \{z \in X \mid \mathrm{dist}(z, \mathcal{K}_m)^* < \mu\}$. Moreover, $\widehat{\Psi}_m$ satisfies the (PS) *condition and has only a finite number of critical points. All nontrivial critical points of $\widehat{\Psi}_m$ are in $N_\mu(\mathcal{K}_m^*)$ and are non-degenerate.*

Proof We follow the idea of [25]. Since $\mathcal{K}_m^*$ is a compact subset of X, we have the following result. For every $\mu > 0$, there exists a C^∞ function $l : X \to [0,1]$, with all its derivatives bounded and

$$l(z) = 1, \quad \forall\, z \in N_\mu(\mathcal{K}_m^*), \tag{2.38}$$
$$l(z) = 0, \quad \forall\, z \in X \setminus N_{2\mu}(\mathcal{K}_m^*). \tag{2.39}$$

Let $M = \sup\limits_{z \in N_{2\mu}(\widehat{\mathcal{K}}_m^*)} \{\|z\|\}$, $C = \|l(z)\|_{C^2}$, $\delta = \inf\limits_{z \in N_{2\mu}(\mathcal{K}_m^*) \setminus N_\mu(\mathcal{K}_m^*)} \{\|\Psi'_m(z)\|\} > 0$. We use the Sard-Smale Theorem to find $y \in X$ such that $\|y\| < \min\{\frac{\varepsilon}{C(2+2M)}, \frac{\delta}{2C(1+2M)}\}$, and $-y$ is a regular value for Ψ'_m. For any $z_0 \in N_{2\mu}(\mathcal{K}_m^*)$, the functional is defined by

$$\widehat{\Psi}_m(z) = \Psi_m(z) + l(z)\langle y, z - z_0\rangle. \tag{2.40}$$

By $\|y\| < \frac{\varepsilon}{C(2+2M)}$ and the definition of $l(z)$, we have (1) and (2), (3). Since $\|y\| < \frac{\delta}{2C(1+2M)}$ and $-y$ is a regular value for Ψ_m, we have that all nontrivial critical points of $\widehat{\Psi}_m$ are in $N_\mu(\mathcal{K}_m^*)$ and are non-degenerate.

In order to prove that $\widehat{\Psi}_m$ satisfies the (PS) condition, assume that there are $\{z_n\} \subset X$ and $\widehat{\Psi}'_m(z_n) \to 0$ $(n \to \infty)$. From the definition of $\widehat{\Psi}_m$, we have $\|\widehat{\Psi}'_m(z)\| > \frac{\delta}{2}$, $\forall z \in N_{2\mu}(\mathcal{K}_m^*) \setminus N_\mu(\mathcal{K}_m^*)$. So $z_n \in (X \setminus N_{2\mu}(\mathcal{K}_m^*)) \cup N_\mu(\mathcal{K}_m^*)$, when n is large enough. From the proposition of $\widehat{\Psi}_m$ and the proof in Lemma 2.3, we have that $\widehat{\Psi}_m$ satisfies the (PS) condition. So it has finitely many critical points.

Proof of Theorem 1.2 We divide the proof into two steps and follow the ideas of [18].

Step 1 Note that $z = 0$ is a critical point of Ψ_m. The Morse index of 0 for Ψ_m is $i_A(B_0 \mid \alpha \cdot \mathrm{Id})$. Since $\gamma \cdot \mathrm{Id} > B_\infty$, we have

$$i_A(\gamma \cdot \mathrm{Id} \mid \alpha \cdot \mathrm{Id}) \geq i_A(B_\infty \mid \alpha \cdot \mathrm{Id}) > i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1. \tag{2.41}$$

Now we claim that Ψ_m has a nontrivial critical point z_m with its Morse index satisfying

$$m^-(z_m) \leq i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1. \tag{2.42}$$

If Ψ_m has only finite critical points, we use the $(i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1)^{\mathrm{th}}$ Morse inequality:

$$\sum_{p=0}^{q} (-1)^{q-p} M_p(a_m, b_m, \Psi_m) \geq \sum_{p=0}^{q} (-1)^{q-p} \beta_p(a_m, b_m, \Psi_m), \tag{2.43}$$

where $q = i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1$, and b_m is large enough such that $\mathcal{K}_m \subset \Psi_m^{-1}[a_m, b_m]$.

Because Ψ_m satisfies the (PS) condition and from Lemma 2.4, we have

$$\beta_p(a_m, b_m, \Psi_m) = \mathrm{rank}(H_p(X, (\Psi_m)_{a_m})) = \delta_{pr}, \tag{2.44}$$

where $r = i_A(\gamma \cdot \mathrm{Id} \mid \alpha \cdot \mathrm{Id})$. Since $i_A(\gamma \cdot \mathrm{Id} \mid \alpha \cdot \mathrm{Id}) > i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1$, the right-hand side of the inequality is equal to 0. If Ψ_m has no nontrivial critical point with its Morse index less than $i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1$, the left-hand side of the inequality is equal to -1, which is a contradiction.

If Ψ_m has infinitely many critical points, assuming that for any $z \in \mathcal{K}_m^*$,

$$m^-(z) > i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1, \tag{2.45}$$

we use Lemma 2.5 and choose μ small enough, such that

(1) $0 \notin N_{2\mu}(\mathcal{K}_m^*)$, so 0 is also an isolated critical point of $\widehat{\Psi}_m$ and has the same Morse index $i_A(B_0 \mid \alpha \cdot \mathrm{Id})$.

(2) For any $z \in N_\mu(\mathcal{K}_m^*)$, $m^-(z)$, which is the dimension of the negative subspace of $\widehat{\Psi}''_m(z)$, satisfies $m^-(z) > i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1$. (Because Ψ_m is C^2 continuous, we can assume this.)

From the proposition (3) in Lemma 2.5, we have if z is a nontrivial critical point of $\widehat{\Psi}_m$, the Morse index $m^-_{\widehat{\Psi}_m}(z)$ satisfies

$$m^-_{\widehat{\Psi}_m}(z) > i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1. \tag{2.46}$$

Then choose $\widetilde{a}_m$ satisfying $N_{2\mu}(\mathcal{K}_m^*) \cap (\Psi_m)_{\widetilde{a}_m} = \emptyset$, that is $(\Psi_m)_{\widetilde{a}_m} = (\widehat{\Psi}_m)_{\widetilde{a}_m}$. So

$$H_q(X, (\Psi_m)_{\widetilde{a}_m}; \mathbb{R}) = H_q(X, (\widehat{\Psi}_m)_{\widetilde{a}_m}; \mathbb{R}) = \delta_{qr}\mathbb{R}. \tag{2.47}$$

Then $\widehat{\Psi}_m$ do not satisfy the $(i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1)^{\mathrm{th}}$ Morse inequality. It is a contradiction.

Step 2 Let $y_m = -\Lambda^{-1} z_m$, since $\Psi'_m(z_m) = 0$, that is $\Lambda^{-1} z_m + {N^*}'_m(z_m) = 0$. So we have $y_m = -\Lambda^{-1} z_m = {N^*}'_m(z_m)$, and y_m satisfies the equation $Ay_m + \Phi'_m(y_m) = 0$. If there is an $R > 0$ such that $\|y_m\| < R$, $m \in \mathbb{N}$, so from the definition of Φ_m, y_m is a nontrivial solution of equation (1.2) when m large enough.

We prove it indirectly and assume $\|y_m\| \to \infty$, as $m \to \infty$. From equation (2.13), we have $N''_m(y_m) \geq B_\infty - \alpha \cdot \mathrm{Id} > 0$ for m large enough. That is ${N_m^*}''(z_m) \leq (B_\infty - \alpha \cdot \mathrm{Id})^{-1}$.

Let $E_\infty^- = E^-(\Psi''_\infty(0))$. We have $\dim(E_\infty^-) = i_A(B_\infty \mid \alpha \cdot \mathrm{Id})$. For any $z \in E_\infty^-$,

$$\begin{aligned} \langle \Psi''_m(z_m)z, z\rangle &= \langle \Lambda^{-1} z, z\rangle + \langle {N_m^*}''(z_m)z, z\rangle \\ &\leq \langle \Lambda^{-1} z, z\rangle + \langle (B_\infty - \alpha \cdot \mathrm{Id})^{-1} z, z\rangle \\ &= \langle \Psi''_\infty(0)z, z\rangle \leq -\delta\|z\|^2. \end{aligned}$$

That is $m^-_{\Psi_m}(z_m) \geq i_A(B_\infty|\alpha \cdot \mathrm{Id})$, and $i_A(B_0 \mid \alpha \cdot \mathrm{Id}) + 1 \geq i_A(B_\infty|\alpha \cdot \mathrm{Id})$, which contradicts the fact that $i_A(B_\infty|\alpha \cdot \mathrm{Id}) - i_A(B_0 \mid \alpha \cdot \mathrm{Id}) = i_A(B_\infty) - i_A(B_0) > 1$. So $\|y_m\|$ are bounded and equation (1.2) has a nontrivial solution.

The proof of Theorem 1.3 is similar to that of Theorem 1.2. The difference is in Step 1. Instead of Morse theory, we make use of minimax arguments for multiplicity of critical points.

Let X be a Hilbert space and assume that $\phi \in C^2(X, \mathbb{R})$ is an even functional, satisfies the (PS) condition and $\phi(0) = 0$. Denote $S_a = \{u \in X \mid \|u\| = a\}$.

Lemma 2.6 (see [12, Corollary 10.19]) *Assume that Y and Z are subspaces of X satisfying $\dim Y = j > k = \mathrm{codim} Z$. If there exist $R > r > 0$ and $\alpha > 0$ such that*

$$\inf \phi(S_r \cap Z) \geq \alpha, \quad \sup \phi(S_R \cap Y) \leq 0,$$

then ϕ has $j - k$ pairs of nontrivial critical points $\{\pm x_1, \pm x_2, \cdots, \pm x_{j-k}\}$, so that $\mu(u_i) \leq k + i$, for $i = 1, 2, \cdots, j - k$.

Since Ψ is even, we have that Ψ_m is also even, and satisfies Lemma 2.1. Let $Y = E_\infty^-$, and $Z = E_m^+ = E^+(\Psi_m''(0))$. We have $\dim Y = i_A(B_\infty|\alpha\cdot \mathrm{Id})$, $\mathrm{codim} Z = i_A(B_0 \mid \alpha\cdot \mathrm{Id})$, $\dim Y >\mathrm{codim} Z$. Then it is easy to prove that Ψ_m satisfies Lemma 2.6 for R, and $\frac{1}{r}$ is large enough. So Ψ_m has $l := i_A(B_\infty) - i_A(B_0)$ pairs of nontrivial critical points

$$\{\pm x_1, \pm x_2, \cdots, \pm x_l\},$$

and $l-1$ pairs of them satisfy

$$m^-(x_i) \le i_A(B_0 \mid \alpha\cdot \mathrm{Id}) + i < i_A(B_\infty \mid \alpha\cdot \mathrm{Id}), \quad i = 1, 2, \cdots, l-1. \tag{2.48}$$

From (2.48), we complete the proof of Theorem 1.3.

3 Applications

We can use the abstract critical point Theorem1.2 and Theorem1.3 to deal with the existence and multiplicity problems of solutions of nonlinear elliptic equations as in [19], the periodic solutions of asymptotically linear Hamiltonian systems as in [18] and the Lagrangian boundary value problems of asymptotically linear Hamiltonian systems as in [14, 17]. To avoid tedious, in the following, we only show an application of the abstract critical point Theorem 1.2 and Theorem 1.3 to the problem of nonlinear Hamiltonian systems with P-periodic boundary conditions.

3.1 First order Hamiltonian systems

In this subsection, we consider the solutions of the nonlinear Hamiltonian systems

$$\begin{cases}\dot z(t) = JH'(t, z(t)), \quad t\in[0,1],\\ z(1) = Pz(0),\end{cases} \tag{3.1}$$

where $z(t)\in\mathbb{R}^{2n}$, $J = \begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}$ is the standard symplectic matrix, $N = \begin{pmatrix} -I_n & 0\\ 0 & I_n\end{pmatrix}$ with I_n the identity in $\mathbb{R}^n$ and $P\in \mathrm{Sp}(2n) = \{M\in \mathrm{GL}(\mathbb{R}^{2n}) \mid M^{\mathrm{T}}JM = J\}$, $H\in C^2([0,1]\times\mathbb{R}^{2n},\mathbb{R})$, and $H'(t,z)$ denotes the gradient of H with respect to the variable z.

Define $L^2 = L^2(0,1;\mathbb{R}^{2n})$, $\widetilde Y = \{z : [0,1]\to\mathbb{R}^{2n} \mid z'\in L^2 \text{ and } z(1) = Pz(0)\}$. Define $\widetilde A z := Jz'$ for every $z\in\widetilde Y$. By the spectral theory, there is a normal orthogonal basis $\{e_n\}$ of L^2, and a sequence $\{\lambda_n\}$, such that $\widetilde A e_n = \lambda_n e_n$, $\forall n\in\mathbb{Z}$. Then for every $z\in L^2$, with $z = \sum\limits_{n\in\mathbb{Z}} z_n e_n$, we have $\|z\|_{L^2}^2 = \sum\limits_{n\in\mathbb{Z}} z_n^2$. Define

$$X = \Big\{z\in L^2 \,\Big|\, \sum_{n\in\mathbb{Z}}(1+|\lambda_n|^{\frac12})|z_n|^2 < \infty\Big\}. \tag{3.2}$$

Then X is a separable Hilbert space with the norm $\|z\|^2 := \sum\limits_{n\in\mathbb{Z}}(1+|\lambda_n|^{\frac12})|z_n|^2$ and the corresponding inner product $(\cdot\,,\,\cdot)$. Define

$$Y = \Big\{z\in L^2 \,\Big|\, \sum_{n\in\mathbb{Z}}(1+|\lambda_n|^{\frac32})|z_n|^2 < \infty\Big\}. \tag{3.3}$$

Then Y is a separable Hilbert space with the norm $\|z\|_Y^2 := \sum\limits_{n\in\mathbb{Z}}(1+|\lambda_n|^{\frac32})|z_n|^2$ and the corresponding inner product $(\cdot\,,\,\cdot)_Y$. Define the operator $A: Y\to X$ by

$$(Ay, x) = \sum_{n\in\mathbb{Z}} \lambda_n x_n y_n, \quad \forall\, x\in X,\ y\in Y, \tag{3.4}$$

where $x = \sum_{n\in\mathbb{Z}} x_n e_n,\ y = \sum_{n\in\mathbb{Z}} y_n e_n$. It is easy to check that (X, Y, A) satisfy the conditions introduced in Section 1. Define $K : L^2 \to X$ by

$$(Kz, x) = (z, x)_{L^2}, \quad \forall\, x \in X,\ z \in L^2, \tag{3.5}$$

and it is easy to see that K is a compact operator. Define $\Phi : X \to \mathbb{R}$ by

$$\Phi(x) = \int_0^1 H(t, x(t))\mathrm{d}t. \tag{3.6}$$

If there exists a constant $C > 0$, such that $\|H''(t, z)\| \le C,\ \forall\, t \in [0, 1], z \in \mathbb{R}^{2n}$, then we have $\Phi \in C^2(X, \mathbb{R})$, and $\Phi'(x) = KH'(t, x(t)),\ \Phi''(x) = KH''(t, x(t))$. If $z \in Y$, satisfying $Az + \Phi'(z) = 0$, we have that z is a solution of (3.1).

Similarly to Theorems 1.2 and 1.3, we have the following results.

Theorem 3.1 *Assume $H \in C^2([0, 1] \times \mathbb{R}^{2n}, \mathbb{R})$ which satisfies*

(H) *There exists an $M > 0$ such that*

$$|H''(t, z)| \le M, \quad \forall (t, z) \in [0, 1] \times \mathbb{R}^{2n},$$

(H_0) *$H'(t, 0) \equiv 0$, $t \in [0, 1]$, $H''(t, 0) = B_0(t)$ and $\nu_A(KB_0(t)) = 0$,*

($\mathrm{H}_\infty^\pm$) *There exists a continuous symmetric matrix function $B_\infty(t)$, and some $R > 0$, such that*

$$H''(t, z) \ge B_\infty (\text{or } H''(t, z) \le B_\infty) \quad \text{for all } t \in [0, 1] \text{ and } |z| > R,$$

(H_t) *$i_A(KB_\infty) > i_A(KB_0) + 1($ or $i_A(KB_\infty) < i_A(KB_0) - 1)$.*

Then (3.1) has at least one nontrivial solution.

Theorem 3.2 *Assume that the conditions in Theorem 3.1 are all satisfied and further more H is even in z. Then (3.1) has at least $|i_A(B_\infty) - i_A(B_0)| - 1$ pairs of nontrivial solutions.*

Remark 3.1 The cases of (H_∞^+) and (H_∞^-) are similar. In fact, the case (H_∞^-) follows from the case (H_∞^+) by applying to the Hamiltonian function $-H(1-t, z)$. So we only consider (H_∞^+) from now on. By Remark 1.1, it does not lose any generality, if we can assume $\nu_A(B_\infty) = 0$.

The existence and multiplicity for nonlinear Hamiltonian systems with P-boundary conditions was first studied by the first author in [13], where the conditions on H are more restricted in some sense.

The proofs of Theorems 3.1 and 3.2 are similar to that of Theorems 1.2 and 1.3. Here we only give a brief statement. Similarly to Lemma 2.1, there exists a sequence of Hamiltonian functions $H_m \in C^2([0, 1] \times \mathbb{R}^{2n})$ satisfying the following properties:

(1) there exists an increasing sequence of real numbers $R_m \to \infty$ $(m \to \infty)$ such that

$$H_m(t, z) \equiv H(t, z), \quad \forall\, t \in [0, 1],\ |z| \le R_m,$$

(2) for each $m \in \mathbb{N}$,

$$H''_m(t, z) \ge B_\infty(t), \quad \forall\, t \in [0, 1],\ |z| \ge R,$$

(3) there exists an $\widetilde{M} > 0$, such that

$$\|H''_m\| \le \widetilde{M}, \quad \forall\, t \in [0, 1],\ z \in \mathbb{R}^{2n},\ m \in \mathbb{N},$$

(4) there exist a $\gamma \in \mathbb{R}$, satisfying $\gamma K > KB_\infty$, $\nu_A(\gamma K) = 0$, and a $C_m > 0$, such that

$$\|H'_m(t,z) - \gamma z\| < C_m \quad \text{for all } t \in [0,1],\ z \in \mathbb{R}^{2n}, \quad H''_m(t,z) - \gamma = o(|z|), \quad \text{as } |z| \to \infty.$$

So we can choose $a, b \in \mathbb{R}$ with $\nu_A(a \cdot \mathrm{Id}) = \nu_A(b \cdot \mathrm{Id}) = 0$, $KB_\infty - a \cdot \mathrm{Id} \geq \mathrm{Id}$, and satisfying

$$b \cdot \mathrm{Id} \geq N''_m(z) \geq \mathrm{Id}, \quad \forall z \in X, \tag{3.7}$$

where $N_m(z) = \int_0^1 H_m(t, z(t))\mathrm{d}t - \frac{a}{2}(z,z)_X$. Define

$$N(z) = \int_0^1 H(t, z(t))\mathrm{d}t - \frac{a}{2}(z,z)_X, \tag{3.8}$$

$$N_\infty(z) = \frac{1}{2}(KB_\infty z, z)_X - \frac{a}{2}(z,z)_X, \tag{3.9}$$

$$\widetilde{N}_\gamma(z) = \frac{1}{2}(\gamma K z, z)_X - \frac{a}{2}(z,z)_X. \tag{3.10}$$

Let $\Lambda = A + a \cdot \mathrm{Id}$. From $\nu_A(a \cdot \mathrm{Id}) = 0$, we have that Λ is inversible and Λ^{-1} is compact. Define

$$\Psi(z) = \frac{1}{2}(\Lambda^{-1}z, z) + N^*(z), \tag{3.11}$$

$$\Psi_m(z) = \frac{1}{2}(\Lambda^{-1}z, z) + N^*_m(z), \tag{3.12}$$

$$\widetilde{\Psi}_\gamma(z) = \frac{1}{2}(\Lambda^{-1}z, z) + \widetilde{N}^*_\gamma(z), \tag{3.13}$$

$$\Psi_\infty(z) = \frac{1}{2}(\Lambda^{-1}z, z) + N^*_\infty(z). \tag{3.14}$$

With similar arguments as in Section 2, we see that N^*_m satisfies Lemmas 2.2, Ψ_m satisfies Lemmas 2.3–2.5. So Ψ_m possesses a nontrivial critical point z_m with Morse index satisfying $m^-(z_m) \leq i_A(KB_0 \mid a \cdot \mathrm{Id}) + 1$. Denote by $-\Lambda^{-1}z_m = y_m$. We have that y_m satisfies the following equations:

$$\begin{cases} \dot{y}_m(t) = JH'_m(t, y_m(t)), \\ y_m(1) = Py_m(0). \end{cases}$$

If there exists a $C > 0$ independent of m such that $\|y_m\|_{L^\infty} \leq C$, then y_m is a nontrivial solution of the original equations (3.1) for m large enough. Otherwise, if $\|y_m\|_{L^\infty} \to \infty$ as $m \to \infty$, by the same arguments as in the last part of [18], we have $\min\limits_{t\in[0,1]} |y_m(t)| \geq R$ for m large enough. So from the definition of H_m, we have $H''_m(t, y_m) \geq B_\infty$. Thus we have $N^{*\prime\prime}(z_m) \leq (KB_\infty - a \cdot \mathrm{Id})^{-1}$. In this case, we have the contradiction as done in Section 2. That is to say $\|y_m\|_{L^\infty}$ is bounded, so y_m is a nontrivial solution of the original equations (3.1).

3.2 Second order Hamiltonian systems

In this subsection, we consider the solutions of the nonlinear Hamiltonian system

$$\begin{cases} (\Lambda(t)x')' + V'(t,x) = 0, \\ x(1) = Mx(0),\ x'(1) = Nx'(0), \end{cases} \tag{3.15}$$

where $M \in \mathrm{GL}(n)$, $M^\tau \Lambda(1) N = \Lambda(0)$, $\Lambda \in C([0,1]; \mathrm{GL}_s(n))$ and $\Lambda(t)$ is positive definite for every $t \in [0,1]$.

By the similar argument, let

$$\widetilde{Y} = \{x : [0,1] \to \mathbb{R}^n \mid (\Lambda(t)x'(t))' \in L^2(0,1;\mathbb{R}^n),\ x(1) = Mx(0),\ x'(1) = Nx'(0)\}. \tag{3.16}$$

We have a normal orthogonal basis $\{f_n\}$ of $L^2(0,1;\mathbb{R}^n)$ and a sequence $\{\eta_n\}$, such that $(\Lambda(t)x'(t))'f_n = \eta_n f_n$, $\forall n \in \mathbb{Z}$. For every $x \in L^2(0,1;\mathbb{R}^n)$, we have $x = \sum\limits_{n\in\mathbb{Z}} x_n f_n$. Define

$$X = \Big\{x \in L^2(0,1;\mathbb{R}^n) \,\Big|\, \sum_{n\in\mathbb{Z}} (1 + |\eta_n|^{\frac{1}{2}}) x_n^2 < \infty\Big\}. \tag{3.17}$$

Then X is a separable Hilbert space with the norm $\|x\|^2 := \sum\limits_{n\in\mathbb{Z}} (1 + |\eta_n|^{\frac{1}{2}})|x_n|^2$ and the corresponding inner product $(\,\cdot\,,\,\cdot\,)$. Define

$$Y = \Big\{y \in (0,1;\mathbb{R}^n) \,\Big|\, \sum_{n\in\mathbb{Z}} (1 + |\eta_n|^{\frac{3}{2}})|y_n|^2 < \infty\Big\}. \tag{3.18}$$

Then Y is a separable Hilbert space with the norm $\|y\|_Y^2 := \sum\limits_{n\in\mathbb{Z}} (1 + |\eta_n|^{\frac{3}{2}})|y_n|^2$ and the corresponding inner product $(\,\cdot\,,\,\cdot\,)_Y$. Define the operator $A : Y \to X$ by

$$(Ay, x) = \sum_{n\in\mathbb{Z}} \eta_n x_n y_n, \quad \forall x \in X,\ y \in Y, \tag{3.19}$$

where $x = \sum\limits_{n\in\mathbb{Z}} x_n f_n$, $y = \sum\limits_{n\in\mathbb{Z}} y_n f_n$. Then we have that (X, Y, A) satisfies the conditions introduced in Section 1. We have the following results.

Theorem 3.3 *Assume that* $V \in C^2([0,1] \times \mathbb{R}^n, \mathbb{R})$ *satisfies*

(V) *there exists an* $M > 0$ *such that*

$$|V''(t,x)| \le M, \quad \forall\,(t,x) \in [0,1] \times \mathbb{R}^n,$$

(V_0) $V'(t,0) \equiv 0$, $t \in [0,1]$, $V''(t,0) = B_0(t)$ *and* $\nu_A(B_0(t)) = 0$,

(V_∞) *there exists a continuous symmetric matrix function* $B_\infty(t)$*, and some* $R > 0$*, satisfying*

$$V''(t,x) \ge B_\infty \quad \text{for all } t \in \mathbb{R} \text{ and } |x| > R,$$

(V_t) $i_A(B_\infty) > i_A(B_0) + 1$.

Then (3.15) *has at least one nontrivial solution.*

Theorem 3.4 *Assume that the conditions in Theorem* 3.3 *are all satisfied and further more* V *is even in* x*. Then* (3.15) *has at least* $i_A(B_\infty) - i_A(B_0) - 1$ *pairs of nontrivial solutions.*

The proofs of the above two results are similar to that of Theorems 3.1 and 3.2, in fact, problem (3.15) can be transferred to problem (3.1).

References

[1] Amann, H. and Zehnder, E., Nontrivial soultions for a class of nonresonance problems and applications to nonlinear differential equations, *Annali Scuola Norm. Sup. Pisa.*, **7**, 1980, 439–603.

[2] Chang, K. C., Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhauser, Basel, 1993.

[3] Chang, K. C., Solutions of asymptotically linear operator equations via Morse theory, *Comm. Pure Appl. Math.*, **34**, 1981, 693–712.

[4] Chang, K. C., Critical Point Theory and Its Application (in Chinese), Shanghai Sci. Tech. Press, Shanghai, 1986.

[5] Conley, C. and Zehnder, E., Morse-type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure Appl. Math.*, **37**, 1984, 207–253.

[6] Dong, Y., Index theory for linear self-adjoint operator equations and nontrivial solutions for asymptotically linear operator equations, *Calc. Var.*, **38**, 2010, 75–109.

[7] Dong, D. and Long, Y., The iteration formula of Maslov-type index theory with applications to nonlinear Hamiltonian systems, *Trans. Amer. Math. Soc.*, **349**, 1997, 2619–2661.

[8] Ekeland, I., Convexity methods in Hamiltonian mechanics, Ergebnisse der Mathematik, Springer, Berlin, 1990.

[9] Ekeland, I., Une theorie de Morse pour les systemes hamiltoniens convexes, *Ann IHP "Analyse non lineaire"*, **1**, 1984, 19–78.

[10] Ekeland, I. and Hofer, H., Periodic solutions with prescribed period for convex autonomous Hamitonian systems, *Invent. Math.*, **81**, 1985, 155–188.

[11] Ekeland, I. and Hofer, H., Convex Hamiltonian energy surfaces and their closed trajectories, *Comm. Math. Phys.*, **113**, 1987, 419–467.

[12] Ghoussoub, N., Duality and Perturbation Methods in Critical Point Theory, Cambridge University Press, Cambridge, 1993.

[13] Liu, C., Maslov-type P-index theory for a symplectic path with applications, *Chin. Ann. Math.*, **27**B(4), 2006, 441–458.

[14] Liu, C., Asymptotically linear Hamiltonian system with Lagrangian boundary conditions, *PJM*, **232**, 2007, 232–254.

[15] Liu, C., Maslov-type index theory for symplectic paths with Lagrangian boundary conditions, *Advanced Nonlinear Studies*, **7**, 2007, 131–161.

[16] Liu, C., Long, Y. and Zhu, C., Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbb{R}^{2n}$, *Math. Ann.*, **323**, 2002, 201–215.

[17] Liu, C., Wang, Q. and Lin, X., An index theory for symplectic paths associated with two Lagrangian subspaces with applications, *Nonlinearity*, **24**, 2011, 43–70.

[18] Liu, Z., Su, J. and Wang, Z., A twist condition and periodic solutions of Hamiltonian system, *Adv. Math.*, **218**, 2008, 1895–1913.

[19] Liu, Z., Su, J. and Wang, Z., Solutions of elliptic problems with nonlinearities of linear growth, *Calc. Var.*, **35**, 2009, 463–480.

[20] Long, Y., Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems, *Sci. China*, **33**, 1990, 1409–1419.

[21] Long, Y., A Maslov-type index theory for symplectic paths, *Topol. Methods Nonlinear Anal.*, **10**, 1997, 47–78.

[22] Long, Y. and Zehnder, E., Morse theory for forced oscillations of asymptotically linear Hamiltonian systems, Stock. Process. Phys. Geom., S. Alberverio et al (eds.), World Scientific Publishing, Teaneck, New Jersey, 1990, 528–563.

[23] Long, Y. and Zhu, C., Closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$, *Ann. Math.*, **155**, 2000, 317–368.

[24] Long, Y. and Zhu, C., Maslov type index theorey for symplectiuc paths and spectral flow (II), *Chin. Ann. Math.*, **21B**(1), 2000, 89–108.

[25] Solimini, S., Morse index estimates in min-max theorems, *Manuscripta Math.*, **63**, 1989, 421–453.

[26] Zhu, C. and Long, Y., Maslov type index theorey for symplectiuc paths and spectral flow (I), *Chin. Ann. Math.*, **20B**(4), 1999, 413–424.